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On almost λ- statistical convergence of fuzzy numbers

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ABSTRACT. The purpose of this paper is to introduce the concepts of almost λ-statistical convergence and strongly almost λ- convergence of fuzzy numbers. We obtain some results related to these concepts. It is also shown that almost λ- statistical convergence and strongly almost λ-convergence are equivalent for almost bounded sequences of fuzzy numbers.

Keywords: fuzzy number, de la Vallee-Poussin mean, statistical convergence.

Sobre a convergência λ-quase estatística de números difusos

RESUMO. Essa pesquisa introduz os conceitos de convergência λ-quase estatística e forte convergência λ-quase estatística de números difusos. Alguns resultados foram obtidos com tais conceitos. Mostra-se que conceitos de convergência λ-quase estatística e forte convergência λ-quase estatística equivalem a quase sequências limitadas de números difusos.

Palavras-chave: números difusos, medias de de la Vallee-Poussin, convergência estatística.

Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh (1965) and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka (1986) introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden (1994), Nanda (1989), Savaş (2000, 2006), Esi (2006), Tripathy and Baruah (2009, 2010a and b), Tripathy and Borgogain (2008, 2011), Tripathy and Dutta (2007, 2010), Tripathy and Sarma (2011) and many others.

Let

$C\left(R^n\right)=\left\{A\subset R^n: \quad A \textit{ is compact and convex set}\right\}$.

The space $C\left(R^n\right)$ has a linear structure induced by the following operations

$A+B=\{a+b: \quad a\in A, b\in B\}$ and $\gamma A=\{\gamma a: \quad a\in A\}$ for $A,B\in C\left(R^n\right)$ and $\gamma\in R$. The Hausdorff distance between A and B in $C\left(R^n\right)$ is defined by

$\delta_\infty(A,B)=\max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}$.

It is well-known that $\left(C\left(R^n\right),\delta_\infty\right)$ is a complete metric space.

Throughout this paper by a fuzzy number we mean a function X from R^n to $[0,1]$ which is normal, fuzzy convex, upper semicontinuous and the closure of $\{t\in R: \quad X(t)>0\}$ is compact. These properties imply that for each $0<\alpha\leq 1$, the $\alpha-$level set $X^\alpha=\{t\in R^n: \quad X(t)\geq\alpha\}$ is a non-empty compact, convex subset of R^n, with support X^0. If R^n is replaced by R, then obviously the set $C\left(R^n\right)$ is reduced to the set of all closed bounded intervals $A=\left[\underline{A},\overline{A}\right]$ on R, and we have

$$\delta_\infty(A,B)=\max\left(\left|\underline{A}-\underline{B}\right|,\left|\overline{A}-\overline{B}\right|\right).$$

Let $L(R)$ denote the set of all fuzzy numbers. The linear structure of $L(R)$ induces the addition $X+Y$ and the scalar multiplication λX in terms of $\alpha-$level sets, by

$$[X+Y]^\alpha=[X]^\alpha+[Y]^\alpha \textit{ and } [\lambda X]^\alpha=\lambda[X]^\alpha$$

for each $0\leq\alpha\leq 1$.

The set R of real numbers can be embedded in $L(R)$ if we define $\overline{r}\in L(R)$ by

$$\overline{r}(t)=\begin{cases}1, & if \quad t=r\\ 0, & if \quad t\neq r\end{cases}.$$

The additive identity and multiplicative identity of $L(R)$ are denoted by $\bar{0}$ and $\bar{1}$, respectively.

For r in R and X in $L(R)$, the product rX is defined as follows:

$$rX(t)=\begin{cases} X(r^{-1}t), & if \quad r\neq 0 \\ 0, & if \quad r=0 \end{cases}.$$

Define a map $d:L(R)xL(R)\rightarrow R$ by $d(X,Y)=\sup_{0\leq\alpha\leq 1}\delta_{\infty}(X^{\alpha},Y^{\alpha})$.

For $X,Y\in L(R)$ define $X\leq Y$ if and only if $X^{\alpha}\leq Y^{\alpha}$ for any $\alpha\in[0,1]$. It is known that $(L(R),d)$ is complete metric space (MATLOKA, 1986).

A sequence $X=(X_k)$ of fuzzy numbers is a function X from the set N of natural numbers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k\in N$ (MATLOKA, 1986).

We denote by w^F denotes the set of all sequences $X=(X_k)$ of fuzzy numbers.

A sequence $X=(X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k:k\in N\}$ of fuzzy numbers is bounded (MATLOKA, 1986).

We denote by ℓ_{∞}^F the set of all bounded sequences $X=(X_k)$ of fuzzy numbers.

A sequence $X=(X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for every $\varepsilon>0$ there is a positive integer k_0 such that $\bar{d}(X_k,X_o)<\varepsilon$ for all $k>k_o$ (MATLOKA, 1986).

We denote by c^F the set of all convergent sequences $X=(X_k)$ of fuzzy numbers.

It is straightforward to note that $c^F\subset\ell_{\infty}^F\subset w^F$.

Nanda (1989) studied the classes of bounded and convergent sequences of fuzzy numbers and showed that these are complete metric spaces.

The metric d has the following properties:

$$d(cX,cY)=|c|d(X,Y), \quad \text{for} \quad c\in R \quad \text{and}$$
$$d(X+Y,Z+W)\leq d(X,Z)+d(Y,W).$$

A metric on $L(R)$ is said to be translation invariant if $d(X+Z,Y+Z)=d(X,Y)$ for all $X,Y,Z\in L(R)$.

The notion of statistical convergence for a sequence of complex numbers was introduced by Fridy (1985) and many others. Over the years and under different names statistical convergence has been discussed in the different theories such as the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy (1985), Salat (1980), Connor (1999) and many others. This concept extends the idea to apply to sequences of fuzzy numbers with Kwon and Shim (2001), Et et al. (2005), Nuray and Savaş (1995) and many others.

Savaş (2006) defined almost convergence for fuzzy numbers as follows:

The sequence $X=(X_k)$ of fuzzy numbers is said to be almost convergent to a fuzzy number X_0 if $\lim_{m\rightarrow\infty}d(t_{km}(X),X_o)=0$ uniformly in m.

where

$$t_{km}(X)=\frac{1}{k+1}\sum_{i=0}^{k}X_{i+m}.$$

This means that for every $\varepsilon>0$, there exists a $k_o\in N$ such that $d(t_{km}(X),X_o)<\varepsilon$ whenever $k\geq k_o$ and for all m.

A sequence $X=(X_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_0 if for every $\varepsilon>0$,

$$\lim_{n\rightarrow\infty}n^{-1}\left|\{k\leq n:d(X_k,X_o)\geq\varepsilon\}\right|=0.$$

The set of all statistically convergent sequences of fuzzy numbers is denoted by S^F.

We note that if a sequence $X=(X_k)$ of fuzzy numbers converges to a fuzzy number X_o, then it is statistically converges to X_0. But the converse statement is not necessarily valid.

Let $\Lambda=(\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_1=1$ and $\lambda_{n+1}\leq\lambda_n+1$, for all $n\in N$.

The generalized de la Vallee-Poussin means is defined by

$$t_n(x)=\lambda_n^{-1}\sum_{k\in I_n}x_k,$$

where $I_n=[n-\lambda_n+1,n]$. A sequence $x=(x_k)$ of complex numbers is said to be (V,λ)-summable to a number l if $t_n(x)\rightarrow L \quad as\ n\rightarrow\infty$, Leindler (1965).

Main Results

Let $X=(X_k)$ be sequence of fuzzy numbers. Then the sequence $X=(X_k)$ of fuzzy numbers is said to be almost λ-statistically convergent to the fuzzy number X_0 if for every $\varepsilon>0$,

$$\lim_{n\to\infty} \lambda_n^{-1}\left|\left\{k \in I_n : d\left(t_{km}(X), X_o\right) \geq \varepsilon\right\}\right| = 0,$$

uniformly in m.

In this case, we write $X_k \to X_o\left(\hat{S}_\lambda^F\right)$ or $\hat{S}_\lambda^F - \lim X_k = X_o$. The set of all almost λ-statistically convergent sequences is denoted by $\hat{S}_\lambda^F$. In the special case $\lambda_n = n$ for all $n \in N$, we shall write $\hat{S}^F$ instead of $\hat{S}_\lambda^F$ and we say that $X = (X_k)$ is almost statistically convergent to the fuzzy number X_o.

Let $X = (X_k)$ be a sequence of fuzzy numbers and $p = (p_k)$ be a sequence of strictly positive real numbers. Then the sequence $X = (X_k)$ is said to be strongly almost λ- convergent if there is a fuzzy number X_o such that

$$\lim_{n\to\infty} \lambda_n^{-1} \sum_{k\in I_n} \left[d\left(t_{km}(X), X_o\right)\right]^{p_k} = 0,$$ uniformly in m.

In this case, we write $X_k \to X_o\left(\hat{w}_\lambda^F, p\right)$ or $\left(\hat{w}_\lambda^F, p\right) - \lim X_k = X_o$. The set of all strongly almost $\lambda-$convergent sequences is denoted by $\left(\hat{w}_\lambda^F, p\right)$. In the special case $\lambda_n = n$ for all $n \in N$, we shall write $\left(\hat{w}^F, p\right)$ instead of $\left(\hat{w}_\lambda^F, p\right)$ and we said that $X = (X_k)$ is strongly almost convergent to the fuzzy number X_0.

Let $X = (X_k)$ be sequence of fuzzy numbers. Then the sequence $X = (X_k)$ of fuzzy numbers is said to be almost bounded if the set $\{t_{km}(X) : k, m \in N\}$ of fuzzy numbers is bounded. By $\hat{l}_\infty^F$, we shall denote the set of all almost bounded sequences of fuzzy numbers.

In this section we give some inclusion relations between strongly almost λ- convergence and almost λ-statistically convergence and show that they are equivalent for almost bounded sequences of fuzzy numbers. We also study the inclusion $\hat{S}^F \subset \hat{S}_\lambda^F$ under certain restrictions on the sequence $\Lambda = (\lambda_n)$.

Theorem 2.1. If $X = (X_k)$, $Y = (Y_k) \in \hat{S}_\lambda^F$ and $c \in R$, then

(a) $\hat{S}_\lambda^F - \lim cX_k = c.\hat{S}_\lambda^F - \lim X_k$.

(b) $\hat{S}_\lambda^F - \lim(X_k + Y_k) = \hat{S}_\lambda^F - \lim X_k + \hat{S}_\lambda^F - \lim Y_k$.

Proof. (a) Let $X = (X_k) \in \hat{S}_\lambda^F$ so that $\hat{S}_\lambda^F - \lim X_k = X_o$, $c \in R$ and $\varepsilon > 0$. Then the proof follows from the following inequality;

$$\lambda_n^{-1}\left|\left\{k \in I_n : d\left(t_{km}(cX), cX_o\right) \geq \varepsilon\right\}\right| \leq \lambda_n^{-1}\left|\left\{k \in I_n : d\left(t_{km}(X), X_o\right) \geq \frac{\varepsilon}{|c|}\right\}\right|$$

for all $m \in N$.

(b) Suppose that $X = (X_k)$, $Y = (Y_k) \in \hat{S}_\lambda^F$ so that $\hat{S}_\lambda^F - \lim X_k = X_o$ and $\hat{S}_\lambda^F - \lim Y_k = Y_o$. By Minkowski's inequality, we get

$$d\left(t_{km}(X+Y), X_o + Y_o\right) \leq d\left(t_{km}(X), X_o\right) + d\left(t_{km}(Y), Y_o\right).$$

Therefore given $\varepsilon > 0$, for all $m \in N$, we have

$$\lambda_n^{-1}\left|\left\{k \in I_n : d\left(t_{km}(X+Y), X_o + Y_o\right) \geq \varepsilon\right\}\right|.$$

$$\leq \lambda_n^{-1}\left|\left\{k \in I_n : d\left(t_{km}(X), X_o\right) \geq \frac{\varepsilon}{2}\right\}\right| + \lambda_n^{-1}\left|\left\{k \in I_n : d\left(t_{km}(Y), Y_o\right) \geq \frac{\varepsilon}{2}\right\}\right|$$

Hence, we obtain the result.

The following theorem shows that almost λ-statistical convergence and strongly almost λ-convergence are equivalent for almost bounded sequences of fuzzy numbers.

Theorem 2.2. Let the sequence $p = (p_k)$ be bounded and $X = (X_k)$ be a sequence of fuzzy numbers. Then

(a) $X_k \to X_o\left(\hat{w}_\lambda^F, p\right)$ implies $X_k \to X_o\left(\hat{S}_\lambda^F\right)$.

(b) $X = (X_k) \in \hat{l}_\infty^F$ and $X_k \to X_o\left(\hat{S}_\lambda^F\right)$ imply $X_k \to X_o\left(\hat{w}_\lambda^F, p\right)$.

(c) $\hat{S}_\lambda^F \cap \hat{l}_\infty^F = \left(\hat{w}_\lambda^F, p\right) \cap \hat{l}_\infty^F$.

Proof. (a) Let $\varepsilon > 0$ and $X_k \to X_o\left(\hat{w}_\lambda^F, p\right)$. For all $m \in N$, we have

$$\lambda_n^{-1} \sum_{k\in I_n} d\left(t_{km}(X), X_o\right)^{p_k} \geq \lambda_n^{-1} \sum_{\substack{k\in I_n \\ d(t_{km}(X), X_o) \geq \varepsilon}} d\left(t_{km}(X), X_o\right)^{p_k}$$

$$\geq \lambda_n^{-1}\left|\left\{k \in I_n : d\left(t_{km}(X), X_o\right) \geq \varepsilon\right\}\right|.\min\left(\varepsilon^h, \varepsilon^H\right).$$

Hence $X = (X_k) \in \hat{S}_\lambda^F$.

(b) Suppose that $X = (X_k) \in \hat{S}_\lambda^F \cap \hat{l}_\infty^F$. Since $X = (X_k) \in \hat{l}_\infty^F$, we write

$d\left(t_{km}(X), X_o\right) \leq T$ for all $m \in N$. Given $\varepsilon > 0$, for all $m \in N$, let

$G_n = \left|\left\{k \in I_n : d\left(t_{km}(X), X_o\right) \geq \varepsilon\right\}\right|$ and $H_n = \left|\left\{k \in I_n : d\left(t_{km}(X), X_o\right) < \varepsilon\right\}\right|$.

Then we have

$$\lambda_n^{-1} \sum_{k\in I_n} d\left(t_{km}(X), X_o\right)^{p_k} = \lambda_n^{-1} \sum_{k\in G_n} d\left(t_{km}(X), X_o\right)^{p_i} + \lambda_n^{-1} \sum_{k\in H_n} d\left(t_{km}(X), X_o\right)^{p_i}$$

$$\leq \max\left(T^h, T^H\right)\lambda_n^{-1} G_n + \max\left(\varepsilon^h, \varepsilon^H\right).$$

Taking the limit as $\varepsilon \to 0$ and $n \to \infty$, it follows that $X = (X_k) \in (\hat{w}_\lambda^F, p)$.

(c) Follows from (a) and (b).

Theorem 2.3. If $\liminf_n \lambda_n n^{-1} > 0$, then $\hat{S}^F \subset \hat{S}_\lambda^F$.

Proof. Let $X = (X_k) \in \hat{S}^F$. For given $\varepsilon > 0$, we get

$$\left|\{k \le n : d(t_{km}(X), X_o) \ge \varepsilon\}\right| \supset G_n$$

where G_n is the same as given in Theorem 2.2.(b). Thus,

$$n^{-1}\left|\{k \le n : d(t_{km}(X), X_o) \ge \varepsilon\}\right| \ge n^{-1}G_n = \lambda_n n^{-1}.$$

Taking limit as $n \to \infty$ and using $\liminf_n \lambda_n n^{-1} > 0$, we get $X = (X_k) \in \hat{S}_\lambda^F$.

Theorem 2.4. Let $0 < p_k \le q_k$ and $(p_k q_k^{-1})$ be bounded. Then $(\hat{w}_\lambda^F, q) \subset (\hat{w}_\lambda^F, p)$.

Proof. Let $X = (X_k) \in (\hat{w}_\lambda^F, q)$. Let $w_k = d(t_{km}(X), X_o)^{q_k}$ for all $m \in N$ and $\lambda_k = p_k q_k^{-1}$ for all $k \in N$. Then $0 < \lambda_k \le 1$ for all $k \in N$. Let b be a constant such that $0 < b \le \lambda_k \le 1$ for all $k \in N$.

Define the sequences (u_k) and (v_k) as follows:

For $w_k \ge 1$, let $(u_k) = (w_k)$ and $v_k = 0$ and for $w_k < 1$, let $u_k = 0$ and $v_k = w_k$. Then it is clear that for all $k \in N$, we have $w_k = u_k + v_k$ and $w_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that $u_k^{\lambda_k} \le u_k \le w_k$ and $v_k^{\lambda_k} \le v_k^{\lambda}$. Therefore

$$\lambda_n^{-1}\sum_{k\in I_n} w_k^{\lambda_k} = \lambda_n^{-1}\sum_{k\in I_n}(u_k + v_k)^{\lambda_k} \le \lambda_n^{-1}\sum_{k\in I_n} w_k + \lambda_n^{-1}\sum_{k\in I_n} v_k^{\lambda}.$$

Now for each n,

$$\lambda_n^{-1}\sum_{k\in I_n} v_k^{\lambda} = \sum_{k\in I_n}\left(\lambda_n^{-1}v_k^{\lambda}\right)^{\lambda}\left(\lambda_n^{-1}\right)^{1-\lambda}$$

$$\le \left(\sum_{k\in I_n}\left(\left(\lambda_n^{-1}v_k^{\lambda}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\lambda}\left(\sum_{k\in I_n}\left(\left(\lambda_n^{-1}\right)^{1-\lambda}\right)^{\frac{1}{1-\lambda}}\right)^{1-\lambda}$$

$$= \left(\lambda_n^{-1}\sum_{k\in I_n} v_k\right)^{\lambda}$$

and so

$$\lambda_n^{-1}\sum_{k\in I_n} w_k^{\lambda_k} \le \lambda_n^{-1}\sum_{k\in I_n} w_k + \left(\lambda_n^{-1}\sum_{k\in I_n} v_k\right)^{\lambda}.$$

Hence $X = (X_k) \in (\hat{w}_\lambda^F, p)$ i.e. $(\hat{w}_\lambda^F, q) \subset (\hat{w}_\lambda^F, p)$.

Theorem 2.5. $\hat{l}_\infty^F = \hat{w}_{\lambda,\infty}^F$,

where:

$$\hat{w}_{\lambda,\infty}^F = \left\{X = (X_k) : \sup_{n,m} \lambda_n^{-1}\sum_{k\in I_n} d(t_{km}(X), \overline{0}) < \infty\right\}.$$

Proof. Let $X = (X_k) \in \hat{w}_{\lambda,\infty}^F$. Then there exists a constant $T_1 > 0$ such that

$$\lambda_1^{-1} d(t_{1m}(X), X_o) \le \sup_{n,m} \lambda_n^{-1}\sum_{k\in I_n} d(t_{km}(X), \overline{0}) \le T_1$$

for all $m \in N$

and so we have $X = (X_k) \in \hat{l}_\infty^F$. Conversely, let $X = (X_k) \in \hat{l}_\infty^F$. Then there exists a constant $T_2 > 0$ such that $d(t_{km}(X), X_o) \le T_2$ for all k and m. So,

$$\lambda_n^{-1}\sum_{k\in I_n} d(t_{km}(X), \overline{0}) \le T_2 \lambda_n^{-1}\sum_{k\in I_n} 1 \le T_2,$$ for all k and m.

Thus $X = (X_k) \in \hat{w}_{\lambda,\infty}^F$.

Conclusion

In this paper, we constructed the concepts of almost λ-statistical convergence of fuzzy numbers and then we obtained effectiveness results in connection with these concepts. Moreover, we showed that almost λ-statistical convergence and strongly almost λ-convergence are equivalent for almost bounded sequences of fuzzy numbers.

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