Acta Scientiarum

http://www.uem.br/acta
ISSN printed: 1806-2563
ISSN on-line: 1807-8664
Doi: 10.4025/actascitechnol.v37i1.16966

Some sequence spaces of fuzzy numbers defined by Orlicz function

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ABSTRACT. In this article we introduce some fuzzy sequence spaces defined by Orlicz function and study different properties of these spaces like completeness, symmetricity etc. We establish some inclusion results among them.

Keywords: completeness, symmetric space, paranorm, nowhere dense set, convex function, Δ_2 -condition.

Alguns espaços sequenciais de números difusos definidos pela função de Orlicz

RESUMO. Os autores apresentam alguns espaços sequenciais de números difusos definidos pela função de Orlicz. Algumas propriedades desses espaços, como completude, simetricidade e outras, são investigadas. Resultados inclusivos são estabelecidos.

Palavras-chave: completude, espaço simétrico, paranorma, conjunto denso em lugar nenhum, função convexa, condição Δ_2.

Introduction

The concept of fuzzy sets was first introduced by Zadeh (1965). It has a wide range of applications in almost all the branches, where mathematics is used. Particularly the computer and IT professionals have frequent use of fuzziness in every aspect. It attracted many workers to introduce different types of fuzzy sequence spaces.

Throughout, a fuzzy sequence will be denoted by $X = \langle X_i \rangle$, where each X_i is a fuzzy real number. Throughout the article w^F will denote the set of all fuzzy sequences.

The concept of paranormed sequences was studied by Nakano (1951) and Simmons (1965) at the initial stage. Later on it was studied by many others.

Some initial works on Orlicz function are done by Et (2001). Some new results of sequence spaces of fuzzy real numbers using Orlicz function are found in Tripathy and Tripathy and Sarma (2011).

The notion was further investigated by many workers on sequence spaces.

Definitions and preliminaries

An 'Orlicz function' M is a mapping $M : [0, \infty) \to [0, \infty)$ such that it is 'continuous, non-decreasing and convex' with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

The initial notion of Orlicz Spaces found in Krasnoselkii and Rutitsky (1961).

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct the sequence space

$$\ell^M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is a Banach space normed by

$$||(x_k)|| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

The space ℓ^M is closely related to the space ℓ^p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \le p < \infty$.

An Orlicz function M is said to satisfy the Δ_2 -condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \le K(Mu)$, $u \ge 0$.

Remark. Let $0 < \lambda < 1$, then $M(\lambda x) \le \lambda M(x)$, for all $x \ge 0$.

Let $p = (p_k)$ be a positive sequence of real numbers. If $0 < p_k \le \sup p_k = H$ and $D = \max(1, 2^{H-1})$, then for $a_k, b_k \in C$ for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \le D\left\{ |a_k|^{p_k} + |b_k|^{p_k} \right\}.$$

Definition. A sequence space E is said to be symmetric if $(X_k) \in E$ implies $(X_{\pi(k)}) \in E$, where π is the permutation of N. This definition for crisp set is found in Kamthan and Gupta (1980).

Let M be an Orlicz function and $p = (p_i)$ be a sequence of strictly positive real numbers. We introduce the following sequence spaces.

$$W^F(M,p)=\left\{<X_i>\in w^F : \lim_m \frac{1}{m}\sum_{i=1}^{m}\left(M\left(\frac{d(X_i,X)}{\rho}\right)\right)^{p_i}=0\right.$$

$$\left.\text{for some } \rho>0 \text{ and } L.\right\}$$

$$W_0^F(M,p)=\left\{<X_i>\in w^F : \lim_m \frac{1}{m}\sum_{i=1}^{m}\left(M\left(\frac{d(X_i,\bar{0})}{\rho}\right)\right)^{p_i}=0, \text{ for some } \rho>0.\right\}$$

$$W_\infty^F(M,p)=\left\{<X_i>\in w : \sup_m \frac{1}{m}\sum_{i=1}^{m}\left(M\left(\frac{d(X_i,\bar{0})}{\rho}\right)\right)^{p_i}<\infty, \text{ for some } \rho>0\right\}$$

Main results

Theorem 3.1. Let $p = (p_i)$ be bounded. The classes of sequences $W^F(M,p)$, $W_0^F(M,p)$ and $W_\infty^F(M,p)$ are complete metric spaces with respect to the metric

$$f(X,Y)=\inf\left\{\rho^{\frac{p_i}{J}}>0 : \sup_i\left(M\left(\frac{d(X_i,Y_i)}{\rho}\right)\right)\le 1.\right\},$$

where: $J = \max(1, \sup p_i)$.

Proof. It can be easily verified that the classes are metric spaces. To prove completeness consider the class $W_\infty^F(M,p)$.

Let $<X^s>$ be a Cauchy sequence in $W_\infty^F(M,p)$. Then $f(X^s,X^t)\to 0$ as s, $t\to\infty$.

For a given ε > 0 choose $r > 0$ and $x_0 > 0$ be such that $\frac{\varepsilon}{rx_0}>0$ and $M\left(\frac{rx_0}{2}\right)\ge 1$.

Then there exists $m_0 \in N$ such that $f(X^s,X^t)<\frac{\varepsilon}{rx_0}$ for all s, $t \ge m_0$.

$$\Rightarrow \inf\left\{\rho^{\frac{p_{ij}}{J}} : \sup_i\left(M\left(\frac{d(X_i^s,X_i^t)}{\rho}\right)\right)\le 1\right\}<\frac{\varepsilon}{rx_0}$$

Now,

$$M\left(\frac{d(X_i^s,X_i^t)}{\rho}\right)\le 1\le M\left(\frac{rx_0}{2}\right).$$

$$\Rightarrow \frac{d(X_i^s,X_i^t)}{f(X^s,X^t)}\le\frac{rx_0}{2}$$

$$\Rightarrow d(X_i^s,X_i^t) < \frac{rx_0}{2}.\frac{\varepsilon}{rx_0}=\frac{\varepsilon}{2}$$

This implies (X_i^t) is a Cauchy sequence of fuzzy real numbers. Since the set of fuzzy real numbers is complete so there exit fuzzy number X_i such that

Let $\lim_{s\to\infty} X_i^t = X_i$ for all $i \in N$.

Now

$$\limsup_{t\to\infty}\sup_i\left(M\left(\frac{d(X_i^s,X_i^t)}{\rho}\right)\right)\le 1 \Rightarrow \sup_i\left(M\left(\frac{d(X_i^s,X_i)}{\rho}\right)\right)\le 1$$

Let s ≥ m_0, then taking infimum of such ρ's we have $f(X_i^s,X_i)<\varepsilon$.

Now using $f(X_i,\bar{0})\le f(X_i,X_i^s)+f(X_i^s,\bar{0})$ we get $(X_i)\in W_\infty^F(M,p)$. Hence $W_\infty^F(M,p)$ is complete.

Proposition 3.2. (*i*) $W^F(M,p)\subset W_\infty^F(M,p)$; (ii) $W_0^F(M,p)\subset W_\infty^F(M,p)$. The inclusions are strict.

Proof. The proof is obvious.

Theorem 3.3. The spaces $W^F(M,p)$ and $W_0^F(M,p)$ are nowhere dense subset of $W_\infty^F(M,p)$.

Proof. The proof is obvious in view of Theorem 3.1 and Proposition 3.2.

Theorem 3.4. (i) If $0<\inf p_i\le p_i<1$, then $W^F(M,p)\subseteq W^F(M)$; (ii) If $1\le p_i<\sup p_i<\infty$, then $W^F(M)\subseteq W^F(M,p)$.

Proof. The first part of the result follows from the inequality

$$\frac{1}{m}\sum_{i=1}^{m}M\left(\frac{d(X_i,X)}{\rho}\right)\le\frac{1}{m}\sum_{i=1}^{m}\left(M\left(\frac{d(X_i,X)}{\rho}\right)\right)^{p_i}$$

and the second part of the result follows from the inequality

$$\frac{1}{m}\sum_{i=1}^{m}\left(M\left(\frac{d(X_i,X)}{\rho}\right)\right)^{p_i}\le\frac{1}{m}\sum_{i=1}^{m}M\left(\frac{d(X_i,X)}{\rho}\right)$$

Theorem 3.5. Let M_1 and M_2 be two Orlicz functions. Then

$$W^F(M_1,p)\cap W^F(M_2,p)\subseteq W^F(M_1+M_2,p).$$

Proof. Let $<X_i>\in W^F(M_1,p)\cap W^F(M_2,p)$. Then

$$\lim_m\frac{1}{m}\sum_{i=1}^{m}\left(M_1\left(\frac{d(X_i,X)}{\rho_1}\right)\right)^{p_i}=0, \text{ for some } \rho_1>0.$$

$$\lim_{m} \frac{1}{m}\sum_{i=1}^{m}\left(M_2\left(\frac{d(X_i,X)}{\rho_2}\right)\right)^{p_i}=0, \text{ for some } \rho_2 > 0.$$

Let ρ = max { ρ_1, ρ_2 }. The result follows from the following inequality.

$$\sum_{i=1}^{m}\left((M_1+M_2)\left(\frac{d(X_i,X)}{\rho}\right)\right)^{p_i} \le D\left\{\sum_{i=1}^{m}\left(M_1\left(\frac{d(X_i,X)}{\rho_1}\right)\right)^{p_i}+\sum_{i=1}^{m}\left(M_2\left(\frac{d(X_i,X)}{\rho_2}\right)\right)^{p_i}\right\}$$

Result 3.6. The sequence spaces ${}_2W(M, \Delta, p)$, ${}_2W_0(M, \Delta, p)$ and ${}_2W_\infty(M, \Delta, p)$ are symmetric.

Proof. The result is obvious.

Conclusion

In this paper we have studied about some paranormed type sequence spaces of fuzzy real numbers and studied related properties. We have proved the completeness property of the introduced sequence space. This notion can be used for further generalization of such sequence spaces.

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Received on April 27, 2012.
Accepted on June 1, 2012.