



# On recognition of simple group $L_2(r)$ by the number of Sylow subgroups

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**ABSTRACT.** Let  $G$  be a finite group and  $n_p(G)$  be the number of Sylow  $p$ -subgroup of  $G$ . In this work it is proved if  $G$  is a finite centerless group and  $n_p(G) = n_p(L_2(r))$  for every prime  $p \in \pi(G) = \pi(L_2(r))$ , where  $r$  is a prime but not Mersenne prime and  $r^2$  does not divide order of  $G$ , then  $L_2(r) \leq G \leq \text{Aut}(L_2(r))$ .

**Keywords:** finite group, simple group, Sylow subgroup.

## Sobre o reconhecimento do grupo simples $L_2(r)$ pelo número dos sub-grupos de Sylow

**RESUMO.** Vamos supor que  $G$  é um grupo finito e  $n_p(G)$  é o número de Sylow  $p$ -sub-grupo de  $G$ . Nessa pesquisa prova-se que, se  $G$  é um grupo finito sem centro e  $n_p(G) = n_p(L_2(r))$  for cada primo  $p \in \pi(G) = \pi(L_2(r))$ , onde  $r$  é um primo mas não um primo de Mersenne e  $r^2$  não divide a ordem de  $G$ , então  $L_2(r) \leq G \leq \text{Aut}(L_2(r))$ .

**Palavras-chave:** grupo finito, grupo simples, sub-grupo de Sylow.

### Introduction

If  $n$  is an integer, then we denote by  $\pi(n)$  the set of all prime divisors of  $n$ . Let  $G$  be a finite group. Denote by  $\pi(G)$  the set of primes  $p$  such that  $G$  contains an element of order  $p$ . Throughout this paper, we denote by  $n_p(G)$  the number of Sylow  $p$ -subgroup of  $G$  that is,  $n_p(G) = |\text{Syl}_p(G)|$ . All other notations are standard and we refer to (CONWAY et al., 1985), for example.

In 1992, Bi (BI, 1992) showed that  $L_2(p^k)$  can be characterized only by the order of normalizer of its Sylow subgroups. In other words, if  $G$  is a group and  $|N_G(P)| = |N_{L_2(p^k)}(Q)|$ , where  $P \in \text{Syl}_r(G)$  and  $Q \in \text{Syl}_r(L_2(p^k))$  for every prime  $r$ , then  $G \cong L_2(p^k)$ . This type of characterization is known for the following simple groups:  $L_2(p^k)$  (BI, 1992),  $L_n(q)$  (BI, 1995),  $S_4(q)$  (BI, 2001), the alternating simple groups (BI, 2001),

$U_n(q)$  (BI, 1990, 2004), the sporadic simple groups (KHOSRAVI; KHOSRAVI, 2005) and  ${}^2D_n(p^k)$  (IRANMANESH; AHANJIDEH, 2008).

Let  $S$  be one of the above simple groups. It is clear that if  $n_p(G) = n_p(S)$  for every prime  $p$  and  $|G| = |S|$ , then  $|N_G(P)| = |N_S(Q)|$ , where  $P \in \text{Syl}_p(G)$  and  $Q \in \text{Syl}_p(S)$ . Thus by the above references,  $G \cong S$ . Now one may ask the following question:

If only  $n_p(G) = n_p(S)$  for every prime  $p$ , then whether  $G$  is isomorphic to  $S$ ? By the following counterexample it is not always true. Consider the simple group  $A_5$ . Let  $G = A_5 \times H$  where  $H$  is a finite nilpotent group with  $\pi(H) \subseteq \{2, 3, 5\}$ . We have  $n_p(G) = n_p(A_5 \times H)$  for every prime  $p$ , but  $G$  is not isomorphic to  $A_5$ .

In Asboei (2013) and Asboei et al. (2011) it is proved if  $G$  is finite centerless group and  $n_p(G) = n_p(L_2(q))$  for every prime  $p \in \pi(G)$ , where  $q \neq 7$  and  $5 \leq q \leq 25$  is prime power,

then  $L_2(r) \leq G \leq \text{Aut}(L_2(r))$ . Also in Asboei et al. (2013) it is proved if  $G$  is a finite centerless group and  $n_p(G) = n_p(M)$ , where  $M$  denotes either of the Mathieu groups  $M_{11}$  or  $M_{12}$  for every prime  $p \in \pi(G)$ , then  $M \leq G \leq \text{Aut}(M)$ . It seems that if  $Z(G) = 1$ , then this type of characterization works. The main theorem of this paper is as follows:

### Main theorem

Let  $G$  be a finite centerless group such that  $n_p(G) = n_p(L_2(r))$  for every prime  $p \in \pi(G) = \pi(L_2(r))$ , where  $r$  is prime but not Mersenne prime and  $r^2$  does not divide order of  $G$ . Then  $L_2(r) \leq G \leq \text{Aut}(L_2(r))$ .

### Preliminary results

In this section we bring two preliminary lemmas used in the proof of the main theorem.

Lemma 1 - (ZHANG, 1995) Let  $G$  be a finite group and  $M$  be a normal subgroup of  $G$ . Then both  $n_p(M)$  and  $n_p(G/M)$  divide  $n_p(G)$  and moreover  $n_p(M)n_p(G/M) \mid n_p(G)$  for every prime  $p$ .

Lemma 2 - (BRAUER; REYNOLD, 1958) (Brauer-Reynolds) Suppose  $G = G'$  and that  $|G|$  is divisible by the prime  $r$  but not by  $r^2$ . If  $n_r(G) = r + 1$ , then  $G$  has a homomorphic image isomorphic to  $L_2(r)$ .

### Proof of the main theorem

First we show that  $n_r(L_2(r)) = r + 1$ . To find the number of Sylow  $r$ -subgroups in  $L_2(r)$  first, look at  $\text{SL}_2(r)$ . The normalizer of a Sylow  $r$ -subgroup is the set of upper triangular matrices with determinant 1, so the order of the normalizer is  $r(r-1)$ . The order of the whole group  $\text{SL}_2(r)$  is  $r(r-1)$ . Therefore the number of Sylow  $r$ -subgroups is  $r + 1$ .

This will be the same as the number of Sylow  $r$ -subgroups of  $L_2(r)$  because the canonical homomorphism from  $\text{SL}_2(r)$  to  $L_2(r)$  yields a bijection on Sylow  $r$ -subgroups. Let  $H$  be minimal among normal subgroups of  $G$  with order divisible by  $r$ . Then  $H$  has exactly  $r + 1$  Sylow  $r$ -subgroups, so by Lemma 2, if  $H = H'$ , then  $H$  has a homomorphic image isomorphic to  $L_2(r)$ . Therefore there exists normal subgroup  $N$  such that  $H/N$  is isomorphic to  $L_2(r)$ . Now set

$\bar{H} := H/N \cong L_2(r)$  and  $\bar{G} := G/N$ . Hence  $L_2(r) \cong \bar{H} \cong \bar{H}C_{\bar{G}}(\bar{H})/C_{\bar{G}}(\bar{H}) \leq \bar{G}/C_{\bar{G}}(\bar{H}) = N_{\bar{G}}(\bar{H})/C_{\bar{G}}(\bar{H}) \leq \text{Aut}(\bar{H})$ . If  $K = \{x \in G \mid xN \in C_{\bar{G}}(\bar{H})\}$ , then  $G/K \cong \bar{G}/C_{\bar{G}}(\bar{H})$ . So  $L_2(r) \leq G/K \leq \text{Aut}(L_2(r))$ . Therefore  $G/K$  is isomorphic to  $L_2(r)$  or  $L_2(r).2 = \text{PGL}_2(r)$ .

First let  $G/K$  isomorphic to  $L_2(r)$ . By Lemma 1,  $n_p(K) = 1$  for every prime  $p \in \pi(G)$ . Thus  $K$  is a nilpotent subgroup of  $G$ .

We claim that  $K = 1$ . Let  $Q$  be a Sylow  $q$ -subgroup of  $K$ , since  $K$  is nilpotent,  $Q$  is normal in  $G$ . Now if  $P \in \text{Syl}_p(G)$ , then  $P$  normalizes  $Q$  and so if  $p \neq q$ , then  $P \leq N_G(Q) = G$ . Also we note that  $KP/K$  is a Sylow  $p$ -subgroup of  $G/K$ . On the other hand, if  $R/K = N_{G/K}(KP/K)$ , then  $R = N_G(P)K$ . We also know that  $n_p(G) = n_p(G/K)$ , so  $|G : R| = |G : N_G(P)|$ . Thus  $R = N_G(P)$  and then  $K \leq N_G(P)$ . Therefore  $Q \leq N_G(P)$ . Since  $P \leq N_G(Q)$  and  $Q \leq N_G(P)$ , this implies that  $[P, Q] \leq P$  and  $[P, Q] \leq Q$ , and so  $[P, Q] \leq P \cap Q = 1$ . Thus  $P \leq C_G(Q)$  and  $Q \leq C_G(P)$ , in the other words,  $P$  and  $Q$  centralize each other. Let  $C = C_G(Q)$ . Then  $C$  contains a full Sylow  $p$ -subgroup of  $G$  for all primes  $p$  different from  $q$ , and thus  $|G : C|$  is a power of  $q$ . Now let  $S$  be a Sylow  $q$ -subgroup of  $G$ . Then  $G = CS$ . Also if  $Q > 1$ , then  $C_Q(S)$  is nontrivial, so  $C_Q(S) \leq Z(G)$ . Since by the assumption  $Z(G) = 1$ , it follows that  $Q = 1$ . Since  $q$  is arbitrary,  $K = 1$ , as claimed. Therefore,  $G$  is isomorphic to  $L_2(r)$ .

Now suppose that  $G/K$  is isomorphic to  $L_2(r).2 = \text{PGL}_2(r)$ . We know that for every  $p$ ,  $n_p(L_2(r)) \mid n_p(L_2(r).2)$ . If  $r \equiv \pm 1 \pmod{8}$ , then  $n_p(L_2(r)) = n_p(L_2(r).2)$  for every  $p$ . So if  $r \equiv \pm 1 \pmod{8}$ , then similar to the above discussion  $G$  is isomorphic to  $L_2(r).2$ .

If  $r \equiv \pm 3 \pmod{8}$ , then  $n_2(L_2(r)) \neq n_2(L_2(r).2)$  (we note that  $n_2(L_2(r).2)$  is greater than  $n_2(L_2(r)) = n_2(G)$ ). Since  $n_2(L_2(r)) = n_2(G)$  and by Lemma 1,  $n_2(L_2(r).2) \mid n_2(G)$ , we get a contradiction. Thus  $G$  is not isomorphic to  $L_2(r)$ .

We also can thus assume that  $H' < H$  and we

work to obtain a contradiction. Now  $r$  does not divide  $|H'|$ , and it follows that  $r$  divides  $|H:H'|$ , and thus  $H$  has a normal subgroup  $N$  with  $|H:N|=r$ . Let  $R$  be a Sylow  $r$ -subgroup of  $H$ , so  $R$  acts coprimely on  $N$ . Let  $p$  be any prime divisor of  $|N|$ , and choose an  $R$ -invariant Sylow  $p$ -subgroup  $P$  of  $N$ . If  $R$  does not centralizes  $P$ , then by Sylow's theorem, the group  $RP$  has at least  $r+1$  Sylow  $r$ -subgroups. Then  $RP$  has exactly  $r+1$  Sylow  $r$ -subgroups, and thus  $r+1$  is a power of  $p$ , and so  $p=2$  and  $r$  is Mersenne, which is a contradiction. It follows that  $R$  centralizes a Sylow  $p$ -subgroup of  $N$  for every prime  $p$ , and thus  $R$  centralizes  $N$  and hence  $R$  is normal in  $H$ . This is a contradiction since  $H$  has  $r+1$  Sylow  $r$ -subgroups, and the proof is complete.

## Conclusion

Our result states: if  $G$  is a finite centerless group such that  $n_p(G) = n_p(L_2(r))$  for every prime  $p$ , where  $r$  is prime but not Mersenne prime and  $r^2$  does not divide order of  $G$ , then  $G$  is isomorphic to  $L_2(r)$  or  $\text{Aut}(L_2(r) = \text{PGL}_2(r))$  when  $r \equiv \pm 1 \pmod{8}$ .

Comparing this type of characterization with characterization by orders of normalizers of Sylow subgroups, it seems that characterization by the number of Sylow subgroups is much stronger than characterization by orders of normalizers of Sylow subgroups.

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