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The $\chi^{2\Delta}{}_{fA}$ defined by modulus

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ABSTRACT. In this paper we introduce $\chi^{2\Delta}{}_{fA}$ sequence spaces defined by modulus function and study general properties of these spaces and also establish some inclusion results among them. Also P – statistical χ^2 sequence spaces is defined and discuss about general properties.

Keywords: gai sequence, analytic sequence, modulus function, double sequences, $\chi^{2\Delta}{}_{fA}$, P-convergence, difference sequence, RH-regular, Prinsheim sense, statistical convergence.

O $\chi^{2\Delta}{}_{fA}$ definido por módulo

RESUMO. Este estudo apresenta espaços de sequência $\chi^{2\Delta}{}_{fA}$ definidos pela função módulo e analisa propriedades gerais desses espaços, estabelecendo alguns resultados de inclusão entre eles. Além disso, espaços de sequência χ^2 p-estatísticos são definidos e discutidos sobre as propriedades gerais.

Palavras-chave: sequência gai, sequência analítica, função módulo, sequências duplas, $\chi^{2\Delta}{}_{fA}$, convergência P, sequência de diferença, regular RH, sentido Prinsheim, convergência estatística.

Introduction

Throughout w, x and $\wedge$ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m,n \in \aleph$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces are found in Bromwich (1965). Later on they were investigated by Hardy (1917), Moricz (1991), Moricz and Rhoades (1988), Basarir and Solancan (1999), Tripathy (2003), Turkmenoglu (1999) and many others.

Let us define the following sets of double sequences:

$$\mathrm{M}_u(t) = \left\{(x_{mn}) \in w^2 : \sup_{mn\in\aleph} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$C_p(t) := \left\{\begin{array}{l}(x_{mn}) \in w^2 : p-\lim_{mn\to\infty} |x_{mn} - L|^{t_{mn}} = 1 \\ \text{for some } L \in C\end{array}\right\},$$

$$C_{0p}(t) := \left\{(x_{mn}) \in w^2 : p-\lim_{mn\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$L_u(t) = \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$c_{bp}(t) := c_p(t) \cap \mathrm{M}_u(t) \text{ and } C_{0bp}(t) = C_{0p}(t) \cap \mathrm{M}_u(t);$$

where:

$t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m,n \in \aleph$ and $p-\lim_{mn\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m,n \in \aleph$; $\mathrm{M}_u(t), C_p(t), C_{0p}(t), L_u(t), C_{bp}(t)$ reduce to the sets $\mathrm{M}_u, C_p, C_{0p}, L_u, C_{bp}$ and C_{0bp}, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gokhan and Colak (2004, 2005) have proved that $\mathrm{M}_u(t)$ and $C_p(t), C_{bp}(t)$ are complete paranormed spaces of double sequences and calculated the $\alpha, \beta, \gamma-$duals of the spaces $\mathrm{M}_u(t)$ and $C_{bp}(t)$. Quite recently, Zeltser has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences Mursaleen et al. (2003) have recently introduced the notion of statistical convergence and statistically Cauchy for double sequences independently and proved a relation between statistical convergent and strongly Cesaro summable double sequences. Mursaleen (2004) and Mursaleen and Edely (2004) have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determned those four dimensional matrices transforming every bounded double sequences $x=(x_{jk})$ into one whose core is a subset of the M-core of x. Altay et al. (2005) have defined the spaces $BS, BS(t), CS_p, CS_{bp}, CS_r$ and BV of double sequences consisting of all double series whose sequence of partial sums are in the spaces $M_u, M_u(t), C_p, C_{bp}, C_r$ and L_u, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$duals of the spaces BS, BV, CS_{bp} and the $\beta(v)$-duals of the spaces CS_{bp} and CS_r of double series. Basar and Sever (2009) have introduced the Banach space L_q of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space L_q Subramanian and Misra (2010) have studied the space $\chi^2{}_M(p,q,u)$ of double sequences and proved some inclusion relations.

Spaces are strongly summable sequences were discussed by Kuttner (1946) and Maddox (1979) and others. The class of sequences which are strongly Cesaro summable with respect to a modulus was introduced by Maddox (1986) as an extension of the definition of strongly Cesaro summable sequences. Cannor (1989) further extended this definition and introduced the notion of strong A – summability with respect to a modulus where $A=(a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A – summability, strong A – summability with respect to a modulus, and A – statistical convergence. In Pringsheim (1900) the notion of convergence of double sequences was presented by a Pringsheim. Also, in Hamilton (1936, 1938, 1939) the four dimensional matrix transformation $(Ax)_{k\ell}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{k\ell}{}^{mn}x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a,b\geq 0$ and $0<p<1$, we have

1.1 $$(a+b)^p \leq a^p + b^p$$

The double series $\sum_{m,n=1}^{\infty}x_{mn}$ is called convergent if and only if the double sequence (S_{mn}) is convergent, where $S_{mn}=\sum_{i,j=1,1}^{m,n}x_{ij}\,(m,n\in\aleph)$.

A sequence $x=(x_{mn})$ is said to be double analytic if $\sup_{mn}|x_{mn}|^{1/m+n}<\infty$. The vector space of all double analytic sequences will be denoted by $\wedge^2$. A sequence $x=(x_{mn})$ is called double gai sequence if $((m+n)!|x_{mn}|)^{1/m+n}\to 0$ as $m,n\to\infty$. The double gai squences will be denoted by χ^2. Let $\phi=\{\text{all finite sequences}\}$.

Consider a double sequence $x=(x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]}=\sum_{i,j=0}^{m,n}x_{ij}\Im_{ij}$ for all $m,n\in\aleph$; where $\Im_{ij}$ denotes the double sequences whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j\in\aleph$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]}\to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x=(x_k)\to(x_{mn})(m,n\in\aleph)$ are also continuous.

Orlicz (1936) used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri (1971) investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p(1\leq p<\infty)$. Subsequently, different classes of sequence spaces were defined by Parashar et al. (1994), Bektas et al. (2003), Altin and Tripathy (2004), Chandrasekhara Rao and Subramanian (2004) and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Krasnoselskii and Rutickii (1961).

Recalling Orlicz (1936) and Krasnoselskii and Rutickii (1961) an Orlicz function Is a function $M:[0,\infty)\to[0,\infty)$ which is continuous, non-decreasing, and convex with $M(0)=0, M(x)>0$, for $x>0$ and $M(x)\to\infty$ as $x\to\infty$. If the convexity of Orlicz function M is r eplaced by subadditivity of M, then this function is called the modulus function, defined by Nakano (1953) and further discussed by Ruckle (1973) and Maddox (1986) and many others.

An modulus function $M$is said to satisfy the Δ_2-condition for small u or at 0 if for each $k\in\aleph$, there exist $R_k>0$ and $u_k>0$ such that $M(ku)\leq R_kM(u)$ for all $u\in(0,u_k]$. Moreover, an modulus function $M$is said to satisfy the Δ_2-condition if and only if

$$\lim_{u\to 0+}\sup\frac{M(2u)}{M(u)}<\infty$$

Two modulus functions M_1 and M_2 are said to be equivalent if there are positive constants

α, β and b

Such that

$$M_1(\alpha u) \leq M_2(u) \leq M_1(\beta u) \text{ for all } u \in [0, b].$$

An modulus function M can always be represented in the following integral form

$$M(u) = \int_0^u \eta(t)dt$$

where:

η, the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$ for $t > 0, \eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$ whenever $\frac{M(u)}{u} \uparrow \infty$ as $u \uparrow \infty$.

Consider the kernel η associated with the modulus function M and let

$$\mu(s) = \sup\{t : \eta(t) \leq s\}$$

Then μ possesses the same properties as the function η. Suppose now

$$\Phi = \int_0^x \mu(s)ds.$$

Then, Φ is an modulus function. The functions M and Φ are called mutually complementary Orlicz functions.

Now, we give the following well-known results.

Let M and Φ be mutually complementary Orlicz functions. Then, we have:

(i) For all $u, y \geq 0$,

1.2 $$uy \leq M(u) + \Phi(y), \text{ (Young's inequality)}$$

(ii)For all $u \geq 0$,

1.3 $$u\eta(u) = M(u) + \Phi(\eta(u)).$$

(iii)For all $u \geq 0$, and $0 < \lambda < 1$,

1.4 $$M(\lambda u) \leq \lambda M(u)$$

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0, \right\}$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leq p < \infty)$, the space ℓ_M coincide with the classical sequence space ℓ_p.

If $X X$ is a sequence space, we procure the following definitions:

(i) X'= the continuous dual of X;

(ii) $$X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \text{ for each } x \in X \right\};$$

(iii) $$X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn} \text{ is convergent, for each } x \in X \right\};$$

(iv) $$X^{\gamma} = \left\{ a = (a_{mn}) : \sup_{mn \geq 1} \left| \sum_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, \text{ for each } x \in X \right\};$$

(v) let X be an FK-space $\supset \Phi$; then $X^f = \left\{ f(\Im_{mn}) : f \in X' \right\}$;

(vi) $$X^{\delta} = \left\{ \begin{array}{l} a = (a_{mn}) : \sup_{mn} [a_{mn} x_{mn}]^{1/m+n} < \infty, \\ \text{for each } x \in X \end{array} \right\};$$

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called $\alpha -$(or Kothe-Toeplitz dual of X, β-(or generalized-Kothe-Toeplitz) dual of $X, \gamma -$dual of $X, \delta -$dual of X respectively. X^{α} is found in Kamthan et al. (1981). It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz (1981) as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\} \text{ for}$$

$$Z = c, c_0 \text{ and } \ell_{\infty},$$

where:

$$\Delta x_k = x_k - x_{k+1}$$

for all $k \in \aleph$.

Here c, c_0 and ℓ_{∞} denote the classes of convergent, null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Basar and Atlay (2003) and in the case $0 < p < 1$ by Altay et al. (2007). The spaces $c(\Delta), c_0(\Delta), \ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$\|x\| = |x_1| + \sup_{k\geq 1}|\Delta x_k|$ and $\|x\|_{bv_p} = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}$ $,(1 \leq p < \infty)$.

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where:

$Z = \wedge^2, \chi^2$ and

$\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \aleph$

Definitions and preliminaries

Definition

A modulus function was introduced by Nakano (1953). We recall that a modulus f is a function from $[0,\infty) \to [0,\infty)$, such that

(1) $f(x) = 0$ if and only if $x = 0$

(2) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0, y \geq 0$,

(3) f is increasing,

(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from here that f is continuouson $[0,\infty)$.

Definition

Let $A = (a_{k\ell}^{mn})$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the k,ℓ – th term to Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$$

Such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Robison (1926) and Toeplitz (1911). Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analogue of regularity for double sequences in which they both added an additional assumption of boundedness. This assumption was made because a double sequence which is $P-$ convergent is not necessarily bounded.

Definition

A double sequence $x = (x_{mn})$ has a Pringsheim limit L(denoted by $P-\lim x = L$) provided that given on $\varepsilon > 0$ there exists $N \in \aleph$ such that $|x_{mn} - L| < \in$ whenever $m, n > N$. We shall describe such an $x = (x_{mn})$ more briefly as '$P-$convergent'.

The four dimensional matrix A is said to be RH-regular if it maps every bounded P – convergent sequence into a P – convergent sequence with the same P – limit. The assumption of boundedness was made because a double sequence which is P – convergent is not necessarily bounded. Using this definition Robison and Hamilton independently, both presented the following Silverman-Toeplity type characterization of RH-regularity.

Theorem

The four dimensional matrix X is RH-regular if and only if

$RH_1 : P-\lim_{jk} a_{jk}^{mn} = 0$ for each m and n,

$RH_2 : P-\lim_{jk} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{jk}^{mn} = 1$,

$RH_3 : P-\lim_{jk} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{jk}^{mn} = 0$, for each n,

$RH_4 : P-\lim_{jk} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{jk}^{mn} = 0$, for each m,

$RH_5 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{jk}^{mn}|$ is $P-$convergent,

$RH_6 :$ There exist finite positive integers A and B such that $\sum_{m,n>B} |a_{jk}^{mn}| < A\cdot$

Definition

Let f be an modulus function and $A = (a_{jk}^{mn})$ be a nonnegative RH-regular summability matrix method. We now define the following new double sequence spaces:

$$\chi_{fA}^{2\Delta} = \left\{ \begin{array}{l} x = (x_{mn}) \in w^2 : P-\lim_{jk} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn} \\ \left(f\left((m+n)!|\Delta x_{mn}|\right)^{1/m+n}\right) = 0 \end{array} \right\},$$

and

$$\wedge_{fA}^{2\Delta} = \left\{ \begin{array}{l} x = (x_{mn}) \in w^2 : \sup_{jk} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn} \left(f(|\Delta x_{mn}|)^{1/m+n}\right) \\ < \infty \end{array} \right\}$$

when

$f(x) = x$, for all $x \in [0,\infty)$ we have $\chi_{fA}^{2\Delta} = \chi_A^{2\Delta}$ and $\wedge_{fA}^{2\Delta} = \wedge_A^{2\Delta}$.

Some spaces are defined by specializing A and f. If $A = (C,1,1)$ the difference sequence spaces defined above become $\chi^2{}_f$ and $\wedge^2{}_f$ which are as follows:

$$\chi_f{}^{2\Delta} = \left\{ \begin{array}{l} x = (x_{mn}) \in w^2 : P - \lim_{jk} \frac{1}{jk} \sum_{m=0}^{j-1} \sum_{n=0}^{k-1} a_{jk}{}^{mn} \\ \left(f\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right) = 0 \end{array} \right\},$$

and

$$\wedge_f{}^{2\Delta} = \left\{ \begin{array}{l} x = (x_{mn}) \in w^2 : \sup_{jk} \frac{1}{jk} \sum_{m=0}^{j-1} \sum_{n=0}^{k-1} a_{jk}{}^{mn} \\ \left(f\left(|\Delta x_{mn}| \right)^{1/m+n} \right) < \infty \end{array} \right\}.$$

Let
$A = (C,1,1)$ and $f(x) = x$, for all $x \in [0, \infty)$ we have
$\chi_{fA}{}^{2\Delta} = \chi_A{}^{2\Delta}$ and $\wedge_{fA}{}^{2\Delta} = \wedge_A{}^{2\Delta}$.

Main results

Theorem

The class of sequences $\chi_{fA}{}^{2\Delta}$ and $\wedge_{fA}{}^{2\Delta}$ are linear spaces.

Proof: It is routine verification. Therefore the proof is omitted

Theorem

If $0 < h \le H < \infty$, where $h = \inf p_{mn}$ and $H = \sup_{mn}$, then any modulus function f and a non-negative RH-regular summability matrix method A, then $\chi_A{}^{2\Delta} \subset \chi_{fA}{}^{2\Delta}$.

Proof: Let $0 < h \le H < \infty$, and $x = (x_{mn}) \in \chi_{fA}{}^{2\Delta}$, let
$0 < \varepsilon < 1$ and δ with $0 < \delta < 1$ such that
$f(t) < \varepsilon$ for $0 \le t < \delta$.

We can write for each j and k.

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{jk}{}^{mn} \left(f\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right) =$$
$$\sum_{m=0}^{\infty} \sum_{n=0 \text{ and } ((m+n)! |\Delta x_{mn}|)^{1/m+n} \le \delta}^{\infty} a_{jk}{}^{mn} \left(f\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right) +$$
$$\sum_{m=0}^{\infty} \sum_{n=0 \text{ and } ((m+n)! |\Delta x_{mn}|)^{1/m+n} > \delta}^{\infty} a_{jk}{}^{mn} \left(f\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right)$$

$$\sum_{m=0}^{\infty} \sum_{n=0 \text{ and } ((m+n)! |\Delta x_{mn}|)^{1/m+n} \le \delta}^{\infty} a_{jk}{}^{mn} \left(f\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right)$$
$$\le \varepsilon^H \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{jk}{}^{mn} \tag{3.1}$$

On the other hand, we use the fact that

$$\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} < 1 + \left[\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right]$$

where $[|t|]$ denotes the integer part of t. Since f is modulus function we have

$$\left(f\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right) \ge f(1)$$

Now, let us consider the second part where the sum is taken over $\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} > \delta$. Thus

$$\sum_{m=0}^{\infty} \sum_{n=0 \text{ and } ((m+n)! |\Delta x_{mn}|)^{1/m+n} > \delta}^{\infty} a_{jk}{}^{mn} \left(f\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right) \le$$
$$\sum_{m=0}^{\infty} \sum_{n=0 \text{ and } ((m+n)! |\Delta x_{mn}|)^{1/m+n} > \delta}^{\infty} a_{jk}{}^{mn} \left(f\left(1 + (m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right)$$
$$\le \left(2 f(1) \delta^{-1} \right)^H \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{jk}{}^{mn} \left(f\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right).$$

From equation (3.1) and RH-regularity of A. Hence $x = (x_{mn}) \in \chi_{fA}{}^{2\Delta}$.

Proposition

$\chi_{fA}{}^{2\Delta} \subset \wedge_{fA}{}^{2\Delta}$.

Proof: The proof is easy, so omitted.

Theorem

(1) $0 < h = \inf p_{mn} \le 1$, then $\chi_{fA}{}^{2\Delta_p} \subset \chi_{fA}{}^{2\Delta}$

(2) If $1 \le p_{mn} \le \sup p_{mn} < \infty$, then $\chi_{fA}{}^{2\Delta} \subset \chi_{fA}{}^{2\Delta_p}$

Proof: (1) Let $x = (x_{mn}) \in \chi_{fA}{}^{2\Delta}$,

since $0 < h = \inf p_{mn} \le 1$, we obtain the following:

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{jk}{}^{mn} \left(f\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right) \le$$
$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{jk}{}^{mn} \left[\left(f\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right) \right]^{p_{mn}}.$$

Thus $x = (x_{mn}) \in \chi_{fA}{}^{2\Delta}$.

Proof: (2) Let $p_{mn} \ge 1$ for each m, n and $\sup_{mn} < \infty$ and $x = (x_{mn}) \in \chi_{fA}{}^{2\Delta}$. Then for each $0 < \varepsilon < 1$ there exists a positive integer K such that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{jk}{}^{mn} \left(f\left((m+n)! |\Delta x_{mn}| \right)^{1/m+n} \right) \le \varepsilon < 1$$

for all $j, k \ge K$.

This implies that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left[\left(f\left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n}\right)\right] \subset .$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left[\left(f\left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n}\right)\right]$$

Thus $x = (x_{mn}) \in \chi_{fA}^{2\Delta_p}$.

Theorem

$\chi_{fA}^{2} \subset \chi_{fA}^{2\Delta}$ and the inclusion is strict where

$$\chi_{fA}^{2\Delta} = \left\{ \begin{array}{l} x = (x_{mn}) \in w^2 : P-\lim_{jk} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn} \\ \left(f\left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n}\right) = 0 \end{array} \right\},$$

Proof: Let $x = (x_{mn}) \in \chi_{fA}^{2}$.

then

3.2 $$P-\lim_{jk} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left(f\left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n}\right) = 0$$

Now for $\left|\Delta x_{mn}\right| = \left|x_{mn} - x_{mn+1} - x_{m+1n} - x_{m+1n+1}\right|$,

we have

$$P-\lim_{jk} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left(f\left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n}\right) \leq$$

$$P-\lim_{jk} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left(f\left((m+n)!\left(\begin{array}{l} \left|x_{mn}\right| + \left|x_{mn+1}\right| + \\ \left|x_{m+1n}\right| + \left|x_{m+1n+1}\right| \end{array} \right) \right)^{1/m+n} \right)$$

$\leq$

$$D^2 P-\lim_{jk} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}$$
$$\left(f\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}\right) + \left(f\left((m+n)!\left|x_{mn+1}\right|\right)^{1/m+n}\right) +$$
$$\left(f\left((m+n)!\left|x_{m+1n}\right|\right)^{1/m+n}\right) + \left(f\left((m+n)!\left|x_{m+1n+1}\right|\right)^{1/m+n}\right)$$

$$\leq D^2 P-\lim_{jk} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}$$
$$\left(f\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}\right) + \left(f\left((m+n)!\left|x_{mn+1}\right|\right)^{1/m+n}\right) +$$
$$\left(f\left((m+n)!\left|x_{m+1n}\right|\right)^{1/m+n}\right) + \left(f\left((m+n)!\left|x_{m+1n+1}\right|\right)^{1/m+n}\right) = 0.$$

where:
$D = \max(1, 2^{H-1})$

Thus $x = (x_{mn}) \in \chi_{fA}^{2\Delta}$.

The inclusion is strict follows from the following example:

Example

Consider the sequence $x = (x_{mn})$ defined by $x_{mn} = \frac{(m+n)^{m+n}}{(m+n)!}$ for all $m, n \in \aleph$. We have $\Delta x_{mn} = 0$ for all m, n. Hence $x = (x_{mn}) \in \chi_{fA}^{2\Delta}$ but $x = (x_{mn}) \notin \chi_{fA}^{2}$.

Theorem

$\wedge_{fA}^{2\Delta}$ are complete linear topological spaces with the paranorm (not necessarily totally)

$$g(x_{mn}) = \sup_m \left|x_{m,1}\right|^{1/m+1} + \sup_n \left|x_{1,n}\right|^{1/1+n} +$$
$$\inf\left\{ \sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left(f\left(\left|\Delta x_{mn}\right|\right)^{1/m+n}\right)\right)^{1/T} \leq 1 \right\}$$

where $T = \max(1, H), H = \sup_{mn} p_{mn}$.

Proof: Clearly $g(0) = 0, g(-x) = g(x)$. Let $x = (x_{mn}), y = (y_{mn}) \in \wedge_{fA}^{2\Delta}$. We have

$$\sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left(f\left(\left|\Delta x_{mn}\right|\right)^{1/m+n}\right)\right)^{1/T}.$$

≤ 1 and

$$\sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left(f\left(\left|\Delta y_{mn}\right|\right)^{1/m+n}\right)\right)^{1/T}$$

≤ 1

Then

$$\sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left(f\left(\left|\Delta(x_{mn} + y_{mn})\right|\right)^{1/m+n}\right)\right)^{1/T} \leq$$

$$\sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left(f\left(\left|\Delta x_{mn} + \Delta y_{mn}\right|\right)^{1/m+n}\right)\right)^{1/T} \leq$$

$$\sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left[\left(f\left(\left|\Delta x_{mn}\right|\right)^{1/m+n}\right) + \left(f\left(\left|\Delta y_{mn}\right|\right)^{1/m+n}\right)\right]\right)^{1/T}$$

$$\leq \sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left(f\left(\left|\Delta x_{mn}\right|\right)^{1/m+n}\right)\right)^{1/T} +$$

$$\sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left(f\left(\left|\Delta y_{mn}\right|\right)^{1/m+n}\right)\right)^{1/T} \leq 1$$

(By Minknowsky's inequality). Now

$$g\left((x_{mn}) + (y_{mn})\right) = \sup_m \left(\left|x_{m,1}\right| + \left|y_{m,1}\right|\right)^{1/m+1} +$$
$$\sup_n \left(\left|x_{1,n}\right| + \left|y_{1,n}\right|\right)^{1/1+n} +$$
$$\inf\left\{ \sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{jk}^{mn}\left(f\left(\left|\Delta(x_{mn} + y_{mn})\right|\right)^{1/m+n}\right)\right)^{1/T} \leq 1 \right\}$$
$$\leq \sup_m \left(\left|x_{m,1}\right| + \left|y_{m,1}\right|\right)^{1/m+1} + \sup_n \left(\left|x_{1,n}\right| + \left|y_{1,n}\right|\right)^{1/1+n} +$$

$$\inf\left\{\sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{jk}{}^{mn}\left(f\left(|\Delta x_{mn}|\right)^{1/m+n}\right)\right)^{1/T}\leq 1\right\}+$$
$$\inf\left\{\sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{jk}{}^{mn}\left(f\left(|\Delta y_{mn}|\right)^{1/m+n}\right)\right)^{1/T}\leq 1\right\}$$
$$= g(x_{mn})+g(y_{mn}).$$

Let $\lambda \in C$, then the continuity of the product follows from the following equality:

$$g(\lambda x_{mn}) = \sup_m |\lambda x_{m,1}|^{1/m+1} + \sup_n |\lambda x_{1,n}|^{1/1+n} +$$
$$\inf\left\{\sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{jk}{}^{mn}\left(f\left(\lambda|\Delta x_{mn}|\right)^{1/m+n}\right)\right)^{1/T}\leq 1\right\}=$$

Now

$$|\lambda|\sup_m |\lambda x_{m,1}|^{1/m+1} + |\lambda|\sup_n |\lambda x_{1,n}|^{1/1+n} +$$
$$\inf\left\{\sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{jk}{}^{mn}\left(f\left(\lambda|\Delta x_{mn}|\right)^{1/m+n}\right)\right)^{1/T}\leq 1\right\}=$$
$$|\lambda| g(x_{mn}).$$

$(x_{mn}{}^s)$ be a Cauchy sequence in $\chi_{fA}{}^{2\Delta}$. Then

$$g\left(\left(x_{mn}{}^s - x_{mn}{}^t\right)\right) \to 0 \text{ as } s,t\to\infty.$$

For given $\varepsilon > 0$, choose $r > 0$ and $x_0 > 0$ be such that $\frac{\varepsilon}{rx_0} > 0$ and $f\left(\frac{\varepsilon}{rx_0}\right) \geq 1$. Now $g\left(\left(x_{mn}{}^s - x_{mn}{}^t\right)\right) \to 0$ as $s,t\to\infty$, implies that there exists $n_0 \in \aleph$ such that

$$g\left(\left(x_{mn}{}^s - x_{mn}{}^t\right)\right) < \frac{\varepsilon}{rx_0} \text{ for all } s,t \geq n_0 .$$

$$\Rightarrow \sup_m \left|x_{m,1}{}^s - x_{m,1}{}^t\right|^{1/m+1} + \sup_n \left|x_{1,n}{}^s - x_{1,n}{}^t\right|^{1/1+n}$$
$$+\inf$$
$$\left\{\sup_{mn}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{jk}{}^{mn}\left(f\left(\left|\Delta x_{mn}{}^s - \Delta x_{mn}{}^t\right|\right)^{1/m+n}\right)\right)^{1/T}\leq 1\right\}$$
$$<\frac{\varepsilon}{rx_0}.$$

This shows that $(x_{m1}{}^s),(x_{m1}{}^t)$ are Cauchy sequences of real numbers. As the set of all real numbers is complete so there exists real numbers $x_{m,1}, x_{1,n}$ such that

$$\lim_{s\to\infty} x_{m,1}{}^s = x_{m,1} \text{ and } \lim_{t\to\infty} x_{1,n}{}^t = x_{1,n}.$$

Now we have,

$$\left(f\left(\left|\Delta x_{mn}{}^s - \Delta x_{mn}{}^t\right|\right)^{1/m+n}\right) \leq 1 \leq f\left(\frac{rx_0}{2}\right)$$

Therefore

$$\frac{\left|\Delta x_{mn}{}^s - \Delta x_{mn}{}^t\right|^{1/m+n}}{g\left(\left(x_{mn}{}^s - x_{mn}{}^t\right)\right)} \leq \frac{rx_0}{2}$$

Hence

$$\left|\Delta x_{mn}{}^s - \Delta x_{mn}{}^t\right|^{1/m+n} < \frac{rx_0}{2}\frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2}.$$

This implies that $(\Delta x_{mn}{}^s)$ is a Cauchy sequence of real numbers. Let $\lim_{s\to\infty} \Delta x_{mn}{}^s = z_{mn}$ for all $m,n \in \aleph$. Now $\Delta x_{11}{}^s = x_{11}{}^s - x_{12}{}^s - x_{21}{}^s + x_{22}{}^s$ and so $\lim_{s\to\infty} x_{22}{}^s = \lim_{s\to\infty}\left(\Delta x_{11}{}^s = x_{11}{}^s + x_{12}{}^s + x_{21}{}^s\right) = z_{11} - x_{11} + x_{12} + x_{21}$.
Hence $\lim_{s\to\infty} x_{22}{}^s$ exists. Proceeding in this way we conclude that $\lim_{s\to\infty} x_{mn}{}^s$ exists and thus $\lim_{s\to\infty} \Delta x_{mn}{}^s$ exists. Using continuity of f, we have $\lim_{t\to\infty}\left(f\left(\left|\Delta x_{mn}{}^s - \Delta x_{mn}{}^t\right|\right)^{1/m+n}\right) \leq 1$

Hence

$$\left(f\left(\left|\Delta x_{mn}{}^s - \Delta x_{mn}\right|\right)^{1/m+n}\right) \leq 1$$

Let $s \geq n_0$, then taking infimum we have $g\left(\left(x_{mn}{}^s - x_{mn}{}^t\right)\right) < \varepsilon$. Thus $\left(x_{mn}{}^s - x_{mn}{}^t\right) \in \wedge_{fA}{}^{2\Delta}$.

By linearity of the space $\wedge_{fA}{}^{2\Delta}$ we have $(x_{mn}) \in \wedge_{fA}{}^{2\Delta}$. Hence $\wedge_{fA}{}^{2\Delta}$ is complete space.

Δ^2 – Statistical convergence

The concept of statistical convergence for single sequences was introduced by Fast in 1951. Later, Mursaleen et al. (2003) and Tripathy (2003) defined statistical analogue for double sequence $x = (x_{mn})$ is said to be $P-$statistical convergence to 0 then,

$$P-\lim_{j,k}\frac{1}{jk}\left\{\begin{matrix}\text{the number of } (m,n): m<j, n<k; \\ \left((m+n)!|x_{mn}|\right)^{1/m+n}\end{matrix}\right\} = 0$$

In this case, we write $st^2 - \lim_{m,n}\left((m+n)!|\Delta x_{mn}|\right)^{1/m+n} = 0$ and we denote the set of all $P-$statistical χ^2 sequence by st^2.

Definition

A real double sequence $x = (x_{mn})$ is said to be $P-$statistical $\Delta-$convergence to 0 then

$$P-\lim_{j,k}\frac{1}{jk}$$
$$\left\{\begin{matrix}\text{the number of } (m,n): m<j, n<k; \\ \left((m+n)!|\Delta x_{mn}|\right)^{1/m+n} = 0\end{matrix}\right\} = 0$$

In this case, we write $st^2 - \lim_{m,n}\left(\left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n} = 0\right.$ and we denote the set of all P – statistical $\chi^{2\Delta}$ sequence by $st^{2\Delta}$.

Theorem

If f be an modulus function, then $\chi_f^{2\Delta} \subset st^{2\Delta}$.

Proof: Suppose that $x = (x_{mn}) \in \chi_f^{2\Delta}$, we obtain the following for every j and k

$$\frac{1}{jk}\sum_{m=0}^{j-1}\sum_{n=0}^{k-1} f\left(\left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n}\right) \geq$$

$$\frac{1}{jk}\sum_{m=0}^{j-1}\sum_{n=0 \text{ and } \left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n}=0}^{k-1} f\left(\left((m+n)!\left|\Delta x_{mn\geq}\right|\right)^{1/m+n}\right) \geq$$

$$\frac{f(0)}{jk}$$

$$\left\{\begin{matrix}\text{the number of } (m,n): m < j, n < k; \\ \left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n} = 0\end{matrix}\right\}.$$

Hence $x = (x_{mn}) \in st^{2\Delta}$,

Theorem

$st^{2\Delta} = \chi_f^{2\Delta}$ if and only if the modulus function f is analytic every where

Proof: Suppose that f is analytic everywhere and $x = (x_{mn}) \in st^{2\Delta}$. Then there exists an integer K such that $f(x) = K$ for all $x \geq 0$. Then for each j and k, we have

$$\frac{1}{jk}\sum_{m=0}^{j-1}\sum_{n=0}^{k-1} f\left(\left(\left|\Delta x_{mn}\right|\right)^{1/m+n}\right) \geq$$

$$\frac{1}{jk}\sum_{m=0}^{j-1}\sum_{n=0 \text{ and } \left|\Delta x_{mn}\right|\geq\varepsilon}^{k-1} f\left(\left(\left|\Delta x_{mn\geq}\right|\right)^{1/m+n}\right) \geq$$

and

$$\frac{1}{jk}\sum_{m=0}^{j-1}\sum_{n=0 \text{ and } \left|\Delta x_{mn}\right|<\varepsilon}^{k-1} f\left(\left(\left|\Delta x_{mn}\right|\right)^{1/m+n}\right) \leq$$

$$\frac{K}{jk}\left\{\text{the number of } (m,n): m < j, n < k; \left(\left|\Delta x_{mn}\right|\right)^{1/m+n}\right\}$$

$$+ f(\varepsilon).$$

thus the Pringsheim's limit on j and k.

Conversely, suppose that f is not an analytic so that there is a positive double sequence (z_{mn}) with $f(z_{jk}) = (jk)^2$ for $j,k = 1,2,\ldots$. Now the sequence $x = (x_{mn})$ defined by

$\left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n} = z_{jk}$ if $m,n = (jk)^2$ for $j,k = 1,2,\ldots$ and $\left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n} = 0$, Other wise. Then we have

$$\frac{1}{jk}\left\{\begin{matrix}\text{the number of } (m,n): m < j, n < k; \\ \left((m+n)!\left|x_{mn}\right|\right)^{1/m+n} = 0\end{matrix}\right\}$$

But $\leq \frac{1}{\sqrt{jk}} \to 0$ as $j,k \to \infty$. Hence $x_{mn} \to 0\left(st^{2\Delta}\right)$.

$x = (x_{mn}) \notin \chi_f^{2\Delta}$, which is a contradiction. Hence $st^{2\Delta} = \chi_f^{2\Delta}$.

Conclusion

Classical ideas of the modulus function of sequence spaces connected with P – statistical and $\chi^{2\Delta}{}_{fA}$.

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