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Asymptotically double lacunry equivalent sequences defined by Orlicz functions

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ABSTRACT. This paper presents the following definition which is natural combination of the definition for asymptotically equivalent and Orlicz function. The two nonnegative double sequences $x=\left(x_{k,l}\right)$ and $y=\left(y_{k,l}\right)$ are said to be M-asymptotically double equivalent to multiple L provided that for every $\varepsilon>0$, $P-\lim_{k,l} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)=0$, for some $\rho>0$, (denoted by $x\overset{M}{\sim}y$) and simply M-asymptotically double equivalent if L=1. Also we give some new concepts related to this definition and some inclusion theorems.

Keywords: asymptotically equivalence, double sequences, P-convergent, double lacunary sequence.

Sequências asintomaticas de espaços duplos definidas pelas funções de Orlicz

RESUMO. Apresenta-se uma definição, a qual é a combinação natural da definição para o equivalente asintótico e função de Orlicz. As duplas sequências nâo-negativas $x=\left(x_{k,l}\right)$ e $y=\left(y_{k,l}\right)$ são o equivalente duplo M-asintótico para o múltiplo L desde que, para cada $\varepsilon>0$, $P-\lim_{k,l} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)=0$, para outros $\rho>0$, (denotado por $x\overset{M}{\sim}y$) e simplesmente equivalente duplo M-asintótico se L=1. Novos conceitos a essa definição e teoremas de inclusão são também proporcionados.

Palavras-chave: equivalência asintótica; sequências duplas; P-convergente, sequência de espaços duplos.

Introduction

In 1993, Marouf (1993) presented definitions for asymptotically equivalent sequences for single sequences and asymptotic regular matrices. In 2003, Patterson (2003) extend these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Later these definitions extended to lacunary sequences by Patterson and Savas (2006) In 2009, Esi (2009) extended these definitions to double lacunary sequences.

Definitions and notations

An Orlicz function is a function $M:\left(0,\infty\right]\to\left(0,\infty\right]$ which is continous, non-decreasing and convex with $M\left(0\right)=0$, $M\left(x\right)>0$ for $x>0$ and $M\left(x\right)\to\infty$ as $x\to\infty$.

An Orlicz function is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K>0$, such that $M(2u)\leq KM(u), u\geq 0$.

The idea of sequence spaces defined by Orlicz functions were introduced and studied by several authors such as Altin et al. (2005) (ALTINOK et al., 2004, 2008), Bektas and Altin (2003), Tripathy and Altin (2008) and many others.

Now we recall some definitions of double sequences.

Definition 2.1. (PRINGSHEIM, 1900) A double sequence $x=\left(x_{k,l}\right)$ has Pringsheim limit L (denoted $P-limx=L$) provided that given $\varepsilon>0$, there exists $N\in\mathrm{N}$ such that $\left|x_{k,l}-L\right|<\varepsilon$, whenever $k,l>N$. We shall describe such an $x=\left(x_{k,l}\right)$ more briefly as "P-convergent". By a bounded double sequence we shall mean there exists a positive number K such that $|x_{k,l}| < K$ for all $k,l \in \mathbb{N}$ and denote such bounded sequence by $\|x\|_{(\infty,2)} = \sup_{k,l}|x_{k,l}| < \infty$. We shall also denote the set of all bounded double sequences by l_{∞}^{ii}. We also note in contrast to the case for single sequence, a P-convergent double sequence need not to be bounded.

Definition 2.2 (SAVAS; PATTERSON, 2011) The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that

$$k_o = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_o = 0, \overline{h_s} = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h_s}$ and $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\},$$

$$\mathrm{q_r} = \frac{\mathrm{k_r}}{\mathrm{k_{r-1}}}, \overline{\mathrm{q_s}} = \frac{\mathrm{l_s}}{\mathrm{l_{s-1}}} \text{ and } \mathrm{q_{r,s}} = \mathrm{q_r}\overline{\mathrm{q_s}}.$$

Definition 2.3. (MURSALEEN; EDELY, 2003) A real double sequence $x = (x_{k,l})$ is said to be statistically convergent *to* L provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn}\left|\{(k,l) : k \leq m, l \leq n; |x_{k,l} - L| \geq \varepsilon\}\right| = 0.$$

In this case, we write $St_2 - \lim_{k,l} x_{k,l} = L$ and we denote the set of all P – statistical convergent double sequences by St_2.

Definition 2.4. (PATTERSON; SAVAS, 2006) The two nonnegative double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be asymptotically double equivalent of multiple L provided that

$$P - \lim_{k,l} \frac{x_{k,l}}{y_{k,l}} = L$$

(denoted by $x \overset{P}{\sim} y$) and simply asymptotically double equivalent if L=1.

Definition 2.5. (ESI; ACIKGOZ, 2014) The two nonnegative double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be asymptotically double statistical equivalent of multiple L provided that for every $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{mn}\left|\left\{(k,l) : k \leq m \text{ and } l \leq n, \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\}\right| = 0.$$

(denoted by $x \overset{S^L}{\sim} y$) and simply asymptotically double statistical equivalent if L=1.

Definition 2.6. (ESI, 2009) Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence; the two nonnegative double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be asymptotically double lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$,

$$P - \lim_{r,s} \frac{1}{h_{r,s}}\left|\left\{(k,l) \in I_{r,s} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\}\right| = 0$$

(denoted by $x \overset{S_{\theta_{r,s}}^L}{\sim} y$) and simply asymptotically double lacunary statistical equivalent if L=1. Furthermore, let $S_{\theta_{r,s}}^L$ denote the set of all sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ such that $x \overset{S_{\theta_{r,s}}^L}{\sim} y$.

Definition 2.7. (ESI, 2009) Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence; the two double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be strong asymptotically double lacunary equivalent of multiple L provided that

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left|\frac{x_{k,l}}{y_{k,l}} - L\right| = 0,$$

(denoted by $x \overset{N_{\theta_{r,s}}^L}{\sim} y$) and simply strong asymptotically double lacunary equivalent if L=1. In addition, let $N_{\theta_{r,s}}^L$ denote the set of all sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ such that $x \overset{N_{\theta_{r,s}}^L}{\sim} y$.

Definition 2.8. Let M be an Orlicz function. The two nonnegative double sequences $x=(x_{k,l})$ and $y=(y_{k,l})$ are said to be M-asymptotically double equivalent of multiple L provided that

$$P-\lim_{k,l} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)=0, \text{ for some } \rho>0,$$

(denoted by $x \overset{M^L}{\sim} y$ and simply M-asymptotically double equivalent if $L=1$.

Since Orlicz function M is continuous and $M(0)=0$ then

$$P-\lim_{k,l} M\left(\left|\frac{x_{k,l}}{y_{k,l}}-L\right|\right)=M\left(P-\lim_{k,l}\left(\left|\frac{x_{k,l}}{y_{k,l}}-L\right|\right)\right)=0$$

$$\Leftrightarrow P-\lim_{k,l}\left(\left|\frac{x_{k,l}}{y_{k,l}}-L\right|\right)=0.$$

Therefore $x \overset{L}{\sim} y \Leftrightarrow x \overset{M^L}{\sim} y$. This means that ordinary asymptotically double equivalence is equivalent to M-asymptotically double equivalence.

Definition 2.9. Let M be an Orlicz function. The two nonnegative double sequences $x=(x_{k,l})$ and $y=(y_{k,l})$ are said to be strong M-asymptotically double equivalent of multiple L provided that

$$P-\lim_{m,n}\frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)=0, \text{ for some } \rho>0, ($$

denoted by $x \overset{[M]^L}{\sim} y$) and simply strong M-asymptotically double equivalent if $L=1$. If $M(x)=x$, then we obtain

$$P-\lim_{m,n}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{k,l}}{y_{k,l}}-L\right|=0,$$

(denoted by $x \overset{[L]}{\sim} y$) which is called strong asymptotically double equivalent of multiple L.

Definition 2.10. Let M be an Orlicz function and $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence; the two double sequences $x=(x_{k,l})$ and $y=(y_{k,l})$ are said to be strong M-asymptotically double lacunary equivalent of multiple L provided that

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)=0, \text{ for some } \rho>0, ($$

denoted by $x \overset{N_{\theta_{r,s}}^{[M]^L}}{\sim} y$) and simply strong M-asymptotically double lacunary equivalent if $L=1$. If $M(x)=x$, then we obtain the set $N_{\theta_{r,s}}^{L}$ which was defined in (ALTINOK et al., 2008).

Main theorems

Theorem 3.1. Let M be an Orlicz function.Then

(a) If $x \overset{[L]}{\sim} y$ then $x \overset{[M]^L}{\sim} y$,

(b) If $\beta=\lim_{t\to\infty}\frac{M\left(\frac{t}{\rho}\right)}{t}\geq 1$, then $x \overset{[L]}{\sim} y \Leftrightarrow x \overset{[M]^L}{\sim} y$.

Proof.(a) Let $x \overset{[L]}{\sim} y$ and $\varepsilon>0$. We choose $0<\delta<1$ such that $M(t)<\varepsilon$ for every t with $0\leq t\leq\delta$. We can write

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right).$$

$$=\frac{1}{mn}\sum_{\substack{k,l=1,1\\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|\leq\delta}}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)+\frac{1}{mn}\sum_{\substack{k,l=1,1\\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|>\delta}}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right).$$

It is clear that

$$\frac{1}{mn}\sum_{\substack{k,l=1,1\\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|\le\delta}}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)<\varepsilon.$$

On the other hand we use the fact that $|a|\le 1+[\![a]\!]$, where $[\![a]\!]$ denotes the integer part of a. Since M is an Orlicz function, we have

$$M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)\ge M(1).$$

Now let us the consider the second part where the sum is taken over $\left|\frac{x_{k,l}}{y_{k,l}}-L\right|>\delta.$ Thus

$$\frac{1}{mn}\sum_{\substack{k,l=1,1\\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|>\delta}}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)$$

$$\le\frac{1}{mn}\sum_{\substack{k,l=1,1\\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|>\delta}}^{m,n} M\left(1+\left[\!\!\left[\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right]\!\!\right]\right)$$

$$\le 2M(1)\delta^{-1}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{k,l}}{y_{k,l}}-L\right|.$$

We have

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)$$

$$\le\varepsilon+2M(1)\delta^{-1}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{k,l}}{y_{k,l}}-L\right|.$$

Therefore $x\overset{[M]^L}{\sim}y$.

(b) Let $\beta\ge 1$ and $x\overset{[M]^L}{\sim}y$ Since $\beta\ge 1$, we have $M(t)\ge\beta t$ for all $t\ge 0.$ It follows that $x\overset{[M]^L}{\sim}y$ implies $x\overset{[L]}{\sim}y$ This and from (a) imply $x\overset{[L]}{\sim}y \Leftrightarrow x\overset{[M]^L}{\sim}y$

Theorem 3.2. Let M be an Orlicz function. Then

(a) If $x\overset{[M]^L}{\sim}y$ then $x\overset{S^L}{\sim}y$,

(b) If $x=(x_{k,l})$ and $y=(y_{k,l})$ are in l_∞^{ii}, then $x\overset{S^L}{\sim}y \Leftrightarrow x\overset{[M]^L}{\sim}y$.

Proof.(a) Given $\varepsilon>0.$ Then

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)\ge\frac{1}{mn}\sum_{\substack{k,l=1,1\\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|\ge\varepsilon}}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)$$

$$\ge M\left(\frac{\varepsilon}{\rho}\right)\frac{1}{mn}\left|\left\{(k,l):k\le m\text{ and }l\le n,\left|\frac{x_{k,l}}{y_{k,l}}-L\right|\ge\varepsilon\right\}\right|$$

from which the result follows.

(b) Suppose that $x=(x_{k,l})$ and $y=(y_{k,l})$ are in l_∞^{ii} and $x\overset{S^L}{\sim}y$ Then we can assume that $\left|\frac{x_{k,l}}{y_{k,l}}-L\right|\le H$ for all k and l. Given $\varepsilon>0.$

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)$$

$$=\frac{1}{mn}\sum_{\substack{k,l=1,1\\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|\ge\varepsilon}}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)+\frac{1}{mn}\sum_{\substack{k,l=1,1\\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|<\varepsilon}}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}}-L\right|}{\rho}\right)$$

$$\le M\left(\frac{H}{\rho}\right)\frac{1}{mn}\left|\left\{(k,l):k\le m\text{ and }l\le n,\left|\frac{x_{k,l}}{y_{k,l}}-L\right|\ge\varepsilon\right\}\right|$$

$$+M\left(\frac{\varepsilon}{\rho}\right).$$

The result follows from this and the proof of (a).

Theorem 3.3. Let M be an Orlicz function and $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence with

$\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$ then

$x \overset{[M]^L}{\sim} y$ implies $x \overset{N_{\theta_{r,s}}^{[M]^L}}{\sim} y$.

Proof. Suppose that $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$ then there exists $\delta > 0$ such that $q_r > 1+\delta$ and $\overline{q}_s > 1+\delta$. This implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\overline{h}_s}{l_s} \geq \frac{\delta}{1+\delta}$. Then for $x \overset{[M]^L}{\sim} y$, we can write

$$A_{r,s} = \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)$$

$$= \frac{1}{h_{r,s}} \sum_{k,l=1,1}^{k_r,l_s} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right) - \frac{1}{h_{r,s}} \sum_{k,l=1,1}^{k_{r-1},l_{s-1}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)$$

$$- \frac{1}{h_{r,s}} \sum_{k=k_{r+1},l=1}^{k_r-1,l_{s-1}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right) - \frac{1}{h_{r,s}} \sum_{k=1,l=l_s+1}^{k_{r-1},l_s-1} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)$$

$$= \frac{k_r l_s}{h_{r,s}} \left[\frac{1}{k_r l_s} \sum_{k,l=1,1}^{k_r,l_s} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)\right]$$

$$- \frac{k_{r-}l_{s-1}}{h_{r,s}} \left[\frac{1}{k_{r-1}l_{s-1}} \sum_{k,l=1,1}^{k_{r-1},l_{s-1}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)\right]$$

$$- \frac{1}{h_r} \sum_{k=k_{r-1}+1}^{k_r} \frac{l_s-1}{h_s} \frac{1}{l_s-1} \sum_{l=1}^{l_s-1} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)$$

$$- \frac{1}{h_s} \sum_{l=l_{r-1}+1}^{l_r} \frac{k_r-1}{h_r} \frac{1}{k_r-1} \sum_{k=1}^{k_r-1} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right).$$

Since $x \overset{[M]^L}{\sim} y$ the last two terms tends zero in the Pringsheim sense, thus we can write

$$A_{r,s} = \frac{k_r l_s}{h_{r,s}} \left[\frac{1}{k_r l_s} \sum_{k,l=1,1}^{k_r,l_s} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)\right]$$

$$- \frac{k_{r-}l_{s-1}}{h_{r,s}} \left[\frac{1}{k_{r-1}l_{s-1}} \sum_{k,l=1,1}^{k_{r-1},l_{s-1}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)\right] + o(1).$$

Since $h_{r,s} = k_r l_s - k_{r-1} l_{s-1}$, we are granted the following

$$\frac{k_r l_s}{h_{r,s}} \leq \frac{1+\delta}{\delta} \text{ and } \frac{k_{r-1} l_{s-1}}{h_{r,s}} \leq \frac{1}{\delta}.$$

So, the terms

$$\frac{1}{k_r l_s} \sum_{k,l=1,1}^{k_r,l_s} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)$$

and

$$\frac{1}{k_{r-1} l_{s-1}} \sum_{k,l=1,1}^{k_{r-1},l_{s-1}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)$$

are both Pringheim null sequences. Therefore $A_{r,s}$ is a Pringsheim null sequence. Thus $x \overset{N_{\theta_{r,s}}^{[M]^L}}{\sim} y$. This completes the proof.

Theorem 3.4. Let M be an Orlicz function and $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence with $\limsup_r q_r < \infty$ and $\limsup_s \overline{q}_s < \infty$ then

$++$

$x \overset{N_{\theta_{r,s}}^{[M]^L}}{\sim} y$ implies $x \overset{[M]^L}{\sim} y$.

Proof. Since $\limsup_r q_r < \infty$ and $\limsup_s \bar{q}_s < \infty$ there exists $H > 0$ such that $q_r < H$ and $\bar{q}_s < H$ for all r and s. Let $x \overset{S^L_{\theta_{r,s}}}{\sim} y$ and $\varepsilon > 0$. Then there exists $r_o > 0$ and $s_o > 0$ such that for every $i \geq r_o$ and $j \geq s_o$

$$B_{i,j} = \frac{1}{h_{i,j}} \sum_{(k,l)\in I_{i,j}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right) < \varepsilon.$$

Let $T = \max\{B_{i,j} : 1 \leq i \leq r_o \text{and} 1 \leq j \leq s_o\}$ and m and n be such that $k_{r-1} < m \leq k_r$ and $l_{s-1} < n \leq l_s$. Thus we obtain the following

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right) \leq \frac{1}{k_{r-1}l_{s-1}}\sum_{k,l=1,1}^{k_r,l_s} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)$$

$$\leq \frac{1}{k_{r-1}l_{s-1}}\sum_{t,u=1,1}^{r,s}\left(\sum_{(k,l)\in I_{t,u}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)\right)$$

$$\leq \frac{1}{k_{r-1}l_{s-1}}\sum_{t,u}^{r_o,s_o} h_{t,u}B_{t,u} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}B_{t,u}$$

$$\leq \frac{T}{k_{r-1}l_{s-1}}\sum_{t,u}^{r_o,s_o} h_{t,u} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}B_{t,u}$$

$$\leq \frac{Tk_{r_o}l_{s_o}r_o s_o}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}B_{t,u}$$

$$\leq \frac{Tk_{r_o}l_{s_o}r_o s_o}{k_{r-1}l_{s-1}} + \left(\sup_{t\geq r_o\cup u\geq s_o} B_{t,u}\right)\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}$$

$$\leq \frac{Tk_{r_o}l_{s_o}r_o s_o}{k_{r-1}l_{s-1}} + \frac{\varepsilon}{k_{r-1}l_{s-1}}\sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}$$

$$\leq \frac{Tk_{r_o}l_{s_o}r_o s_o}{k_{r-1}l_{s-1}} + \varepsilon H^2.$$

Since k_r and l_s both approaches infinity as both m and n approaches infinity, it follows that $x \overset{[M]^L}{\sim} y$. This completes the proof.

The following theorem is an immediate consequence of Theorem 3.3 and Theorem 3.4.

Theorem 3.5. Let M be an Orlicz function and $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence with $1 < \liminf_{r,s} q_{rs} \leq \limsup_{r,s} q_{rs} < \infty$, then

$$x \overset{[M]^L}{\sim} y \Leftrightarrow x \overset{N^{[M]^L}_{\theta_{r,s}}}{\sim} y.$$

Theorem 3.6. Let M be an Orlicz function and $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. Then

(a) If $x \overset{N^{[M]^L}_{\theta_{r,s}}}{\sim} y$ then $x \overset{S^L_{\theta_{r,s}}}{\sim} y$.

(b) If $x = (x_{k,l})$ and $y = (y_{k,l})$ are in l^{ii}_∞, then

$$x \overset{N^{[M]^L}_{\theta_{r,s}}}{\sim} y \Leftrightarrow x \overset{S^L_{\theta_{r,s}}}{\sim} y.$$

Proof. (a) Given $\varepsilon > 0$. Then for some $\rho > 0$

$$\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right) \geq \frac{1}{h_{r,s}}\sum_{\substack{(k,l)\in I_{r,s}\\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|\geq\varepsilon}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)$$

$$\geq M\left(\frac{\varepsilon}{\rho}\right)\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\}\right|$$

from which the result follows.

(b) Given $\varepsilon > 0$. Suppose that $x = (x_{k,l})$ and $y = (y_{k,l})$ are in l^{ii}_∞ and $x \overset{S^L_{\theta_{r,s}}}{\sim} y$. Then we can assume that $\left|\frac{x_{k,l}}{y_{k,l}} - L\right| \leq H$, for all k and l. Now for some $\rho > 0$

$$\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right) = \frac{1}{h_{r,s}}\sum_{\substack{(k,l)\in I_{r,s}\\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|\geq\varepsilon}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)$$

$$+ \frac{1}{h_{r,s}}\sum_{\substack{(k,l)\in I_{r,s}\\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|<\varepsilon}} M\left(\frac{\left|\frac{x_{k,l}}{y_{k,l}} - L\right|}{\rho}\right)$$

$$\leq M\left(\frac{H}{\rho}\right)\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}:\left|\frac{x_{k,l}}{y_{k,l}}-L\right|\geq\varepsilon\right\}\right|+M\left(\frac{\varepsilon}{\rho}\right).$$

This and from (a) the result follows.

Theorem 3.7. Let M be an Orlicz function. Then

(a) If $x \overset{N_{\theta_{r,s}}^{L}}{\sim} y$ then $x \overset{N_{\theta_{r,s}}^{[M]^L}}{\sim} y$,

(b) If $\beta=\lim_{t\to\infty}\frac{M\left(\frac{t}{\rho}\right)}{t}\geq 1$, then $x \overset{N_{\theta_{r,s}}^{L}}{\sim} y \Leftrightarrow x \overset{N_{\theta_{r,s}}^{[M]^L}}{\sim} y$

Proof. The proof is similar to the Theorem 3.1., so we omit it.

Conclusion

The definition of asymptotically equivalent sequences was introduced by Marouf (1993) in 1993. Later on it was further investigated from sequence space point of view and linked with summability theory by several authors. The results obtained in this study are much more general than those obtained earlier.

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