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Some extended Tauberian theorems for $(A)^{(k)}(C,\alpha)$ summability method

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ABSTRACT. In this paper, some new Tauberian conditions are introduced for $(A)^{(k)}(C,\alpha)$ summability method.

Keywords: Abel summability, $(A)(C,\alpha)$ summability, $(A)^{(k)}(C,\alpha)$ summability, Tauberian conditions and theorems.

Alguns teoremas tauberiano estendidas para $(A)^{(k)}(C,\alpha)$ método de somabilidade

RESUMO. Neste artigo algumas novas condições de tauberiano são introduzidas para $(A)^{(k)}(C,\alpha)$ método de somabilidade.

Palavras chave: somabilidade de Abel, somabilidade de $(A)(C,\alpha)$, somabilidade de $(A)^{(k)}(C,\alpha)$, condições de tauberiano e teoremas.

Introduction

Let $\sum a_n$ be a given infinite series of real numbers with the sequence of n-th partial sums $(s_n)=(\sum_{k=0}^{n}a_k)$. For a sequence (s_n), we define $\Delta s_n = s_n - s_{n-1}$, with $\Delta s_0 = s_0$. Let A_n^{α} be defined by the generating function $(1-x)^{-\alpha-1}=\sum_{n=0}^{\infty}A_n^{\alpha}x^n \quad (|x|<1)$, where $\alpha > -1$. A sequence (s_n) is said to be (C,α) summable to s and we write $s_n \to s\,(C,\alpha)$, if

$$s_n^{\alpha}=\frac{1}{A_n^{\alpha}}\sum_{k=0}^{n}A_{n-k}^{\alpha-1}s_k \to s$$

as $n\to\infty$. Note that $(C,0)$ summability is the ordinary convergence. We write $\tau_n = na_n$ and define τ_n^{α} as the (C,α) mean of τ_n.

A sequence (s_n) is said to be Abel summable to s, and we write $s_n \to s\,(A)$, if the series $\sum_{n=0}^{\infty}a_n x^n$ is convergent for $0\le x<1$ and tends to s as $x\to 1^-$. A sequence (s_n) is said to be $(A)(C,\alpha)$ summable to s and we write $s_n \to s\,(A)(C,\alpha)$, if $(1-x)\sum_{n=0}^{\infty}s_n^{\alpha}x^n$ is convergent for $0\le x<1$ and tends to s as $x\to 1^-$. If we take $\alpha=0$, then $(A)(C,\alpha)$ summability reduces to Abel summability.

A generalization of Abel summability is introduced by (LITTLEWOOD, 1967) as follows. Let $f(x)=\sum_{n=0}^{\infty}a_n x^n, 0\le x<1$. Let

$$f_1(x)=\frac{1}{1-x}\int_x^1 f(t)dt,$$

and suppose that $\int_0^1 f_1(t)dt$ exists as $\lim_{\xi\to 1^-}\int_0^{\xi}f(t)dt$. Let

$$f_2(x)=\frac{1}{1-x}\int_x^1 f_1(t)dt,$$

an so on. We write

$$f_k(x)=\frac{1}{1-x}\int_x^1 f_{k-1}(t)dt$$

for positive integer k. The $f_k(x)$ is called the k-tuple average of f as $x\to 1^-$ by (LITTLEWOOD, 1967). If $\lim_{x\to 1^-}f_k(x)=s$ for some positive integer k, we say that (s_n) is $(A)^{(k)}$ summable to s.

Let $g(x)=(1-x)\sum_{n=0}^{\infty}s_n^{\alpha}x^n, 0\le x<1, \alpha>-1$. If $\lim_{x\to 1^-}g_k(x)=s$ for some positive integer k, we say that (s_n) is $(A)^{(k)}(C,\alpha)$ summable to s.

A sequence (s_n) is said to be slowly oscillating (STANOJEVIČ, 1998) if,

$$\lim_{\lambda \to 1^+} \limsup_{n} \max_{n+1 \le k \le [\lambda n]} | s_k - s_n | = 0.$$

A sequence (s_n) is said to be (C,α) slowly oscillating if (s_n^α) is slowly oscillating.

We use the symbols $s_n = o(1)$, $s_n = O(1)$ to mean respectively that $s_n \to 0$ as $n \to \infty$ and that (s_n) is bounded for large enough n. We also write $s_n = o(1)(C,\alpha)$ to mean that $s_n^\alpha = o(1)$.

Hardy (1910) proved that $na_n = O(1)$ is a Tauberian condition for (C,α), $\alpha > 0$, summability of (s_n). Later, Littlewood (1911) proved that (C,α) summability of (s_n) in Hardy's theorem (HARDY, 1910) can be replaced by the Abel summability of (s_n). (HARDY; LITTLEWOOD, 1913) replaced the condition $na_n = O(1)$ by the one-sided Tauberian condition $na_n \ge -H$ for some positive constant H. Littlewood (1911) proved that if (s_n) is Abel summable to s and $s_n = O(1)$, then (s_n) is $(C,1)$ summable to s. Szasz (1935) generalized Littlewood's theorem (LITTLEWOOD, 1911) which states that if (s_n) is Abel summable to s and $\tau_n^1 \ge -H$ for some positive constant H, then (s_n) is $(C,1)$ summable to s. Pati (2002) obtained a more general theorem which states that if (s_n) is $(A)(C,\alpha)$ summable for some $\alpha \ge 0$ to s and $\tau_n^\alpha \ge -H$ for some positive constant H, then (s_n) is (C,α) summable to s. Quite recently, several new Tauberian conditions for $(A)(C,\alpha)$ summability method have been obtained in Çanak et al. (2010), Erdem and Çanak (2010), and Çanak and Erdem, (2011).

Littlewood (1967) proved that $na_n \ge -H$ for some positive constant H is a Tauberian condition for $(A)^{(k)}$, where k is a positive integer k, summability of (s_n). Pati (2007) established two Tauberian theorems which are more general than a theorem of Pati (2002) and a theorem of Littlewood (1967).

Our aim in this paper is to introduce some new conditions in terms of τ_n^α to recover (C,α) convergence of (τ_n) from its $(A)^{(k)}(C,\alpha)$ summability. Namely, we prove the following Tauberian theorems.

Theorem 1.1

If, for some positive integer k and $\alpha \ge 0$, (τ_n) is $(A)^{(k)}(C,\alpha)$ summable to s and

$$n\Delta \tau_n^\alpha = o(1) \quad (1)$$

then (τ_n) is $(C,\alpha-1)$ summable to s and (s_n) is $(C,\alpha-1)$ slowly oscillating.

Theorem 1.2

If, for some positive integer k and $\alpha \ge 0$, (τ_n) is $(A)^{(k)}(C,\alpha)$ summable to s and for some positive constant H

$$n\Delta \tau_n^\alpha \ge -H \quad (2)$$

then (τ_n) is (C,α) summable to s and (s_n) is (C,α) slowly oscillating.

Theorem 1.3

If, for some positive integer k and $\alpha \ge 0$, (τ_n) is $(A)^{(k)}(C,\alpha)$ summable to s and for some positive constant H

$$n\Delta \tau_n^\alpha = O(1) \quad (3)$$

then (τ_n) is $(C,\alpha+\delta-1)$ summable to s for every $\delta > 0$.

Proofs of our Theorems depend on the following Tauberian theorem due to Littlewood (1967).

Theorem 1.4

If for some positive integer k, (s_n) is $A^{(k)}$ summable to s, then $na_n \ge -H$ for some positive constant H is a Tauberian condition for the convergence of (s_n) to s.

Lemmas

For the proof of our theorems, we need the following lemmas.

Lemma 2.1

Kogbetliantz (1925, 1931) For $\alpha > -1$, $\tau_n^\alpha = n\Delta s_n^\alpha = n(s_n^\alpha - s_{n-1}^\alpha)$.

Lemma 2.2

Çanak et al. (2010) For

$$\alpha > -1,\ n\Delta\tau_n^{\alpha+1} = (\alpha+1)(\tau_n^{\alpha} - \tau_n^{\alpha+1}) \quad (1)$$

Lemma 2.3

(HARDY, 1991) If $s_n^{\alpha} \to s$ as $n \to \infty$, $\alpha > -1$, then $s_n^{\alpha+\delta} \to s$ as $n \to \infty$ for every $\delta > 0$.

Lemma 2.4

(HARDY, 1991) If $s_n^{\alpha} \to s(C,\beta)$, then $s_n^{\alpha+\beta} \to s$ for $\alpha \ge 0$, $\beta \ge 0$, and conversely.

Lemma 2.5

(PEYERIMHOFF, 1969) All the Cesàro methods of positive order are equivalent for bounded sequences. More precisely, if $s_n = O(1)$ and $s_n^{\alpha} \to s$ as $n \to \infty$ for some $\alpha > 0$, then $s_n^{\beta} \to s$ as $n \to \infty$ for some $\beta > 0$.

Proofs of Theorems

Proof of Theorem 1.1

By hypothesis, we have $f_k(x) \to s$ as $x \to 1^-$, where $f_k(x)$ is the k-tuple average of:

$$f(x) = (1-x)\sum_{n=0}^{\infty}\tau_n^{\alpha}x^n = \sum_{n=0}^{\infty}(\tau_n^{\alpha} - \tau_{n-1}^{\alpha})x^n, 0 \le x < 1, (\tau_{-1}^{\alpha} = 0). \quad (4)$$

The condition (1) implies that $n\Delta\tau_n^{\alpha} \ge -H$ for some positive constant H. By Theorem 1.4, we get

$$\sum_{n=0}^{\infty}(\tau_n^{\alpha} - \tau_{n-1}^{\alpha}), (\tau_{-1}^{\alpha} = 0) \quad (5)$$

is convergent to s, i.e.,

$$\tau_n^{\alpha} \to s, n \to \infty. \quad (6)$$

This means that (τ_n) is (C,α) summable to s. By Lemma 2.2, we have

$$n\Delta\tau_n^{\alpha} = \alpha(\tau_n^{\alpha-1} - \tau_n^{\alpha}). \quad (7)$$

It follows from (1) and (6) that

$$\tau_n^{\alpha-1} \to s, n \to \infty, \quad (8)$$

which means that (τ_n) is $(C,\alpha-1)$ summable to s. By Lemma 2.1, we have

$$s_n^{\alpha-1} = \sum_{k=1}^{n}\frac{\tau_k^{\alpha-1}}{k}. \quad (9)$$

Since $(\tau_n^{\alpha-1})$ converges to s, there exists $M > 0$ such that

$$|\tau_n^{\alpha-1}| \le M \quad (10)$$

for all n. For any $n < k < \infty$, we have

$$|s_k^{\alpha-1} - s_n^{\alpha-1}| \le \sum_{k=n+1}^{[\lambda n]}\left|\frac{\tau_k^{\alpha-1}}{k}\right| \le M\sum_{k=n+1}^{[\lambda n]}\frac{1}{k} \le M\frac{[\lambda n]-n}{n}, \quad (11)$$

whence we conclude that

$$\limsup_{n}\max_{n+1\le k\le[\lambda n]}|s_k^{\alpha-1} - s_n^{\alpha-1}| \le M(\lambda-1). \quad (12)$$

Letting $\lambda \to 1^+$, we obtain (s_n) is $(C,\alpha-1)$ slowly oscillating. This completes the proof of Theorem 1.1.

Corollary 3.1

If, for some positive integer k, (τ_n) is $(A)^{(k)}(C,1)$ summable to s, and (1) holds, then (τ_n) is convergent to s and (s_n) is slowly oscillating.

Proof

Take $\alpha = 1$ in Theorem 1.1.

Proof of Theorem 1.2

We have (τ_n) is (C,α) summable to s by Theorem 1.4. That (s_n) is (C,α) slowly oscillating follows from Lemma 2.2.

Proof of Theorem 1.3

The condition (3) implies that

$$n\Delta\tau_n^{\alpha} \ge -H \quad (13)$$

for some positive constant H. By Theorem 1.2, we have

$$\tau_n \to s(C,\alpha). \quad (14)$$

By Lemma 2.3,

$$\tau_n \to s(C,\alpha+1) \quad (15)$$

and by Lemma 2.2,

$$n\Delta\tau_n^{\alpha+1} = \alpha(\tau_n^{\alpha} - \tau_n^{\alpha+1}) = o(1), \quad (16)$$

which is equivalent to

$$n\Delta\tau_n^{\alpha} = o(1)(C,1) \quad (17)$$

by Lemma 2.4. Since $n\Delta\tau_n^{\alpha} = O(1)$ by hypothesis, we have, by Lemma 2.5,

$$n\Delta\tau_n^{\alpha} \to 0\,(C,\delta) \quad (18)$$

for every $\delta > 0$, which is equivalent to

$$n\Delta\tau_n^{\alpha+\delta} = o(1) \quad (19)$$

by Lemma 2.4.

By Lemma 2.2, we have

$$n\Delta\tau_n^{\alpha+\delta} = (\alpha+\delta)(\tau_n^{\alpha+\delta-1} - \tau_n^{\alpha+\delta}) = o(1). \quad (20)$$

By Lemma 2.3,

$$\tau_n^{\alpha+\delta} \to s, n \to \infty \quad (21)$$

It now follows from (20) that

$$\tau_n^{\alpha+\delta-1} \to s, n \to \infty, \quad (22)$$

which is equivalent to

$$\tau_n \to s\,(C,\alpha+\delta-1). \quad (23)$$

This completes the proof of Theorem 1.3.

Corollary 3.2

If, for some positive integer k, (τ_n) is $(A)^{(k)}(C,1)$ summable to s, and (3) holds, then (τ_n) is (C,δ) summable to s for every $\delta > 0$.

Proof

Take $\alpha = 1$ in Theorem 1.3.

Corollary 3.3

If, for some positive integer k and $0 < \alpha < 1$, (τ_n) is $(A)^{(k)}(C,\alpha)$ summable to s, and (3) holds, then (τ_n) is convergent to s.

Proof

Take $\delta = 1-\alpha$ ($0 < \alpha < 1$) in Theorem 1.3.

Corollary 3.4

If, for some positive integer k, (τ_n) is $(A)^{(k)}$ summable to s, and

$$n\Delta(na_n) = O(1), \quad (24)$$

then (τ_n) is $(C,\delta-1)$ summable to s for every $\delta > 0$.

Proof

Take $\alpha = 0$ in Theorem 1.3.

Conclusion

New Tauberian theorems for the product $(A)^{(k)}$ and (C,α) summability methods have been established. Some new Tauberian conditions in terms of (C,α) mean of (τ_n) have been obtained to recover (C,α) convergence of (τ_n) and slow oscillation of (C,α) mean from $(A)^{(k)}(C,\alpha)$ summability of (τ_n).

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