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One parameter family of S-tangent surfaces

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ABSTRACT. In this paper, we study S – tangent surfaces according to Sabban frame in the Lorentzian Heisenberg group H. We obtained differential equations in terms of their geodesic curvatures in the Lorentzian Heisenberg group H. Finally, we found explicit parametric equations of one parameter family of S – tangent surfaces according to Sabban Frame.

Keywords: biharmonic curve, Heisenberg group, geodesic curvature, fluid flows.

Uma família de parâmetros de superfícies tangenciais em S

RESUMO. Investigam-se as superfícies tangenciais em S conforme o esquema de Sabban no grupo H de Lorentz-Heisenberg. Os autores obtiveram equações diferenciadas em termos de curvaturas geodésicas no grupo H de Lorentz-Heisenberg. Encontraram-se equações paramétricas explicitas de uma família parametrica de superfícies tangenciais em S de acordo com o esquema de Sabban.

Palavras-chave: curva bi-harmônica, grupo de Heisenberg, curvatura geodésica, fluxos fluidos.

Introduction

Construction of fluid flows constitutes an active research field with a high industrial impact. Corresponding real-world measurements in concrete scenarios complement numerical results from direct simulations of the Navier-Stokes equation, particularly in the case of turbulent flows, and for the understanding of the complex spatio-temporal evolution of instationary flow phenomena. More and more advanced imaging devices (lasers, highspeed cameras, control logic, etc.) are currently developed that allow to record fully timeresolved image sequences of fluid flows at high resolutions. As a consequence, there is a need for advanced algorithms for the analysis of such data, to provide the basis for a subsequent pattern analysis, and with abundant applications across various areas, (O'NEIL, 1983; KWON et al., 2005).

Paper, sheet metal, and many other materials are approximately unstretchable. The surfaces obtained by bending these materials can be flattened onto a plane without stretching or tearing. More precisely, there exists a transformation that maps the surface onto the plane, after which the length of any curve drawn on the surface remains the same. Such surfaces, when sufficiently regular, are well known to mathematicians as developable surfaces. While developable surfaces have been widely used in engineering, design and manufacture, they have been less popular in computer graphics, despite the fact that their isometric properties make them ideal primitives for texture mapping, some kinds of surface modelling, and computer animation (CARMO, 1976; O'NEIL, 1983; KWON et al., 2005).

A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left| \mathsf{T}(\phi) \right|^2 dv_h,$$

where $\mathsf{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ, (EELLS; SAMPSON, 1964; EELLS; LEMAIRE, 1978; JIANG, 1986; DIMITRIC, 1992; CADDEO et al., 2004, 2005).

The Euler-Lagrange equation of the bienergy is given by $\mathsf{T}_2(\phi) = 0$. Here the section $\mathsf{T}_2(\phi)$ is defined by

$$\mathsf{T}_2(\phi) = -\Delta_{\phi} \mathsf{T}(\phi) + \mathrm{tr}R\big(\mathsf{T}(\phi), d\phi\big) d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organised as follows: Firstly, we study S – tangent surfaces according to Sabban frame in the Lorentzian Heisenberg group H. Secondly, we obtain differential equations about in terms of their geodesic curvatures. Finally, we find explicit parametric equations of one parameter family of S – tangent surfaces according to Sabban Frame and we illustrate our results in Figures 1,2,3.

The Lorentzian Heisenberg Group H

Heisenberg group H can be seen as the space R^3 endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y})$$

H is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group, (RAHMANI, 1992; KORPINAR; TURHAN, 2011, 2012,TURHAN; KORPINAR, 2010, 2011).

The identity of the group is (0,0,0) and the inverse of (x,y,z) is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on H is

$$g=-dx^2+dy^2+(xdy+dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathrm{e}_1=\frac{\partial}{\partial z},\mathrm{e}_2=\frac{\partial}{\partial y}-x\frac{\partial}{\partial z},\mathrm{e}_3=\frac{\partial}{\partial x}\right\}. \qquad (2.1)$$

The characterising properties of this algebra are the following commutation relations:

$$g(\mathrm{e}_1,\mathrm{e}_1)=g(\mathrm{e}_2,\mathrm{e}_2)=1,\ g(\mathrm{e}_3,\mathrm{e}_3)=-1.$$

Proposition 2.1.

For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:

$$\nabla=\frac{1}{2}\begin{pmatrix}0 & \mathrm{e}_3 & \mathrm{e}_2\\ \mathrm{e}_3 & 0 & \mathrm{e}_1\\ \mathrm{e}_2 & -\mathrm{e}_1 & 0\end{pmatrix}, \qquad (2.2)$$

where the (i,j)-element in the table above equals $\nabla_{\mathrm{e}_i}\mathrm{e}_j$ for our basis $\{\mathrm{e}_k, k=1,2,3\}$.

The unit pseudo-Heisenberg sphere (Lorentzian Heisenberg sphere) is defined by

$$\left(\mathsf{S}_1^2\right)_{\mathsf{H}}=\{\beta\in\mathsf{H}:g(\beta,\beta)=1\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W)=-g(R(X,Y)Z,W).$$

Moreover we put

$$R_{ijk}=R(\mathrm{e}_i,\mathrm{e}_j)\mathrm{e}_k,\ R_{ijkl}=R(\mathrm{e}_i,\mathrm{e}_j,\mathrm{e}_k,\mathrm{e}_l),$$

where the indices i,j,k and l take the values 1, 2 and 3.

$$R_{121}=\frac{1}{4}\mathrm{e}_2,\ R_{131}=\frac{1}{4}\mathrm{e}_3,\ R_{232}=-\frac{3}{4}\mathrm{e}_3$$

and

$$R_{1212}=-\frac{1}{4},\ R_{1313}=\frac{1}{4},\ R_{2323}=-\frac{3}{4}. \qquad (2.3)$$

Timelike Biharmonic S-Curves According To Sabban Frame In The $\left(\mathsf{S}_1^2\right)_{\mathsf{H}}$

Let $\gamma:I\to\mathsf{H}$ be a timelike curve in the Lorentzian Heisenberg group H parametrized by arc length. Let $\{\mathrm{T},\mathrm{N},\mathrm{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group H along γ defined as follows:

T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_{\mathrm{T}}\mathrm{T}$ (normal to γ), and B is chosen so that $\{\mathrm{T},\mathrm{N},\mathrm{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}&\nabla_{\mathrm{T}}\mathrm{T}=\kappa\mathrm{N},\\ &\nabla_{\mathrm{T}}\mathrm{N}=\kappa\mathrm{T}+\tau\mathrm{B},\\ &\nabla_{\mathrm{T}}\mathrm{B}=-\tau\mathrm{N},\end{aligned} \qquad (3.1)$$

where κ is the curvature of γ and τ is its torsion,

$$g(\mathrm{T},\mathrm{T})=-1, g(\mathrm{N},\mathrm{N})=1, g(\mathrm{B},\mathrm{B})=1,$$
$$g(\mathrm{T},\mathrm{N})=g(\mathrm{T},\mathrm{B})=g(\mathrm{N},\mathrm{B})=0.$$

Now we give a new frame different from Frenet frame, (BISHOP, 1975; BABAARSLAN; YAYLI, 2011). Let $\alpha : I \to (\mathrm{S}_1^2)_{\mathrm{H}}$ be unit speed spherical timelike curve. We denote σ as the arc-length parameter of α. Let us denote $\mathrm{t}(\sigma) = \alpha'(\sigma)$, and we call $\mathrm{t}(\sigma)$ a unit tangent vector of α. We now set a vector $\mathrm{s}(\sigma) = \alpha(\sigma) \times \mathrm{t}(\sigma)$ along α. This frame is called the Sabban frame of α on $(\mathrm{S}_1^2)_{\mathrm{H}}$. Then we have the following spherical Frenet-Serret formulae of α :

$$
\begin{aligned}
\nabla_{\mathrm{t}}\alpha &= \mathrm{t}, \\
\nabla_{\mathrm{t}}\mathrm{t} &= \alpha + \kappa_g \mathrm{s}, \\
\nabla_{\mathrm{t}}\mathrm{s} &= \kappa_g \mathrm{t},
\end{aligned} \tag{3.2}
$$

where κ_g is the geodesic curvature of the timelike curve α on the $(\mathrm{S}_1^2)_{\mathrm{H}}$ and
$g(\mathrm{t},\mathrm{t}) = -1, g(\alpha,\alpha) = 1, g(\mathrm{s},\mathrm{s}) = 1,$
$g(\mathrm{t},\alpha) = g(\mathrm{t},\mathrm{s}) = g(\alpha,\mathrm{s}) = 0.$

With respect to the orthonormal basis $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$, we can write

$$
\begin{aligned}
\alpha &= \alpha_1 \mathrm{e}_1 + \alpha_2 \mathrm{e}_2 + \alpha_3 \mathrm{e}_3, \\
\mathrm{t} &= t_1 \mathrm{e}_1 + t_2 \mathrm{e}_2 + t_3 \mathrm{e}_3, 3.3 \\
\mathrm{s} &= s_1 \mathrm{e}_1 + s_2 \mathrm{e}_2 + s_3 \mathrm{e}_3.
\end{aligned} \tag{3}
$$

To separate a biharmonic curve according to Sabban frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as biharmonic S-curve.

Theorem 3.1.

α is a timelike biharmonic S-curve if and only if

$$
\begin{aligned}
\kappa_g &= \text{constant} \neq 0, \\
1 + \kappa_g^2 &= [-\frac{1}{4} + \frac{1}{2}s_1^2] + \kappa_g[\alpha_1 s_1], \\
\kappa_g^3 &= \alpha_3 s_3 - \kappa_g[\frac{1}{4} - \frac{1}{2}\alpha_1^2].
\end{aligned} \tag{3.4}
$$

Proof.

Using Equation (2.1) and Sabban formulas Equation (3.2), we have Equation (3.4).

Corollary 3.2.

All of timelike biharmonic S-curves are helices.

Inextensible Flows Of S – Tangent Surfaces Of Timelike Biharmonic S-Curves According To Sabban Frame In The H

To separate a tangent surface according to Sabban frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for this surface as S – tangent surface.

The purpose of this section is to study S – tangent surfaces of timelike biharmonic S-curve in the Lorentzian Heisenberg group Heis^3.

The S – tangent surface of γ is a ruled surface

$$
\mathrm{R}^{\mathrm{S}}(\sigma,u) = \alpha(\sigma) + u\alpha'(\sigma). \tag{4.1}
$$

Definition 4.1.

A surface evolution $\mathrm{R}^{\mathrm{S}}(\sigma,u,t)$ and its flow $\frac{\partial \mathrm{R}^{\mathrm{S}}}{\partial t}$ are said to be inextensible if its first fundamental form $\{\mathrm{E},\mathrm{F},\mathrm{G}\}$ satisfies

$$
\frac{\partial \mathrm{E}}{\partial t} = \frac{\partial \mathrm{F}}{\partial t} = \frac{\partial \mathrm{G}}{\partial t} = 0. \tag{4.2}
$$

Definition 4.2.

We can define the following one-parameter family of developable ruled surface $\mathrm{R}^{\mathrm{S}}(\sigma,u,t) = \alpha(\sigma,t) + u\alpha'(\sigma,t).$

Hence, we have the following theorem.

Theorem 4.3.

Let R^{S} be one-parameter family of the S – tangent surface of a unit speed non-geodesic timelike biharmonic S-curve. Then $\frac{\partial \mathrm{R}^{\mathrm{S}}}{\partial t}$ is inextensible if and only if

$$
\begin{aligned}
&\frac{\partial}{\partial t}[\sinh \mathrm{A}(t) + u[\sinh \mathrm{A}(t)\sigma \\
&+ \frac{\cosh^2 \mathrm{A}(t)}{2\mathrm{B}_0^2}[\mathrm{B}_0(t)\sigma + \mathrm{B}_1(t)] \\
&- \frac{\cosh^2 \mathrm{A}(t)}{4\mathrm{B}_0^2(t)}\sinh 2[\mathrm{B}_0(t)\sigma + \mathrm{B}_1(t)] - \frac{\mathrm{B}_2(t)}{\mathrm{B}_0(t)}
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
&\cosh \mathrm{A}(t)\cosh[\mathrm{B}_0(t)\sigma + \mathrm{B}_1(t)] + \mathrm{B}_4(t) \\
&+ [\frac{\cosh \mathrm{A}(t)}{\mathrm{B}_0(t)}\sinh[\mathrm{B}_0(t)\sigma + \mathrm{B}_1(t)] + \mathrm{B}_2(t)] \\
&[\frac{\cosh \mathrm{A}(t)}{\mathrm{B}_0(t)}\cosh[\mathrm{B}_0(t)\sigma + \mathrm{B}_1(t)] + \mathrm{B}_3(t)]]
\end{aligned} \tag{4.3}
$$

$$+u[-\sinh A(t)\sigma-\frac{\cosh^2 A(t)}{2B_0^2(t)}[B_0(t)\sigma+B_1(t)]$$
$$+\frac{\cosh^2 A(t)}{4B_0^2(t)}\sinh 2[B_0(t)\sigma+B_1(t)]$$
$$+\frac{B_2(t)}{B_0(t)}\cosh A(t)\cosh[B_0(t)\sigma+B_1(t)]+B_4(t)$$
$$-[\frac{\cosh A(t)}{B_0(t)}\sinh[B_0(t)\sigma+B_1(t)]+B_2(t)]$$
$$[\frac{\cosh A(t)}{B_0(t)}\cosh[B_0(t)\sigma+B_1(t)]+B_3(t)]]]^2$$
$$+\frac{\partial}{\partial t}[\cosh A(t)\sinh[B_0(t)\sigma+B_1(t)]$$
$$+u[\frac{\cosh A(t)}{B_0(t)}\cosh[B_0(t)\sigma+B_1(t)]+B_3(t)]$$
$$+B_3(t)]^2+\frac{\partial}{\partial t}[\cosh A(t)\cosh[B_0(t)\sigma+B_1(t)]$$
$$+u[\frac{\cosh A(t)}{B_0(t)}\sinh[B_0(t)\sigma+B_1(t)]+B_2(t)]$$
$$+u[\cosh A(t)\cosh[B_0(t)\sigma+B_1(t)](B_0(t)$$
$$+\sinh A(t))-\frac{\cosh A(t)}{B_0(t)}\cosh[B_0(t)\sigma+B_1(t)]$$
$$+u[\cosh A(t)\sinh[B_0(t)\sigma+B_1(t)](B_0(t)+\sinh A$$
$$-\frac{\cosh A(t)}{B_0(t)}\sinh[B_0(t)\sigma+B_1(t)]+B_2(t)]]^2=0$$

where B_1,B_2,B_3,B_4 are smooth functions of time and

$$B_0(t)=\frac{\sqrt{1+\kappa_g^2(t)}}{\cosh A(t)}-\sinh A(t).$$

Proof.

Assume that α is a unit speed non-geodesic timelike biharmonic S-curve.

From definition of S-helix, we obviously obtain

$$\begin{aligned}t&=\sinh A(t)e_1+\cosh A(t)\sinh[B_0(t)\sigma\\&+B_1(t)]e_2\\&+\cosh A(t)\cosh[B_0(t)\sigma+B_1(t)]e_3.\end{aligned}\tag{4.4}$$

Using the formula of the Sabban, we write a relation:

$$\nabla_t t=t_1'e_1+(t_2'+t_1t_3)e_2+(t_3'+t_1t_2)e_3.\tag{4.5}$$

Since, we immediately arrive at

$$\begin{aligned}\nabla_t t&=\cosh A(t)\cosh[B_0(t)\sigma+B_1(t)]\\&(B_0(t)+\sinh A(t))e_2\\&+\cosh A(t)\sinh[B_0(t)\sigma+B_1(t)](B_0(t)+\sinh A(t))e_3.\end{aligned}$$

Obviously, we also obtain

$$\begin{aligned}s(\sigma)&=\frac{1}{\kappa_g(t)}[-\sinh A(t)\sigma-\frac{\cosh^2 A(t)}{2B_0^2(t)}\\&[B_0(t)\sigma+B_1(t)]\\&+\frac{\cosh^2 A(t)}{4B_0^2(t)}\sinh 2[B_0(t)\sigma+B_1(t)]+\frac{B_2(t)}{B_0(t)}\\&\cosh A(t)\cosh[B_0(t)\sigma+B_1(t)]\\&+B_4(t)-[\frac{\cosh A(t)}{B_0(t)}\sinh[B_0(t)\sigma+B_1(t)]\\&+B_2(t)][\frac{\cosh A(t)}{B_0(t)}\cosh[B_0(t)\sigma+B_1(t)]\\&+B_3(t)]]e_1+\frac{1}{\kappa_g(t)}[\cosh A(t)\cosh[B_0(t)\sigma\\&+B_1(t)](B_0(t)+\sinh A(t))\\&-\frac{\cosh A(t)}{B_0(t)}\cosh[B_0(t)\sigma+B_1(t)]+B_3(t)]e_2\\&+\frac{1}{\kappa_g(t)}[\cosh A(t)\sinh[B_0(t)\sigma+B_1(t)]\\&(B_0(t)+\sinh A(t))-\frac{\cosh A(t)}{B_0(t)}\\&\sinh[B_0(t)\sigma+B_1(t)]+B_2(t)]e_3.\end{aligned}\tag{4.6}$$

Also, the position vector of α is

$$\begin{aligned}\alpha(\sigma)&=[\sinh A(t)\sigma+\frac{\cosh^2 A(t)}{2B_0^2}[B_0(t)\sigma+B_1(t)]\\&-\frac{\cosh^2 A(t)}{4B_0^2(t)}\sinh 2[B_0(t)\sigma+B_1(t)]\\&-\frac{B_2(t)}{B_0(t)}\cosh A(t)\cosh[B_0(t)\sigma+B_1(t)]\\&+B_4(t)+[\frac{\cosh A(t)}{B_0(t)}\sinh[B_0(t)\sigma+B_1(t)]\\&+B_2(t)][\frac{\cosh A(t)}{B_0(t)}\cosh[B_0(t)\sigma+B_1(t)]\\&+B_3(t)]]e_1+[\frac{\cosh A(t)}{B_0(t)}\\&\cosh[B_0(t)\sigma+B_1(t)]+B_3(t)]e_2\\&+[\frac{\cosh A(t)}{B_0(t)}\sinh[B_0(t)\sigma+B_1(t)]+B_2(t)]e_3,\end{aligned}\tag{4.7}$$

where B_1,B_2,B_3,B_4 are smooth functions of time and

$$\mathsf{B}_0(t)=\frac{\sqrt{1+\kappa_g^2(t)}}{\cosh\mathsf{A}(t)}-\sinh\mathsf{A}(t).$$

Furthermore, we have the natural frame $\{(\mathsf{R}^s)_\sigma,(\mathsf{R}^s)_u\}$ given by

$$\begin{aligned}(\mathsf{R}^s)_\sigma &= [\sinh\mathsf{A}(t)+u[\sinh\mathsf{A}(t)\sigma\\
&+\frac{\cosh^2\mathsf{A}(t)}{2\mathsf{B}_0^2}[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&-\frac{\cosh^2\mathsf{A}(t)}{4\mathsf{B}_0^2(t)}\sinh 2[\mathsf{B}_0(t)\sigma\\
&+\mathsf{B}_1(t)]-\frac{\mathsf{B}_2(t)}{\mathsf{B}_0(t)}\cosh\mathsf{A}(t)\\
&\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_4(t)\\
&+[\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+\mathsf{B}_2(t)][\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\cosh\\
&[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_3(t)]]]\\
&+u[-\sinh\mathsf{A}(t)\sigma-\frac{\cosh^2\mathsf{A}(t)}{2\mathsf{B}_0^2(t)}[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+\frac{\cosh^2\mathsf{A}(t)}{4\mathsf{B}_0^2(t)}\sinh 2[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\frac{\mathsf{B}_2(t)}{\mathsf{B}_0(t)}\cosh\mathsf{A}(t)\\
&\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_4(t)\\
&-[\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+\mathsf{B}_2(t)][\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_3(t)]]]]e_1\\
&+[\cosh\mathsf{A}(t)\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+u[\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+\mathsf{B}_3(t)]+u[\cosh\mathsf{A}(t)\cosh[\mathsf{B}_0(t)\sigma\\
&+\mathsf{B}_1(t)](\mathsf{B}_0(t)+\sinh\mathsf{A}(t))\\
&-\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_3(t)]]e_2\\
&+[\cosh\mathsf{A}(t)\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+u[\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+\mathsf{B}_2(t)]+u[\cosh\mathsf{A}(t)\sinh[\mathsf{B}_0(t)\sigma\\
&+\mathsf{B}_1(t)](\mathsf{B}_0(t)+\sinh\mathsf{A}(t))\\
&-\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_2(t)]]e_3.\end{aligned}$$

and

$$\begin{aligned}(\mathsf{R}^s)_u &= \sinh\mathsf{A}(t)e_1+\cosh\mathsf{A}(t)\sinh[\mathsf{B}_0(t)\sigma\\
&+\mathsf{B}_1(t)]e_2+\cosh\mathsf{A}(t)\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]e_3.\end{aligned}$$

The components of the first fundamental form are

$$\begin{aligned}\frac{\partial \mathrm{E}}{\partial t} &= \frac{\partial}{\partial t}g((\mathsf{R}^s)_s,(\mathsf{R}^s)_s)=\frac{\partial}{\partial t}[\sinh\mathsf{A}(t)+u[\sinh\mathsf{A}(t)\sigma\\
&+\frac{\cosh^2\mathsf{A}(t)}{2\mathsf{B}_0^2}[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]-\\
&\frac{\cosh^2\mathsf{A}(t)}{4\mathsf{B}_0^2(t)}\sinh 2[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&-\frac{\mathsf{B}_2(t)}{\mathsf{B}_0(t)}\cosh\mathsf{A}(t)\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+\mathsf{B}_4(t)+[\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\\
&\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_2(t)][\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\\
&\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_3(t)]]\\
&+u[-\sinh\mathsf{A}(t)\sigma-\frac{\cosh^2\mathsf{A}(t)}{2\mathsf{B}_0^2(t)}[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+\frac{\cosh^2\mathsf{A}(t)}{4\mathsf{B}_0^2(t)}\sinh 2[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+\frac{\mathsf{B}_2(t)}{\mathsf{B}_0(t)}\cosh\mathsf{A}(t)\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+\mathsf{B}_4(t)-[\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+\mathsf{B}_2(t)][\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_3(t)]]]^2\\
&+\frac{\partial}{\partial t}[\cosh\mathsf{A}(t)\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&+u[\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_3(t)]\\
&+u[\cosh\mathsf{A}(t)\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\\
&(\mathsf{B}_0(t)+\sinh\mathsf{A}(t))-\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\cosh[\mathsf{B}_0(t)\sigma\\
&+\mathsf{B}_1(t)]+\mathsf{B}_3(t)]]^2+\frac{\partial}{\partial t}[\cosh\mathsf{A}(t)\\
&\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+u[\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\sinh[\mathsf{B}_0(t)\sigma\\
&+\mathsf{B}_1(t)]+\mathsf{B}_2(t)]+u[\cosh\mathsf{A}(t)\\
&\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)](\mathsf{B}_0(t)+\sinh\mathsf{A}(t))\\
&-\frac{\cosh\mathsf{A}(t)}{\mathsf{B}_0(t)}\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_2(t)]]^2,\end{aligned}$$

$$\frac{\partial \mathrm{F}}{\partial t}=0,\ \frac{\partial \mathrm{G}}{\partial t}=0.$$

Hence, $\frac{\partial \mathsf{R}^{\mathsf{S}}}{\partial t}$ is inextensible if and only if Equation (4.3) is satisfied. This concludes the proof of theorem.

Theorem 4.4.

Let R^{S} be one-parameter family of the S – tangent surface of a unit speed non-geodesic timelike biharmonic S-curve. Then, the parametric equations of this family are

$$\begin{aligned}
x_{\mathsf{R}^{\mathsf{S}}}(\sigma,u,t) &= \frac{\cosh \mathsf{A}(t)}{\mathsf{B}_0(t)}\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)] \\
&+ u\cosh \mathsf{A}(t)\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_2(t), \\
y_{\mathsf{R}^{\mathsf{S}}}(\sigma,u,t) &= \frac{\cosh \mathsf{A}(t)}{\mathsf{B}_0(t)}\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)] \\
&+ u\cosh \mathsf{A}(t)\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_3(t), \\
z_{\mathsf{R}^{\mathsf{S}}}(\sigma,u,t) &= \sinh \mathsf{A}(t)\sigma+\frac{\cosh^2 \mathsf{A}(t)}{2\mathsf{B}_0^2(t)} \\
&[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)] \\
&-\frac{\cosh^2 \mathsf{A}(t)}{4\mathsf{B}_0^2(t)}\sinh 2[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)] \\
&-\frac{\mathsf{B}_2(t)}{\mathsf{B}_0(t)}\cosh \mathsf{A}(t)\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)] \\
&+u\sinh \mathsf{A}(t)+u\cosh \mathsf{A}(t)(\frac{\cosh \mathsf{A}(t)}{\mathsf{B}_0(t)} \\
&\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)] \\
&+\mathsf{B}_2(t))\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]+\mathsf{B}_4(t),
\end{aligned} \tag{4.8}$$

where $\mathsf{B}_1,\mathsf{B}_2,\mathsf{B}_3,\mathsf{B}_4$ are constants of integration and

$$\mathsf{B}_0(t)=\frac{\sqrt{1+\kappa_g^2(t)}}{\cosh \mathsf{A}(t)}-\sinh \mathsf{A}(t).$$

Proof.

By the Sabban formula, we have the following equation

$$\begin{aligned}
\mathrm{t} &= \sinh \mathsf{A}(t)\mathrm{e}_1+\cosh \mathsf{A}(t)\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\mathrm{e}_2 \\
&+\cosh \mathsf{A}(t)\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]\mathrm{e}_3.
\end{aligned} \tag{4.9}$$

Using Equation (2.1) in Equation (4.9), we obtain

$$\begin{aligned}
\mathrm{t} = (&\cosh \mathsf{A}(t)\cosh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)], \\
&\cosh \mathsf{A}(t)\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)], \\
&\sinh \mathsf{A}(t)+\cosh \mathsf{A}(t)(\frac{\cosh \mathsf{A}(t)}{\mathsf{B}_0(t)}\sinh[\mathsf{B}_0(t)\sigma \\
&+\mathsf{B}_1(t)]+\mathsf{B}_2(t))\sinh[\mathsf{B}_0(t)\sigma+\mathsf{B}_1(t)]),
\end{aligned} \tag{4.10}$$

where $\mathsf{B}_1,\mathsf{B}_2$ are constants of integration.

Consequently, the parametric equations of R^{S} can be found from Equation (4.1), Equation (4.10). This concludes the proof of Theorem.

We can use Mathematica in above theorem, yields

The equation (4.8) is illustrated colour Red, Blue, Purple, Orange, Magenta, Cyan, Yellow, Green at the time $t=1$, $t=1.2$, $t=1.4$, $t=1.6$, $t=1.8$, $t=2$, $t=2.2$, $t=2.4$.

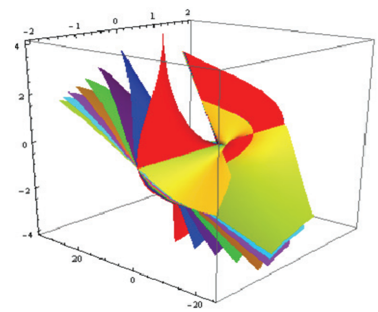

Figure 1. For $B_1 = B_2 = B_3 = B_4 = 1$.

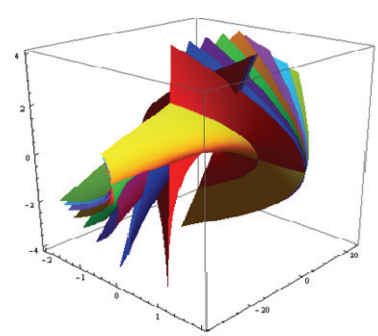

Figure 2. For $B_1 = B_2 = B_3 = B_4 = -1$.

Corollary 4.5.

Let $\alpha : I \to (\mathsf{S}_1^2)_{\mathsf{H}}$ be a unit speed non-geodesic timelike biharmonic S-curve. Then, the parametric equations of α are

$$\begin{aligned}
x^{\mathsf{S}}(\sigma) &= \frac{\cosh \mathsf{A}}{\mathsf{B}_0}\sinh[\mathsf{B}_0\sigma+\mathsf{B}_1]+\mathsf{B}_2, \\
y^{\mathsf{S}}(\sigma) &= \frac{\cosh \mathsf{A}}{\mathsf{B}_0}\cosh[\mathsf{B}_0\sigma+\mathsf{B}_1]+\mathsf{B}_3, \\
z^{\mathsf{S}}(\sigma) &= \sinh \mathsf{A}\sigma+\frac{\cosh^2 \mathsf{A}}{2\mathsf{B}_0^2}[\mathsf{B}_0\sigma+\mathsf{B}_1] \\
&-\frac{\cosh^2 \mathsf{A}}{4\mathsf{B}_0^2}\sinh 2[\mathsf{B}_0\sigma+\mathsf{B}_1]
\end{aligned}$$

$$-\frac{B_2}{B_0}\cosh A\cosh[B_0\sigma+B_1]+B_4,$$

where B_1, B_2, B_3, B_4 are constants of integration and

$$B_0=\frac{\sqrt{1+\kappa_g^2}}{\cosh A}-\sinh A.$$

We can use Mathematica in above theorem, yields

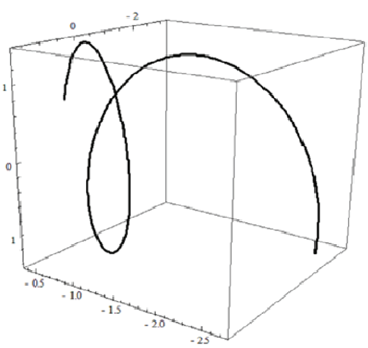

Figure 3. Biharmonic S-curve for $B_1 = B_2 = B_3 = B_4 = 1$.

Conclusion

The flow of a curve or surface is said to be inextensible if, in the former case, the arclength is preserved, and in the latter case, if the intrinsic curvature is preserved. Physically, inextensible curve and surface flows are characterized by the absence of any strain energy induced from themotion.

In this work, we study S – tangent surfaces according to Sabban frame in the Lorentzian Heisenberg group H. We find explicit parametric equations of one parameter family of S – tangent surfaces according to Sabban Frame.

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