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On some *I*-convergent generalized difference lacunary double sequence spaces defined by orlicz functions

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ABSTRACT. In this article, we introduce the lacunary generalized difference double paranormed sequence spaces $[w^2(M,\Delta^u,p,q)]_\theta^I$, $[w_0^2(M,\Delta^u,p,q)]_\theta^I$ and $[w_\infty^2(M,\Delta^u,p,q)]_\theta^I$ defined over a seminormed sequence space (X, q) using ideal convergence. The authors also study their properties and inclusion relations between them.

Keywords: ideal, *I*-convergent, *P*-convergent, difference sequence, Orlicz function.

Espaços sequenciais duplos com diferença lacunar generalizada I-convergentes definidos pela função de Orlicz

RESUMO. Neste artigo apresentamos espaços sequenciais para-normalizados duplos com diferença lacunar generalizada $[w^2(M,\Delta^u,p,q)]_\theta^I$, $[w_0^2(M,\Delta^u,p,q)]_\theta^I$ e $[w_\infty^2(M,\Delta^u,p,q)]_\theta^I$ definidos sobre um espaço sequencial semi-normalizado (X, q) utilizando convergência ideal. Os autores também analisaram suas propriedades e relações de inclusão entre eles.

Palavras-chave: ideal, I-convergente, P-convergente, sequência de diferenças, função de Orlicz.

Introduction

Let ℓ_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x=(x_k)$ with the usual norm $\|x\|=\sup_k |x_k|$.

The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$. A subset of $\mathbb{N}$ is said to have density (natural or asymptotic) $\delta(E)$ if

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{\infty}\chi_E(k)$$

exists.

A single sequence $x=(x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon>0$

$$\delta(\{k\in\mathbb{N} : |x_k-L|\geq\varepsilon\})=0.$$

The notion of statistical convergence for single sequences was introduced by Fast (1951) and Schoenberg (1959) independently. Later on it was studied by Fridy and Orhan (1979), Maddox (1989), Šalát (1980), Fridy (1985), Tripathy (1988) and many others.

Any concept involving statistical convergence plays a vital role not only in pure mathematics but also in other branches of mathematics especially in information theory, computer science, and biological science.

Let X be a non-empty set, then a family of sets $I\subset 2^X$ (the class of all subsets of X) is called an *ideal* if and only if for each $A, B\in I$, we have $A\cup B\in I$ and for each $A\in I$, and each $B\subset A$, we have $B\in I$. A non-empty family of sets $F\subset 2^X$ is a *filter* on X if and only if $\phi\notin F$ for each A, $B\in F$, we have $A\cap B\in F$ and for each $A\in F$, and each $B\supset A$, we have $B\in F$. An ideal I is called *non-trivial* ideal if $I\neq\phi$ and $X\neq I$. Clearly $I\subset 2^X$ is a non-trivial ideal if and only if $F=F(I)=\{X\text{-}A: A\in I\}$ is a filter on X. A non-trivial ideal $I\subset 2^X$ is called 'admissible' if and only if $\{\{x\}: x\in X\}\subset I$. A non-trivial ideal I is *maximal* if there cannot exists any non-trivial ideal $J\neq I$ containing I as a subset. Further details on ideals can be found in Kostyrko et al. (2000-2001).

The notion of I-convergence initially introduced by Kostyrko et al. (2000-2001). Later on, it was further investigated from the sequence space point of view and linked with the summability theory by Šalát et al. (2004, 2005), Tripathy and Hazarika (2008, 2009, 2011a and b), Hazarika (2011), Savas (2010), Kumar (2007) and others. The notion of *I*-convergence of double sequences initially introduced by Tripathy and Tripathy (2005).

A sequence (x_k) real numbers is said to be *I-convergent* to a real number ℓ if for each $\varepsilon > 0$ such that the set $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}$ belongs to *I*. In this case we write *I*-lim $x_k = \ell$ (KOSTYRKO et al., 2000-2001.)

Kizmaz (1981) introduced the notion of difference sequence spaces as follows:

$$X(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\}$$

for $X = \ell_\infty$, c and c_0.

Later on, the notion was generalized by Et and Colak (1995) as follows:

$$X(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

for $X = \ell_\infty$, c and c_0,

where:

$$\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}),$$

$$\Delta^0 x = x$$

and also this generalized difference notion has the following binomial representation:

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i} \text{ for all } k \in \mathbb{N}.$$

Subsequently, difference sequence spaces were studied by Esi (2009a and b), Esi and Tripathy (2008), Tripathy et al. (2005) and many others.

An Orlicz function M is a function M: $[0, \infty) \to [0, \infty)$, which is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0)=0$, $M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called the modulus function and characterized by Ruckle (1973). An Orlicz function M is said to satisfy Δ_2-*condition* for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

Remark 1.1. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct the sequence space.

$$l_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) < \infty, \text{ for some } r > 0 \right\}$$

which is a Banach space normed by

$$\|(x_k)\| = \inf\left\{ r > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

In a later stage different Orlicz sequence spaces were introduced and studied by Tripathy and Mahanta (2004), Esi (1999), Esi and Et (2000), Parashar and Choudhary (1994) and many others.

Let w^2 denote the set of all double sequences of complex numbers. By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by P-lim x $=L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$ (PRINGSHEIM, 1900). We shall describe such an $x = (x_{k,l})$ more briefly as '*P*- convergent'. We shall denote the space of all *P*-'convergent' sequences by c^2. The double sequence $x = (x_{k,l})$ is bounded if and only if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. We shall denote all bounded double sequences by l_∞^2.

The notion of statistical convergence for double sequences was introduced by Tripathy (2003). For this he introduced the notion of density of subsets of $\mathbb{N} \times \mathbb{N}$ as follows: A subset E of $\mathbb{N} \times \mathbb{N}$ is said to have density $\rho(E)$ if

$$\rho(E) = \lim_{p,s\to\infty} \frac{1}{ps} \sum_{n\leq p} \sum_{k \leq s} \chi_E(n,k) \text{ exists.}$$

The notion of double sequences studied by Esi (2010, 2011), Morciz (1991), Morciz and Rhoades (1988), Kumar (2007) and many others. Tripathy and Sarma (2006) introduced the statistically convergent double sequence spaces.

Example 2.1. If we take $I_2 = I_2(f) = \{A \subseteq \mathbb{N} \times \mathbb{N} :$ A is a finite subset$\}$. Then $I_2(f)$ is a non-trivial admissible ideal of $\mathbb{N} \times \mathbb{N}$ and the corresponding convergence coincide with the usual convergence.

Example 2.2. If we take $I_2 = I_2(\rho) = \{A \subseteq \mathbb{N} \times \mathbb{N} :$ $\rho(A) = 0\}$ where $\rho(A)$ denote the double asymptotic density of the set A. Then $I_2(\rho)$ is a non-trivial admissible ideal of $\mathbb{N} \times \mathbb{N}$ and the corresponding convergence coincide with the statistical convergence.

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that (SAVAS; PATTERSON, 2006).

$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \ as \ r \to \infty$

and,

$l_0 = 0, \overline{h}_s = l_s - l_{s-1} \to \infty \ as \ s \to \infty$.

Notations: $k_{r,s} = k_r l_s, h_{r,s} = h_r \overline{h}_s$ and $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l) : k_{r-1} < k \le k_r \ and \ l_{s-1} < l \le l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \overline{q}_s = \frac{l_s}{l_{s-1}} \ and \ q_{r,s} = q_r \overline{q}_s .$$

The set of all double lacunary sequences denoted by

$$N_{\theta_{r,s}} = \left\{ x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} | x_{k,l} - L | = 0, \ for \ some \ L \right\}$$

Definitions and results

In this presentation our goal is to extend a few results known in the literature from ordinary (single) difference sequences to difference double sequences. Some studies on double sequence spaces can be found in Gokhan and Colak (2004, 2005, 2006).

It is quite natural to expect that some new sequence spaces by double lacunary summability method can be defined by combining the concept of Orlicz function and *I*-convergence. We now ready to present the multidimensional sequence spaces.

Definition 2.1. Let I_2 be an admissible ideal of □ × □ . Let M be an Orlicz function and $p = (p_{k,l})$ be a factorable double sequence of strictly positive real numbers and $\theta_{r,s}$ be a double lacunary sequence. Let X be a seminormed space over the complex field □ with the seminorm q. We now define the following new generalized difference lacunary sequence spaces:

$$[w^2(M,\Delta^u,p,q)]_\theta^I = \left\{ x = (x_{k,l}) \in w^2 : \left\{ (r,s) \in □ \times □ : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{q(\Delta^u x_{k,l} - L)}{\rho} \right) \right]^{p_{k,l}} \ge \varepsilon \right\} \in I_2, \ \text{for some } \rho > 0 \text{ and L} \right\}$$

$$[w_0^2(M,\Delta^u,p,q)]_\theta^I = \left\{ x = (x_{k,l}) \in w^2 : \left\{ (r,s) \in □ \times □ : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{q(\Delta^u x_{k,l})}{\rho} \right) \right]^{p_{k,l}} \ge \varepsilon \right\} \in I_2, \ \text{for some } \rho > 0 \right\}$$

$$[w_\infty^2(M,\Delta^u,p,q)]_\theta^I = \left\{ x = (x_{k,l}) \in w^2 : \exists K > 0 \ s.t. \left\{ (r,s) \in □ \times □ : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{q(\Delta^u x_{k,l})}{\rho} \right) \right]^{p_{k,l}} \ge K \right\} \in I_2, \ \text{for some } \rho > 0 \right\}$$

and

$$[w_\infty^2(M,\Delta^u,p,q)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{q(\Delta^u x_{k,l})}{\rho} \right) \right]^{p_{k,l}} < \infty, \ \text{for some } \rho > 0 \right\},$$

where:

$$\Delta^u x = (\Delta^u x_{k,l}) = (\Delta^{u-1} x_{k,l} - \Delta^{u-1} x_{k,l+1} - \Delta^{u-1} x_{k+1,l} + \Delta^{u-1} x_{k+1,l+1}),$$

$$(\Delta^1 x_{k,l}) = (\Delta x_{k,l}) = (x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}), \Delta^0 x_{k,l} = x_{k,l}$$

and also this generalized difference double notion has the following binomial representation:

$$\Delta^u x_{k,l} = \sum_{i=0}^{u}\sum_{j=0}^{u}(-1)^{i+j}\binom{u}{i}\binom{u}{j}x_{k+i,l+j}$$

Some double spaces are obtained by specializing I_2, $\theta_{r,s}$, M, p, q and u. Here are some examples:

(i) If $\theta_{r,s} = \{(k_r, l_s)\} = \{(2^r, 2^s)\}$, $M(x) = x, u = 0, p_{k,l} = 1$ for all $k,l \in \mathbb{N}$, $I_2 = I_2(f)$ and $q(x) = |x|$, then we obtain ordinary double sequence spaces $[w^2], [w_0^2]$ and $[w_\infty^2]$.

(ii) If $M(x) = x, u = 0, p_{k,l} = 1$ for all $k,l \in \mathbb{N}$, $I_2 = I_2(f)$ and $q(x) = |x|$, then we obtain ordinary double lacunary sequence spaces $[w^2]_\theta, [w_0^2]_\theta$ and $[w_\infty^2]_\theta$.

(iii) If $M(x) = x, u = 0$ and $q(x) = |x|$, then we obtain new double lacunary sequence spaces as follows

$$[w^2(p)]_\theta^I = \left\{ x = (x_{k,l}) \in w^2 : \left\{ (r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |\Delta^u x_{k,l} - L|^{p_{k,l}} \geq \varepsilon \right\} \in I_2, \text{ for some } \rho > 0 \text{ and } L \right\}$$

$$[w_0^2(p)]_\theta^I = \left\{ x = (x_{k,l}) \in w^2 : \left\{ (r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |\Delta^u x_{k,l}|^{p_{k,l}} \geq \varepsilon \right\} \in I_2, \text{ for some } \rho > 0 \right\}$$

$$[w_\infty^2(p)]_\theta^I = \left\{ x = (x_{k,l}) \in w^2 : \exists K > 0 \ s.t. \left\{ (r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |\Delta^u x_{k,l}|^{p_{k,l}} \geq K \right\} \in I_2, \text{ for some } \rho > 0 \right\}$$

and

$$[w_\infty^2(p)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |\Delta^u x_{k,l}|^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \right\}$$

(iv) If $u = 0$ and and $q(x) = |x|$, then we obtain new double lacunary sequence spaces as follows:

$$[w^2(M,p)]_\theta^I = \left\{ x = (x_{k,l}) \in w^2 : \left\{ (r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|x_{k,l} - L|}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2, \text{ for some } \rho > 0 \text{ and } L \right\}$$

$$[w_0^2(M,p)]_\theta^I = \left\{ x = (x_{k,l}) \in w^2 : \left\{ (r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2, \text{ for some } \rho > 0 \right\}$$

$$[w_\infty^2(M,p)]_\theta^I = \left\{ x = (x_{k,l}) \in w^2 : \exists K > 0 \ s.t. \left\{ (r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} \geq K \right\} \in I_2, \text{ for some } \rho > 0 \right\}$$

and

$$[w_\infty^2(M,p)]_\theta = \left\{ x=(x_{k,l})\in w^2 : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{|x_{k,l}|}{\rho}\right)\right]^{p_{k,l}} < \infty, \text{ for some } \rho>0 \right\}$$

(v) If $u = 1$ and and $q(x)=|x|$, then we obtain new double lacunary sequence spaces as follows:

$$[w^2(M,\Delta,p)]_\theta^I = \left\{ x=(x_{k,l})\in w^2 : \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{|\Delta x_{k,l}-L|}{\rho}\right)\right]^{p_{k,l}} \geq \varepsilon\right\}\in I_2, \text{ for some } \rho>0 \text{ and } L \right\}$$

$$[w_0^2(M,\Delta,p)]_\theta^I = \left\{ x=(x_{k,l})\in w^2 : \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{|\Delta x_{k,l}|}{\rho}\right)\right]^{p_{k,l}} \geq \varepsilon\right\}\in I_2, \text{ for some } \rho>0 \right\}$$

$$[w_\infty^2(M,\Delta,p)]_\theta^I = \left\{ x=(x_{k,l})\in w^2 : \exists K>0 \text{ s.t.} \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{|\Delta x_{k,l}|}{\rho}\right)\right]^{p_{k,l}} \geq K\right\}\in I_2, \text{ for some } \rho>0 \right\}$$

and

$$[w_\infty^2(M,\Delta,p)]_\theta = \left\{ x=(x_{k,l})\in w^2 : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{|\Delta x_{k,l}|}{\rho}\right)\right]^{p_{k,l}} < \infty, \text{ for some } \rho>0 \right\}$$

Main results

Theorem 3.1. *Let* $p=(p_{k,l})$ be bounded double sequence. The classes $[w^2(M,\Delta^u,p,q)]_\theta^I$, $[w_0^2(M,\Delta^u,p,q)]_\theta^I$, $[w_\infty^2(M,\Delta^u,p,q)]_\theta^I$ and $[w_\infty^2(M,\Delta^u,p,q)]_\theta$ are linear spaces over the complex field $\mathbb{C}$.

Proof. We give the proof only for $[w_0^2(M,\Delta^u,p,q)]_\theta^I$. The others can be treated similarly. Let $x=(x_{k,l})$, $y=(y_{k,l})\in[w_0^2(M,\Delta^u,p,q)]_\theta^I$. Then there exists $\rho_1>0$ and $\rho_2>0$ such that

$$A_{\frac{\varepsilon}{2}} = \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^u x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}} \geq \frac{\varepsilon}{2}\right\}\in I_2 \tag{3.1}$$

and

$$B_{\frac{\varepsilon}{2}} = \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^u y_{k,l})}{\rho_2}\right)\right]^{p_{k,l}} \geq \frac{\varepsilon}{2}\right\}\in I_2 \tag{3.2}$$

Let $\alpha,\beta\in\mathbb{C}$ be scalars. By the continuity of the function M the following inequality holds:

$$\frac{1}{h_{r.s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^u(\alpha x_{k,l}+\beta y_{k,l}))}{|\alpha|\rho_1+|\beta|\rho_2}\right)\right]^{p_{k,l}} \leq$$

$$\frac{D}{h_{r.s}}\sum_{(k,l)\in I_{r,s}}\left\{\left[\frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2}M\left(\frac{q(\Delta^u x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}} + \left[\frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}M\left(\frac{q(\Delta^u y_{k,l})}{\rho_2}\right)\right]^{p_{k,l}}\right\}$$

$$\leq \frac{D}{h_{r.s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^u x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}} + \frac{D}{h_{r.s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^u y_{k,l})}{\rho_2}\right)\right]^{p_{k,l}}$$

Now, from the above relations and the equations (3.1) and (3.2), we have the following

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^u(\alpha x_{k,l}+\beta y_{k,l}))}{\rho}\right)\right]^{p_{k,l}} \geq \varepsilon\right\}$$

$$\subseteq \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{D}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{q(\Delta^u x_{k,l})}{\rho_1} \right) \right]^{p_{k,l}} \geq \frac{\varepsilon}{2} \right\}$$

$$\cup \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{D}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{q(\Delta^u y_{k,l})}{\rho_2} \right) \right]^{p_{k,l}} \geq \frac{\varepsilon}{2} \right\}$$

Therefore $\alpha x + \beta y \in [w_0^2(M, \Delta^u, p, q)]_\theta^I$. Hence $[w_0^2(M, \Delta^u, p, q)]_\theta^I$ is a linear space.

Theorem 3.2. The double sequence spaces $[w^2(M, \Delta^u, p, q)]_\theta^I$, $[w_0^2(M, \Delta^u, p, q)]_\theta^I$, $[w_\infty^2(M, \Delta^u, p, q)]_\theta^I$ and $[w_\infty^2(M, \Delta^u, p, q)]_\theta$ are seminormed spaces, seminormed by

$$f((x_{k,l})) = \sum_{k=1}^{u} q(x_{k,1}) + \sum_{l=1}^{u} q(x_{1,l}) + \inf\left\{ \rho^{\frac{p_{k,l}}{H}} > 0 : \sup_{k,l}\left[M\left(\frac{q(\Delta^u x_{k,l})}{\rho} \right) \right] \leq 1 \right\}$$

where:
$H = \max\{1, \sup_{k,l} p_{k,l}\}$

Proof. Since q is a seminorm, so we have $f((x_{k,l})) \geq 0$ for all $x = (x_{k,l})$; $f(\theta^2) = 0$ and $f((\lambda x_{k,l})) = |\lambda| f((x_{k,l}))$ for all scalars λ.

Now, let $x = (x_{k,l}), y = (y_{k,l}) \in [w_0^2(M, \Delta^u, p, q)]_\theta^I$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sup_{k,l}\left[M\left(\frac{q(\Delta^u x_{k,l})}{\rho_1} \right) \right] \leq 1 \text{ and } \sup_{k,l}\left[M\left(\frac{q(\Delta^u y_{k,l})}{\rho_2} \right) \right] \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\sup_{k,l}\left[M\left(\frac{q(\Delta^u (x_{k,l} + y_{k,l}))}{\rho} \right) \right]$$

$$\leq \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_{k,l}\left[M\left(\frac{q(\Delta^u x_{k,l})}{\rho_1} \right) \right] + \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_{k,l}\left[M\left(\frac{q(\Delta^u y_{k,l})}{\rho_2} \right) \right] \leq 1$$

Since $\rho_1, \rho_2 > 0$, so we have

$$f((x_{k,l} + y_{k,l})) = \sum_{k=1}^{u} q(x_{k,1} + y_{k,1}) + \sum_{l=1}^{u} q(x_{1,l} + y_{1,l}) + \inf\left\{ \rho^{\frac{p_{k,l}}{H}} > 0 : \sup_{k,l}\left[M\left(\frac{q(\Delta^u (x_{k,l} + y_{k,l}))}{\rho} \right) \right] \leq 1 \right\}$$

$$+ \leq \sum_{k=1}^{u} q(y_{k,1}) + \sum_{l=1}^{u} q(y_{1,l}) + \inf\left\{ \rho^{\frac{p_{k,l}}{H}} > 0 : \sup_{k,l}\left[M\left(\frac{q(\Delta^u (y_{k,l}))}{\rho} \right) \right] \leq 1 \right\} = f((x_{k,l})) + f((y_{k,l})).$$

Therefore f is a seminorm.

Theorem 3.3. Let (X, q) be a complete seminormd space. Then the spaces $[w^2(M, \Delta^u, p, q)]_\theta^I$, $[w_0^2(M, \Delta^u, p, q)]_\theta^I$, $[w_\infty^2(M, \Delta^u, p, q)]_\theta^I$ and $[w_\infty^2(M, \Delta^u, p, q)]_\theta$ are complete seminormed spaces, seminormed by f.

Proof. We prove the theorem for the space $[w_0^2(M, \Delta^u, p, q)]_\theta^I$. The other cases can be established following similar technique. Let $x^i = (x_{k,l}^i)$ be a Cauchy sequence in $[w_0^2(M, \Delta^u, p, q)]_\theta^I$. Let $\varepsilon > 0$ be given and for $r > 0$, choose x_0 fixed such that $M\left(\frac{rx_0}{2}\right) \geq 1$ and there exists $m_0 \in \mathbb{N}$ such that

$$f\left(\left(x_{k,l}^i - x_{k,l}^j\right)\right) < \frac{\varepsilon}{rx_0} \text{ for all } i, j \geq m_0$$

By definition of seminorm, we have

3.3 $$\sum_{k=1}^{u} q(x_{k,1}^i) + \sum_{l=1}^{u} q(x_{1,l}^j) + \inf\left\{ \rho^{\frac{p_{k,l}}{H}} > 0 : \sup_{k,l}\left[M\left(\frac{q(\Delta^u x_{k,l}^i - \Delta^u x_{k,l}^j)}{\rho} \right) \right] \leq 1 \right\} < \frac{\varepsilon}{rx_0}$$

This shows that $(x_{k,1}^i)$ and $(x_{1,l}^j)(k, l \leq u)$ are Cauchy sequences in (X, q). Since (X, q) is complete, so there exist $x_{k,1}, x_{1,l} \in X$ such that

$\lim_{i\to\infty} x_{k,1}^i = x_{k,1}$ and $\lim_{j\to\infty} x_{1,l}^j = x_{1,l} (k, l \leq u)$.

Now from (3.3), we have

3.4 $$M\left(\frac{q(\Delta^u (x_{k,l}^i - x_{k,l}^j))}{f\left(x_{k,l}^i - x_{k,l}^j\right)} \right) \leq 1 \leq M\left(\frac{rx_0}{2} \right)$$

for all $i, j \geq m_0$.

This implies

$q(\Delta^u (x_{k,l}^i - x_{k,l}^j)) \leq \frac{rx_0}{2} \cdot \frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2}$ for all $i, j \geq m_0$.

So, $\left(\Delta^u (x_{k,l}^i)\right)$ is a Cauchy sequence in (X, q). Since (X, q) is complete, there exists $x_{k,l} \in X$ such that $\lim_i \Delta^u (x_{k,l}^i) = x_{k,l}$ fo r all $k, l \in \mathbb{N}$. Since M is continuous, so for $i \geq m_0$, on taking limit as $j \to \infty$, we have from (3.4),

$$M\left(\frac{q(\Delta^u(x_{k,l}^i)-\lim_j \Delta^u x_{k,l}^j)}{\rho}\right)\le 1 \Rightarrow M\left(\frac{q(\Delta^u(x_{k,l}^i)-x_{k,l})}{\rho}\right)\le 1$$

On taking the infimum of such ρ', we have

$f\left(x_{k,l}^i - x_{k,l}\right) < \varepsilon$ for all $i \ge m_0$.

Thus $\left(x_{k,l}^i - x_{k,l}\right) \in [w_0^2(M,\Delta^u,p,q)]_\theta^I$. By linearity of the space $[w_0^2(M,\Delta^u,p,q)]_\theta^I$, we have for all $i \ge m_0$, $(x_{k,l}) = (x_{k,l}^i) + \left(x_{k,l}^i - x_{k,l}\right) \in [w_0^2(M,\Delta^u,p,q)]_\theta^I$.

Thus $[w_0^2(M,\Delta^u,p,q)]_\theta^I$ is a complete space.

Proposition 3.4. **(a)** $[w^2(M,\Delta^u,p,q)]_\theta^I \subset [w_\infty^2(M,\Delta^u,p,q)]_\theta^I$

(b) $[w_0^2(M,\Delta^u,p,q)]_\theta^I \subset [w_\infty^2(M,\Delta^u,p,q)]_\theta^I$. The inclusions are strict.

Proof. It is easy, so omitted.

To show that the inclusions are strict, consider the following example.

Example 3.1. Let $\theta_{r,s} = \{(k_r,l_s)\} = \{(2^r,2^s)\}, M(x) = x^p, p \ge 1, u = 1,$ $q(x) = |x|$, $p_{k,l} = 2$ for all $k,l \in \mathbb{N}$ and consider the double sequence

$$x_{k,l} = \begin{cases} 0, \text{ if } k+l \text{ is odd} \\ k, \text{ otherwise} \end{cases}$$

Then

$$\Delta^u x_{k,l} = \begin{cases} 2k+1, \text{ if } k+l \text{ is even} \\ -2k-1, \text{ otherwise} \end{cases}$$

Hence $x = (x_{k,l}) \in [w_\infty^2(M,\Delta^u,p,q)]_\theta^I$, but $x = (x_{k,l}) \notin [w^2(M,\Delta^u,p,q)]_\theta^I$.

Theorem 3.5. The double sequence spaces $[w^2(M,\Delta^u,p,q)]_\theta^I$ and $[w_0^2(M,\Delta^u,p,q)]_\theta^I$, are nowhere dense subsets of $[w_\infty^2(M,\Delta^u,p,q)]_\theta$.

Proof. The proof is obvious in view of Theorem 3.3 and Proposition 3.4.

Theorem 3.6. Let $u \ge 1$ then for all $0 < i \le u$, $[Z^2(M,\Delta^i,p,q)]_\theta^I \subset [Z^2(M,\Delta^u,p,q)]_\theta^I$, where $Z^2 = w^2, w_0^2$ and w_∞^2. The inclusions are strict.

Proof. We establish it for only $[w_\infty^2(M,\Delta^{u-1},p,q)]_\theta^I \subset [w_\infty^2(M,\Delta^u,p,q)]_\theta^I$. Let $x = (x_{k,l}) \in [w_\infty^2(M,\Delta^{u-1},p,q)]_\theta^I$. Then there exists $K > 0$ and $\rho > 0$, we have

3.5 $$\left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{q(\Delta^{u-1}x_{k,l})}{\rho}\right)\right]^{p_{k,l}} \ge K\right\} \in I_2$$

Since M is non-decreasing, continuous and convex and for

$$\Delta^u x = (\Delta^u x_{k,l}) = (\Delta^{u-1}x_{k,l} - \Delta^{u-1}x_{k,l+1} - \Delta^{u-1}x_{k+1,l} + \Delta^{u-1}x_{k+1,l+1}),$$

we have

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{q(\Delta^{u-1}x_{k,l})}{4\rho}\right)\right]^{p_{k,l}} \le \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{q(\Delta^{u-1}x_{k,l} - \Delta^{u-1}x_{k,l+1} - \Delta^{u-1}x_{k+1,l} + \Delta^{u-1}x_{k+1,l+1})}{4\rho}\right)\right]^{p_{k,l}}$$

$$\le D^2 \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \{\left[\frac{1}{4}M\left(\frac{q(\Delta^{u-1}x_{k,l})}{\rho}\right)\right]^{p_{k,l}} + \left[\frac{1}{4}M\left(\frac{q(\Delta^{u-1}x_{k,l+1})}{\rho}\right)\right]^{p_{k,l}}$$

$$+\left[\frac{1}{4}M\left(\frac{q(\Delta^{u-1}x_{k+1,l})}{\rho}\right)\right]^{p_{k,l}} + \left[\frac{1}{4}M\left(\frac{q(\Delta^{u-1}x_{k+1,l+1})}{\rho}\right)\right]^{p_{k,l}}\}$$

$$\leq D^2\{\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^{u-1}x_{k,l})}{\rho}\right)\right]^{p_{k,l}}+\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^{u-1}x_{k,l+1})}{\rho}\right)\right]^{p_{k,l}}$$

$$+\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^{u-1}x_{k+1,l})}{\rho}\right)\right]^{p_{k,l}}+\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^{u-1}x_{k+1,l+1})}{\rho}\right)\right]^{p_{k,l}}\}$$

Hence we have

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^{u}x_{k,l})}{4\rho}\right)\right]^{p_{k,l}}\geq K\right\}$$

$$\subseteq\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{D^2}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^{u-1}x_{k,l})}{\rho}\right)\right]^{p_{k,l}}\geq \frac{K}{4}\right\}$$

$$\cup\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{D^2}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^{u-1}x_{k,l+1})}{\rho}\right)\right]^{p_{k,l}}\geq \frac{K}{4}\right\}$$

$$\cup\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{D^2}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^{u-1}x_{k+1,l})}{\rho}\right)\right]^{p_{k,l}}\geq \frac{K}{4}\right\}$$

$$\cup\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{D^2}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^{u-1}x_{k+1,l+1})}{\rho}\right)\right]^{p_{k,l}}\geq \frac{K}{4}\right\}$$

Consequently we get

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^{u}x_{k,l})}{4\rho}\right)\right]^{p_{k,l}}\geq K\right\}\in I_2$$

It follows that $x=(x_{k,l})\in[w_\infty^2(M,\Delta^u,p,q)]_\theta^I$ and hence $[w_\infty^2(M,\Delta^{u-1},p,q)]_\theta^I \subset[w_\infty^2(M,\Delta^u,p,q)]_\theta^I$. On applying the principle of induction, it follows that $[w_\infty^2(M,\Delta^i,p,q)]_\theta^I\subset[w_\infty^2(M,\Delta^u,p,q)]_\theta^I$ for $i=1,2,3,....,u-1$. The proof for the rest cases are similar. To show that the inclusions are strict, consider the following example.

Example 3.2. Let $\theta_{r,s}=\{(k_r,l_s)\}=\{(2^r,2^s)\}, M(x)=x^p, p\geq 1, u=1,$ $q(x)=|x|$, $p_{k,l}=1$ for all *k* odd and for all $l\in\mathbb{N}$ and $p_{k,l}=2$ otherwise. Consider the sequence $x=(x_{k,l})$ defined by $x_{k,l}=k+l$ for all $k,l\in\mathbb{N}$. We have $\Delta x_{k,l}=0$ for all $k,l\in\mathbb{N}$. Hence $x=(x_{k,l})\in[w_\infty^2(M,\Delta,p,q)]_\theta^I$ but $x=(x_{k,l})\notin[w_\infty^2(M,p,q)]_\theta^I$.

Theorem 3.7. **(a)** *If* $0<\inf_{k,l}p_{k,l}\leq p_{k,l}<1$, *then* $[Z^2(M,\Delta^u,p,q)]_\theta^I\subset[Z^2(M,\Delta^u,q)]_\theta^I$,

(b) If $1<p_{k,l}\leq\sup_{k,l}p_{k,l}<\infty$ then $[Z^2(M,\Delta^u,q)]_\theta^I\subset[Z^2(M,\Delta^u,p,q)]_\theta^I$, where $Z^2=w^2,w_0^2$ and w_∞^2.

Proof. The first part of the result follows from the inequality

$$M\left(\frac{q(\Delta^u x_{k,l})}{\rho}\right)\leq\left[M\left(\frac{q(\Delta^u x_{k,l})}{\rho}\right)\right]^{p_{k,l}}$$

and the second part of the result follows from the inequality

$$\left[M\left(\frac{q(\Delta^u x_{k,l})}{\rho}\right)\right]^{p_{k,l}}\leq M\left(\frac{q(\Delta^u x_{k,l})}{\rho}\right).$$

Theorem 3.8. Let M_1 and M_2 be Orlicz functions satisfying Δ_2- condition. If $\beta=\lim_{t\to\infty}\frac{M_2(t)}{t}\geq 1$, then $[Z^2(M_1,\Delta^u,p,q)]_\theta^I\subset[Z^2(M_1 M_2,\Delta^u,p,q)]_\theta^I$, where $Z^2=w^2,w_0^2$ and w_∞^2.

Proof. We prove it for $Z^2=w_0^2$ and the other cases will follows on applying similar techniques. Let $x=(x_{k,l})\in[w_0^2(M_1,\Delta^u,p,q)]_\theta^I$, then

$$I_2-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M_1\left(\frac{q(\Delta^u x_{k,l})}{\rho}\right)\right]^{p_{k,l}}=0\cdot$$

Let $0<\varepsilon<1$ and δ with $0<\delta<1$ such that $M_2(t)<\varepsilon$ for $0\leq t<\delta$. Let

$$y_{k,l}=M_1\left(\frac{q(\Delta^u x_{k,l})}{\rho}\right)$$

and consider

3.6 $[M_2(y_{k,l})]^{p_{k,l}}=[M_2(y_{k,l})]^{p_{k,l}}+[M_2(y_{k,l})]^{p_{k,l}}$

where the first term is over $y_{k,l} \le \delta$ and the second is over $y_{k,l} > \delta$. From the first term in (3.6) and using the Remark 1.1 we have

$$[M_2(y_{k,l})]^{p_{k,l}} < [M_2(2)]^H + [(y_{k,l})]^{p_{k,l}} \tag{3.7}$$

On the other hand, we use the fact that

$$y_{k,l} < \frac{y_{k,l}}{\delta} < 1 + \frac{y_{k,l}}{\delta}$$

Since M_2 is non-decreasing and convex, it follows that

$$M_2(y_{k,l}) < M_2\left(1 + \frac{y_{k,l}}{\delta}\right) < \frac{1}{2}M_2(2) + \frac{1}{2}M_2\left(\frac{2y_{k,l}}{\delta}\right)$$

Since M_2 satisfies Δ_2- condition, we have

$$M_2(y_{k,l}) < \frac{1}{2}K\frac{y_{k,l}}{\delta}M_2(2) + \frac{1}{2}K\frac{y_{k,l}}{\delta}M_2(2) = K\frac{y_{k,l}}{\delta}M_2(2).$$

Hence, from the second term in (3.6)

$$[M_2(y_{k,l})]^{p_{k,l}} \le \max\left(1, (KM_2(2)\delta^{-1})^H\right)[(y_{k,l})]^{p_{k,l}} \tag{3.8}$$

By (3.7) and (3.8), taking I_2-limit in the Pringsheim sense, we have $x = (x_{k,l}) \in [w_0^2(M_1 \circ M_2, \Delta^u, p, q)]_\theta^I$. Observe that in this part of the proof we do not need $\beta \ge 1$. Now, let $\beta \ge 1$ and $x = (x_{k,l}) \in [w_0^2(M_1, \Delta^u, p, q)]_\theta^I$. Since $\beta \ge 1$ we have $M_2(t) \ge \beta t$ for all $t \ge 0$. It follows that $x = (x_{k,l}) \in [w_0^2(M_1 \circ M_2, \Delta^u, p, q)]_\theta^I$ implies $x = (x_{k,l}) \in [w_0^2(M_1, \Delta^u, p, q)]_\theta^I$. This implies that $[w_0^2(M_1 \circ M_2, \Delta^u, p, q)]_\theta^I = [w_0^2(M_1, \Delta^u, p, q)]_\theta^I$.

Theorem 3.9. Let M, M_1 and M_2 be Orlicz functions, q, q_1 and q_2 be seminorms. Then

(i)
$$[Z^2(M_1, \Delta^u, p, q)]_\theta^I \cap [Z^2(M_2, \Delta^u, p, q)]_\theta^I \subset [Z^2(M_1 + M_2, \Delta^u, p, q)]_\theta^I$$

(ii)
$$[Z^2(M, \Delta^u, p, q_1)]_\theta^I \cap [Z^2(M, \Delta^u, p, q_2)]_\theta^I \subset [Z^2(M, \Delta^u, p, q_1 + q_2)]_\theta^I$$

(iii) If q_1 is stronger than q_2, then $[Z^2(M, \Delta^u, p, q_1)]_\theta^I \subset [Z^2(M, \Delta^u, p, q_2)]_\theta^I$, where $Z^2 = w^2, w_0^2$ and w_∞^2.

Proof. (i) We establish it for only $Z^2 = w_0^2$. The rest cases are similar. Let $x = (x_{k,l}) \in [w_0^2(M_1, \Delta^u, p, q)]_\theta^I \cap [w_0^2(M_2, \Delta^u, p, q)]_\theta^I$.
Then for each $\varepsilon > 0$, there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M_1\left(\frac{q(\Delta^u x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}} \ge \frac{\varepsilon}{2}\right\} \in I_2$$

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M_2\left(\frac{q(\Delta^u x_{k,l})}{\rho_2}\right)\right]^{p_{k,l}} \ge \frac{\varepsilon}{2}\right\} \in I_2$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The result follows from the following inequality

$$\frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[(M_1 + M_2)\left(\frac{q(\Delta^u x_{k,l})}{\rho}\right)\right]^{p_{k,l}}$$

$$\le D\frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left\{\left[M_1\left(\frac{q(\Delta^u x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}} + \left[M_2\left(\frac{q(\Delta^u x_{k,l})}{\rho_2}\right)\right]^{p_{k,l}}\right\}$$

The proofs of (ii) and (iii) follow obviously.

The proof of the following result is also routine work.

Proposition 3.10. For any Orlicz function M, IF $q_1 \cong$ (equivalent to) q_2 then, $[Z^2(M, \Delta^u, p, q_1)]_\theta^I = [Z^2(M, \Delta^u, p, q_2)]_\theta^I$, where $Z^2 = w^2, w_0^2$ and w_∞^2.

Conclusion

In this article we defined some new sequence spaces by double lacunary summability method by combining the concept of Orlicz function and *I*-convergence. Further, we proved some topological and algebraic properties of the resulting spaces.

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