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B-tubular surfaces in Lorentzian Heisenberg Group H^3

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ABSTRACT. In this paper, B-tubular surfaces in terms of biharmonic spacelike new type B-slant helices according to Bishop frame in the Lorentzian Heisenberg group H^3 are studied. The Necessary and sufficient conditions for new type B-slant helices to be biharmonic are obtained. B-tubular surfaces in the cLorentzian Heisenberg group H^3 are characterized. Additionally, main results in Figures 1, 2, 3 and 4 are illustrated.

Keywords: bienergy, Bishop frame, general helices Lorentzian Heisenberg group, new type B-slant helix, tubular surface.

Superfícies B-tubulares no Grupo H^3 de Lorentz-Heisenberg

RESUMO. Analisam-se superfícies B-tubulares nos termos de novos tipos de hélices B-inclinados, semelhantes ao espaço bi-harmônico, de acordo com o esquema de Bishop no grupo H^3 de Lorentz-Heisenberg. Obtêm-se as condições necessárias e suficientes para o novo tipo de hélices B-inclinados. As superfícies B-tubulares no grupo H^3 de Lorentz-Heisenberg são caracterizadas e os resultados nas Figuras 1, 2, 3 e 4 são analisadas.

Palavras-chave: Bi-energia, esquema de Bishop, hélices do grupo Lorentz-Heisenberg, novos tipos de hélices B-inclinados, superfície tubular.

Introduction

Tubular surfaces are very useful for representing long thin objects, for instance, poles, 3D fonts, brass instrument or internal organs of the body in solid modeling. It includes natural quadrics (cylinder, cone and sphere), revolute quadrics, tori, pipes and Dupin cyclide. Also, canal surfaces are among the surfaces which are easier to describe both analytically and operationally, (CARMO, 1976), (O'NEIL, 1983), (FAROUKI; NEFF, 1990), (LU, 1994), (GRAY, 1998), (PETERNELL; POTTMANN, 1997), (ZHU et al., 2005), (XU et al., 2006), (KORPINAR; TURHAN, 2011), (TURHAN; KORPINAR, 2011a).

We remind that, if γ is a space curve, a tubular surface associated to this curve is a surface swept by a family of spheres of constant radius (which will be the radius of the tube), having the center on the given curve. Alternatively, as we shall see in the next section, for them we can construct quite easily a parameterization using the Frenet frame associate to the curve. The tubular surfaces are used quite often in computer graphics, but we think they deserve more attention for several reasons. For instance, there is the problem of representing the curves themselves. Usually, the space curves are represented by using solids rather then tubes. There are, today, several very good computer algebra system (such as Maple, or Mathematica) which allow the vizualisation of curves and surfaces, in different kind of representations, (KORPINAR; TURHAN, 2012), (POTTMANN; PETERNELL, 1998).

The aim of this paper is to study tubular surfaces surrounding biharmonic spacelike B-slant helices according to Bishop frame in the Lorentzian Heisenberg group H^3.

Let (M,g) and (N,h) be Lorentzian manifolds and $\phi: M \to N$ a smooth map. Denote by ∇^{ϕ} the connection of the vector bundle $\phi^{*}TN$ induced from the Levi-Civita connection ∇^{h} of (N,h). The 'second fundamental form' $\nabla d\phi$ is defined by

$$(\nabla d\phi)(X,Y)=\nabla^{\phi}_{X} d\phi(Y)-d\phi(\nabla_{X}Y),\ X,Y\in\Gamma(TM).$$

Here ∇ is the Levi-Civita connection of (M,g). The tension field $\tau(\phi)$ is a section of $\phi^{*}TN$ defined by $\tau(\phi)=tr\nabla d\phi$.

A smooth map ϕ is said to be *harmonic* if its tension field vanishes. It is well known that ϕ is

harmonic if and only if ϕ is a critical point of the 'energy':

$$E(\phi)=\frac{1}{2}\int h(d\phi,d\phi)dv_g$$

over every compact region of M. Now let $\phi:M\to N$ be a harmonic map. Then the Hessian H of E is given by

$$\mathsf{H}_\phi(V,W)=\int h(\mathsf{J}_\phi(V),W)dv_g, V,W\in\Gamma(\phi^*TN).$$

Here the *Jacobi operator* J_ϕ is defined by

$$\mathsf{J}_\phi(V):=\Delta^\phi V-\mathsf{R}_\phi(V), V\in\Gamma(\phi^*TN),$$

$$\Delta^\phi:=-\sum_{i=1}^m\left(\nabla^\phi_{e_i}\nabla^\phi_{e_i}-\nabla^\phi_{\nabla_{e_i}e_i}\right),$$
$$\mathsf{R}_\phi(V)=\sum_{i=1}^m R^N(V,d\phi(e_i))d\phi(e_i),$$

where R^N and $\{e_i\}$ are the Riemannian curvature of N, and a local orthonormal frame field of M, respectively, (EELLS; SAMPSON, 1964), (JIANG, 1986a, 1986b), (TURHAN; KORPINAR, 2010).

Let $\phi:(M,g)\to(N,h)$ be a smooth map between two Lorentzian manifolds. The *bienergy* $E_2(\phi)$ of ϕ over compact domain $\Omega\subset M$ is defined by

$$E_2(\phi)=\int_\Omega h(\tau(\phi),\tau(\phi))dv_g.$$

A smooth map $\phi:(M,g)\to(N,h)$ is said to be 'biharmonic' if it is a critical point of the $E_2(\phi)$.

The section $\tau_2(\phi)$ is called the *bitension field* of ϕ and the Euler-Lagrange equation of E_2 is

$$\tau_2(\phi):=-\mathsf{J}_\phi(\tau(\phi))=0.$$

This study is organised as follows: Firstly, we give necessary and sufficient conditions for new type B-slant helices to be biharmonic. We characterize B-tubular surfaces in the Lorentzian Heisenberg group H^3. Secondly, we study B-tubular surfaces in terms of biharmonic spacelike new type B-slant helices according to Bishop frame in the Lorentzian Heisenberg group H^3. Finally, we illustrate our results in Figures 1, 2, 3 and 4.

The Lorentzian Heisenberg Group H^3

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to R^3 and the group operation is defined as

$$(x,y,z)\cdot(\bar{x},\bar{y},\bar{z})=(x+\bar{x},y+\bar{y},z+\bar{z}-\bar{x}y+x\bar{y}).$$

The identity of the group is (0,0,0) and the inverse of (x,y,z) is given by $(-x,-y,-z)$ (RAHMANI; RAHMANI, 2006). The left-invariant Lorentz metric on Heis^3 is

$$g=-dx^2+dy^2+(xdy+dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathrm{e}_1=\frac{\partial}{\partial z},\mathrm{e}_2=\frac{\partial}{\partial y}-x\frac{\partial}{\partial z},\mathrm{e}_3=\frac{\partial}{\partial x}\right\}. \quad (2.1)$$

The characterising properties of this algebra are the following commutation relations:

$$g(\mathrm{e}_1,\mathrm{e}_1)=g(\mathrm{e}_2,\mathrm{e}_2)=1, g(\mathrm{e}_3,\mathrm{e}_3)=-1.$$

Proposition 2.1.

For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:

$$\nabla=\begin{pmatrix}0 & \mathrm{e}_3 & \mathrm{e}_2\\ \mathrm{e}_3 & 0 & \mathrm{e}_1\\ \mathrm{e}_2 & -\mathrm{e}_1 & 0\end{pmatrix}, \quad (2.2)$$

where the (i,j)-element in the table above equals $\nabla_{\mathrm{e}_i}\mathrm{e}_j$ for our basis

$$\{\mathrm{e}_k,k=1,2,3\}.$$

Moreover we put

$$R_{ijk}=R(\mathrm{e}_i,\mathrm{e}_j)\mathrm{e}_k,\ R_{ijkl}=R(\mathrm{e}_i,\mathrm{e}_j,\mathrm{e}_k,\mathrm{e}_l),$$

where the indices i,j,k and l take the values 1, 2 and 3.

$$R_{121}=-\mathrm{e}_2, R_{131}=-\mathrm{e}_3,\ R_{232}=3\mathrm{e}_3$$

and

$$R_{1212}=-1, R_{1313}=1, R_{2323}=-3.$$

Spacelike biharmonic new type B-slant helices with Bishop frame ın the Lorentzian Heisenberg group H^3

Let $\gamma: I \to H^3$ be a non geodesic spacelike curve on the Lorentzian Heisenberg group H^3 parametrized by arc length. Let $\{t,n,b\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group H^3 along γ defined as follows:

t is the unit vector field γ' tangent to γ, n is the unit vector field in the direction of $\nabla_t t$ (normal to γ), and b is chosen so that $\{t,n,b\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas, (TURHAN; KORPINAR, 2011b):

$$\begin{aligned} \nabla_t t &= \kappa n, \\ \nabla_t n &= \kappa t + \tau b, \\ \nabla_T B &= \tau n, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} &g(t,t)=1, g(n,n)=-1, g(b,b)=1, \\ &g(t,n)=g(t,b)=g(n,b)=0. \end{aligned}$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative, (BISHOP, 1975). The Bishop frame is expressed as

$$\begin{aligned} \nabla_t t &= k_1 m_1 - k_2 m_2, \\ \nabla_t m_1 &= k_1 t, \\ \nabla_t m_2 &= k_2 t, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &g(t,t)=1, g(m_1,m_1)=-1, g(m_2,m_2)=1, \\ &g(T,M_1)=g(t,m_2)=g(m_1,m_2)=0. \end{aligned}$$

Here, we shall call the set $\{t,m_1,m_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures.

Also, $\tau(s)=\psi'(s)$ and $\kappa(s)=\sqrt{\left|k_2^2-k_1^2\right|}$.

With respect to the orthonormal basis $\{e_1,e_2,e_3\}$ we can write

$$\begin{aligned} t &= t^1 e_1 + t^2 e_2 + t^3 e_3, \\ m_1 &= m_1^1 e_1 + m_1^2 e_2 + m_1^3 e_3, \\ m_2 &= m_2^1 e_1 + m_2^2 e_2 + m_2^3 e_3. \end{aligned} \tag{3.3}$$

Theorem 3.1.

$\gamma: I \to H^3$ is a spacelike biharmonic curve with Bishop frame if and only if

$$\begin{aligned} &k_1^2 - k_2^2 = \text{constant} = C \neq 0, \\ &k_1'' + \left[k_1^2 - k_2^2\right]k_1 = -k_1\left[1+4\left(m_2^1\right)^2\right] \\ &+ 4k_2 m_1^1 m_2^1, \\ &k_2'' + \left[k_1^2 - k_2^2\right]k_2 = -4k_1 m_1^1 m_2^1 \\ &- k_2\left[-1+4\left(m_1^1\right)^2\right] \end{aligned} \tag{3.4}$$

Proof.

Using Equation(2.2) and Jacobi operator, we obtain above system. This completes the proof.

Definition 3.2.

A regular spacelike curve $\gamma: I \to H^3$ is called a new type slant helix provided the spacelike unit vector m_2 of the curve γ has constant angle Q with spacelike vector u, that is

$$g(m_2(s),u)=\cos Q \text{ for all } s \in I. \tag{3.5}$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

To separate a spacelike new type slant helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as spacelike new type B-slant helix.

B-tubular surfaces in terms of spacelike biharmonic new type B-Slant helices according to Bishop frame in the Heisenberg group H^3

The envelope of a 1-parameter family of the spheres in the Lorentzian Heisenberg group H^3 is called a tubular surface in the Lorentzian Heisenberg group H^3. The curve formed by the

centers of the spheres is called center curve of the tubular surface. The radius of the tubular surface is the function r such that r is the constant radius of the sphere.

To separate a tubular surface according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the surface defined above as B-tubular surface.

Theorem 4.1.

Let $\gamma : I \to \mathsf{H}^3$ be a unit speed biharmonic spacelike new type $\mathsf{B}-$slant helix with non-zero curvatures. Then the equation of $\mathsf{B}-$tubular surface $\mathsf{M}^{\mathsf{B}}(s,t)$ is

$$\begin{aligned}\mathsf{M}^{\mathsf{B}}(s,t) &= [\sin \mathsf{Q}s - \frac{\mathsf{C}_2}{\mathsf{C}_0}\cos \mathsf{Q}\sinh[\mathsf{C}_0 s + \mathsf{C}_1] \\ &- \frac{1}{4\mathsf{C}_0}\cos^2 \mathsf{Q}(2[\mathsf{C}_0 s + \mathsf{C}_1] + \sinh 2[\mathsf{C}_0 s + \mathsf{C}_1]) \\ &+ r\cosh\ t\cos\ \mathsf{Q}]\mathrm{e}_1 \\ &+ [\frac{1}{\mathsf{C}_0}\cos\ \mathsf{Q}\sinh\ [\mathsf{C}_0 s + \mathsf{C}_1] + \mathsf{C}_3 \\ &+ r\sinh t\sinh[\mathsf{C}_0 s + \mathsf{C}_1] \\ &+ r\cosh t\sin \mathsf{Q}\cosh[\mathsf{C}_0 s + \mathsf{C}_1]]\mathrm{e}_2 \\ &+ [\frac{1}{\mathsf{C}_0}\cos\ \mathsf{Q}\cosh\ [\mathsf{C}_0 s + \mathsf{C}_1] \\ &+ \mathsf{C}_2 + r\sinh\ t\cosh\ [\mathsf{C}_0 s + \mathsf{C}_1] \\ &- r(s)r'(s)\cos\ \mathsf{Q}\sinh\ [\mathsf{C}_0 s + \mathsf{C}_1] \\ &+ r\cosh\ t\sin\ \mathsf{Q}\sinh\ [\mathsf{C}_0 s + \mathsf{C}_1]]\mathrm{e}_3,\end{aligned} \tag{4.1}$$

where $\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3$ are constants of integration and

$$\mathsf{C}_0 = \frac{\sqrt{k_2^2 - k_1^2}}{\cos \mathsf{Q}} - 2\sin \mathsf{Q}.$$

Proof.

Assume that the center curve of $\mathsf{B}-$tubular surface $\mathsf{M}^{\mathsf{B}}(s,t)$ is a unit speed spacelike biharmonic $\mathsf{B}-$slant helix γ and M^{B} denote a patch that parametrizes the envelope of the spheres defining the tubular surface.

Thus, it is seen that

$$\begin{aligned}\mathsf{M}^{\mathsf{B}}(s,t) &= \gamma(s) + \mathsf{a}(s,t)\mathrm{t}(s) \\ &+ \mathsf{b}(s,t)\mathrm{m}_1(s) + \mathsf{c}(s,t)\mathrm{m}_2(s),\end{aligned} \tag{4.2}$$

where a, b and c are differentiable on the interval on which γ is defined.

So, without loss of generality, we take the axis of γ is parallel to the spacelike vector e_1. Then,

$$g(\mathrm{m}_2, \mathrm{e}_1) = m_2^1 = \cos \mathsf{Q}, \tag{4.3}$$

where Q is constant angle.

On the other hand, the vector m_2 is a unit spacelike vector, we reach

$$\begin{aligned}\mathrm{m}_2 &= \cos\ \mathsf{Q}\,\mathrm{e}_1 + \sin\ \mathsf{Q}\cosh\ \mathsf{A}(s)\mathrm{e}_2 \\ &+ \sin\ \mathsf{Q}\sinh\ \mathsf{A}(s)\mathrm{e}_3.\end{aligned} \tag{4.4}$$

On the other hand, using Bishop formulas Equation (3.2) and Equation (2.1), we have

$$\mathrm{m}_1 = \sinh \mathsf{A}(s)\mathrm{e}_2 + \cosh \mathsf{A}(s)\mathrm{e}_3. \tag{4.5}$$

It is apparent that

$$\begin{aligned}\mathrm{t} &= \sin \mathsf{Q}\mathrm{e}_1 + \cos \mathsf{Q}\cosh\ \mathsf{A}(s)\mathrm{e}_2 \\ &+ \cos \mathsf{Q}\sinh\ \mathsf{A}(s)\mathrm{e}_3.\end{aligned} \tag{4.6}$$

A straightforward computation shows that

$$\nabla_t \mathrm{t} = (t_1')\mathrm{e}_1 + (t_2' + 2t_1 t_3)\mathrm{e}_2 + (t_3' + 2t_1 t_2)\mathrm{e}_3.$$

Therefore, we use Bishop formulas Equation (3.2) and above equation we get

$$\mathsf{A}(s) = [\frac{\sqrt{k_2^2 - k_1^2}}{\cos \mathsf{Q}} - 2\sin \mathsf{Q}]s + \mathsf{C}_1, \tag{4.7}$$

where C_1 is a constant of integration.

From Equation(4.6), we get

$$\begin{aligned}\mathrm{t} = (&\cos\ \mathsf{Q}\sinh\ [\mathsf{C}_0 s + \mathsf{C}_1], \\ &\cos\ \mathsf{Q}\cosh\ [\mathsf{C}_0 s + \mathsf{C}_1], \\ &\sin\ \mathsf{Q} - x\cos\ \mathsf{Q}\cosh\ [\mathsf{C}_0 s + \mathsf{C}_1]).\end{aligned} \tag{4.8}$$

Since,

$$\gamma(s) = [\sin Qs - \frac{C_2}{C_0}\cos Q \sinh[C_0 s + C_1]$$
$$-\frac{1}{4C_0}\cos^2 Q(2[C_0 s + C_1]$$
$$+ \sinh 2[C_0 s + C_1]) + C_4$$
$$+[\frac{1}{C_0}\cos Q \cosh[C_0 s + C_1]$$
$$+ C_2][\frac{1}{C_0}\cos Q \sinh[C_0 s + C_1] + C_3]]e_1$$
$$+[\frac{1}{C_0}\cos Q \sinh[C_0 s + C_1] + C_3]e_2$$
$$+[\frac{1}{C_0}\cos Q \cosh[C_0 s + C_1] + C_2]e_3,$$

where $C_0 = \frac{\sqrt{k_2^2 - k_1^2}}{\cos Q} - 2\sin Q.$

On the other hand, using definition of tubular surface, we have

$$g(M^B(s,t) - \gamma(s), M^B(s,t) - \gamma(s)) = r^2(s). \quad (4.9)$$

Since $M^B(s,t) - \gamma(s)$ is a normal vector to the tubular surface, we get

$$g(M^B(s,t) - \gamma(s), M_s^B(s,t)) = 0. \quad (4.10)$$

Using now Equation (4.9) and Equation (4.10), we get

$$\begin{aligned} &a^2(s) - b^2(s) + c^2(s) = r^2(s), \\ &a(s)a_s(s) - b(s)b_s(s) + c(s)c_s(s) = 0. \end{aligned} \quad (4.11)$$

When we differentiate Equation (4.2) with respect to s and use the Bishop formulas, we obtain

$$\begin{aligned} M_s^B(s,t) &= (1 + a_s(s) + b(s)k_1 + c(s)k_2)t(s) \\ &+ (a(s)k_1 + b_s)m_1(s) \\ &+ (c_s(s) - a(s)k_2)m_2(s). \end{aligned} \quad (4.12)$$

Combining Equation (4.9) and Equation (4.10) we have

$$a(s) = -r(s)r'(s). \quad (4.13)$$

A further computation gives

$$-b^2(s) + c^2(s) = r^2(s)(1 - (r'(s))^2). \quad (4.14)$$

The solution of Equation (4.14) can be written in the following form

$$\begin{aligned} b(s) &= r\sinh t, \\ c(s) &= r\cosh t. \end{aligned} \quad (4.15)$$

Thus Equation (4.2) becomes

$$M^B(s,t) = \gamma(s) + r\sinh t m_1(s) + r\cosh t m_2(s). \quad (4.16)$$

Therefore, by Equations.(4.4) - (4.6) and taking into account Equation(4.16), we obtain the system Equation(4.1). This completes the proof.

Theorem 4.2.

Let $\gamma : I \to H^3$ be a unit speed spacelike biharmonic new type $B-$slant helix with non-zero Bishop curvatures. Then, the parametric equations of $B-$tubular surface $M^B(s,t)$ are

$$\begin{aligned} x_{M^B}(s,t) &= [\frac{1}{C_0}\cos Q \cosh[C_0 s + C_1] \\ &+ r\sinh t \cosh[C_0 s + C_1] \\ &+ r\cosh t \sin Q \sinh[C_0 s + C_1] + C_2], \\ y_{M^B}(s,t) &= [\frac{1}{C_0}\cos Q \sinh[C_0 s + C_1] \\ &+ r\sinh t \sinh[C_0 s + C_1] \\ &+ r\cosh t \sin Q \cosh[C_0 s + C_1] + C_3], \\ z_{M^B}(s,t) &= [\sin Qs - \frac{C_2}{C_0}\cos Q \sinh[C_0 s + C_1] \\ &- \frac{1}{4C_0}\cos^2 Q(2[C_0 s + C_1] \\ &+ \sinh 2[C_0 s + C_1]) + C_4 \\ &+ [\frac{1}{C_0}\cos Q \cosh[C_0 s + C_1] + C_2] \\ &[\frac{1}{C_0}\cos Q \sinh[C_0 s + C_1] + C_3] \\ &+ r\cosh t \cos Q] \\ &- [\frac{1}{C_0}\cos Q \cosh[C_0 s + C_1] + C_2 \\ &+ r\sinh t \cosh[C_0 s + C_1] \\ &+ r\cosh t \sin Q \sinh[C_0 s + C_1]] \\ &[\frac{1}{C_0}\cos Q \sinh[C_0 s + C_1] + C_3 \\ &+ r\sinh t \sinh[C_0 s + C_1] \\ &+ r\cosh t \sin Q \cosh[C_0 s + C_1]] \end{aligned} \quad (4.17)$$

where C_0, C_1, C_2, C_3 are constants of integration and

$$C_0 = \frac{\sqrt{k_2^2 - k_1^2}}{\cos Q} - 2\sin Q .$$

Proof.

Using Equation (2.1) and Equation (4.1), we have Equation (4.17). Thus proof is complete.

The obtained parametric equations for Equation (4.17) is illustrated in Figures 1, 2 and 3 with helping the programme of Mathematica as follow:

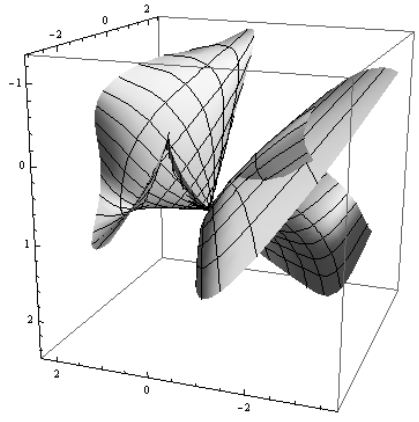

Figure 1. For $C_0 = C_1 = -C_2 = C_3 = 1$.

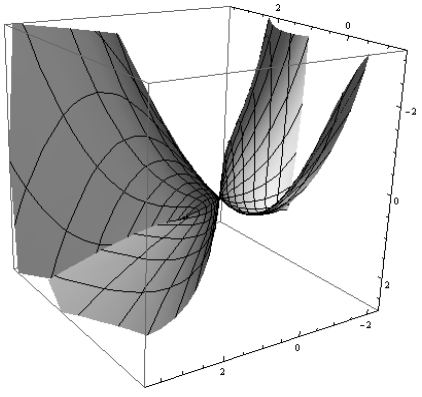

Figure 2. For $C_0 = -C_1 = C_2 = -C_3 = 1$.

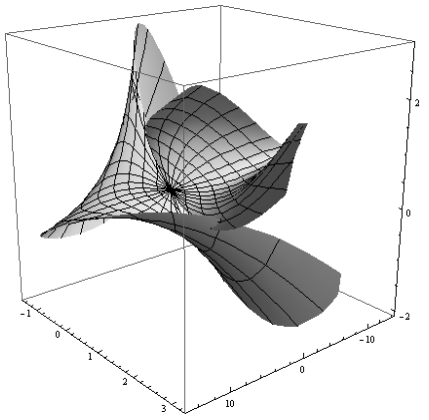

Figure 3. For $-C_0 = C_1 = -C_2 = C_3 = -1$.

Corollary 4.3.

Let $\gamma : I \to H^3$ be a unit speed biharmonic spacelike new type $B-$slant helix with non-zero Bishop curvatures. Then the equations of γ are

$$\mathbf{x}(s) = \frac{1}{C_0}\cos Q \cosh[C_0 s + C_1] + C_2,$$

$$y(s) = \frac{1}{C_0}\cos Q \sinh[C_0 s + C_1] + C_3,$$

$$z(s) = \sin Q s - \frac{C_2}{C_0}\cos Q \sinh[C_0 s + C_1]$$

$$-\frac{1}{4C_0}\cos^2 Q(2[C_0 s + C_1] + \sinh 2[C_0 s + C_1]) + C_4,$$

where C_0, C_1, C_2, C_3 are constants of integration and

$$C_0 = \frac{\sqrt{k_2^2 - k_1^2}}{\cos Q} - 2\sin Q.$$

If we use Mathematica in above system, we get (Figure 4):

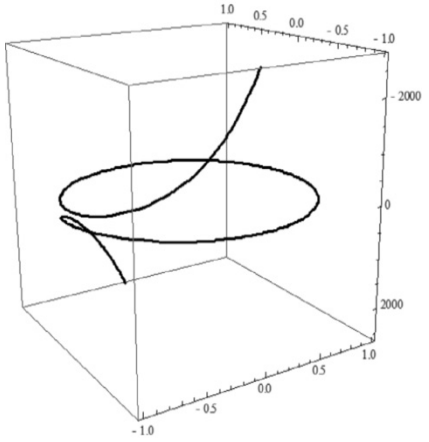

Figure 4. For $-C_0 = C_1 = -C_2 = C_3 = -1$.

Conclusion

A tubular surface associated to a curve is a surface swept by a family of spheres of constant radius (which will be the radius of the tube), having the center on the given curve. Alternatively, as we shall see in the next section, for them we can construct quite easily a parameterization using the Frenet frame associate to the curve.

In this work, B-tubular surfaces in the Lorentzian Heisenberg group H^3 are characterized.

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