Acta Scientiarum

http://www.uem.br/acta
ISSN printed: 1806-2563
ISSN on-line: 1807-8664
Doi: 10.4025/actascitechnol.v37i2.17273

The Inverse Nodal problem for the fractional diffusion equation

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ABSTRACT. In this paper, on a general finite interval, the inverse problem of recovering the potential function for a fractional diffusion equation with new spectral parameter, called the nodal point, is given. Furthermore, using Mittag Leffler function, asymptotic formulas for nodal points and nodal length for a fractional diffusion equation are also found.

Keywords: fractional calculus, fractional diffusion equation, inverse nodal problem, Mittag Leffler function, asymptotic formulas.

O Inverse Nodal problema para a equação de difusão fracionária

RESUMO. Neste trabalho, em um intervalo finito geral, o problema inverso de recuperar a função potencial para uma equação de difusão fracionária com novo parâmetro espectral, chamado de ponto nodal, é dado. Além disso, usando a função de Mittag Leffler, fórmulas assintóticas para os pontos nodais e comprimento nodal para a equação de difusão fracionária também são encontrados.

Palavras-chave: cálculo fracionário, equação de difusão fracionária, problema nodal inversa, função Mittag Leffler, fórmulas assintóticas.

Introduction

Fractional calculus is 'the theory of derivatives and integrals of any arbitrary real or complex order which unify and generalize the notions of integer-order differentiation and $n-$ fold integration' Kilbas et al. (2006), Podlubny (1999). It has been in the minds of mathematicians for 315 years and still contains many questions. Firstly, the idea of this area appeared in a letter by Leibniz to L' Hospital in 17th century, Podlubny (1999). In the following three hundred years a lot of mathematicians contribute to the fractional calculus: Johann Bernoulli, John Wallis, L. Euler, J. L. Lagrange, P. S. Laplace, S. F. Lacroix, J. B. J. Fourier, N. H. Abel, J. Liouville, S. S. Greatheed, A. Morgan, B. Riemann, W. Center, H. Holmgren, A. K. Grünwald, A. V. Letnikov, H. Laurent, O. Heaviside, G. H. Hardy, H. Weyl, E. L. Post, H. T. Davis, A. Erdélyi, H. Kober, A. Zygmund, M. Riesz, I. M. Gel'fand, G. E. Shilov, I. N. Sneddon, S. G. Samko, T. J. Osler, E. R. Love, and many others Boumenir and Tuan, (2010a and b), Chechkin et al. (2003), Freiling and Yurko (2001) and Gorenflo et al. (2002). Fractional diffusion equations have been investigated in a lot of different physical situations. These equations are widely applicable because many scenarios exist in which they find relevance. Furthermore, inverse spectral analysis involves the problem of restoring a linear operator from some of its spectral parameters Isakov (1993, 2006), Levitan and Gasymov (1964), Levitan and Sargsjan (1975), Levitan (1987), Jaulent and Jean (1972). Currently, inverse problems are being studied for certain special classes of ordinary differential operators. In recent years, Hald and McLaughlin (1989) have taken an inverse problem approach to the following problem:

$$L(y) = \lambda y, \tag{1}$$

$$y'(0) - hy(0) = 0, \tag{2}$$

$$y'(1) + Hy(1) = 0. \tag{3}$$

The inverse nodal problem lies in the use of nodal points of the eigenfunctions of (1)-(3) as spectral parameters. Hald and McLaughlin (1989) and Browne and Sleeman (1996) proved that one can use the nodal points to determine the potential function of regular Sturm-Liouville problem. In the last years, the inverse nodal problem and fractional calculus for Sturm Liouville problem has been studied by several authors Browne and Sleeman (1996), Yang (1997), Cheng et al. (2000), McLaughlin (1988), Bas (2013), Koyunbakan and Panakhov (2007), Gasymov and Guseinov (1981).

Tuan (2011) proved that by taking suitable initial distributions only finitely many measurements on the boundary were required to recover uniquely the diffusion coefficient of one dimensional fractional diffusion equation. His method was based on possibility of extracting the full boundary spectral data from special lateral measurements. The purpose of our study is to give the inverse problem of recovering the potential function for the fractional diffusion differential equation by using nodal datas Hald and McLaughlin (1989).

Now, let's give the preliminaries and notations regarding to the fractional calculus Kilbas et al. (2006), Podlubny (1999).

Preliminaries and notations

1) Definition

A real function

$f(r),\ r>0,$

where:

is said to be in space $C_\alpha,\ \alpha\varepsilon\Re$, if there exists a real number $p(>\alpha)$, such that

$f(r)=r^p,\ f_1(r)$

where:

$f_1(r)\in C[0,\infty).$

2) Definition

A real function

$f(r),\ r>0,$

where:

is said to be in space

$C_\alpha^m,\ m\in \mathrm{N}\cup\{0\},\ \text{if } f^{(m)}\in C_\alpha$

3) Definition

Let $f \in C_\alpha$ and α?-1, in the following, the expression two of the most commonly encountered tools in the theory and applications of fractional calculus are provided by the Riemann-Liouville operator R_z^α

$$R_z^\alpha f(z)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1} f(t)dt, & (\mathrm{Re}(\alpha)>0),\\ \frac{d^n}{dz^n}R_z^{\alpha+n}f(z), & (-n<\mathrm{Re}(\alpha)\le 0;\ n\in \mathrm{N}),\end{cases}$$

and the Weyl operator W_z^α which are defined by

$$W_z^\alpha f(z)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_z^\infty (t-z)^{\upsilon-1} f(t)dt, & (\mathrm{Re}(\alpha)>0),\\ \frac{d^n}{dz^n}W_z^{\alpha+n}f(z), & (-n<\mathrm{Re}(\alpha)\le 0;\ n\in \mathrm{N}).\end{cases}$$

4) Definition

The Caputo derivative of fractional order α of function $f(t)$ is defined as

$$ {}_0^C D_t^\alpha f(t)=\frac{1}{\Gamma(\alpha-n)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau \quad (n-1<\alpha<n). \quad (4)$$

5) Definition

One and two-parameter function of the Mittag-Leffler is defined by the series expansion, in the following form, respectively,

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$$

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad (\alpha>0,\quad \beta>0).$$

As mentioned above, Vu Kim Tuan obtained inverse problem for fractional diffusion equation in Tuan (2011). In the present study, we will define the inverse problem for the same equation by using nodal datas. In the following for the operation and content integrity, firstly we will give the basic concepts for the equation.

Consider the 1-dimensional fractional diffusion equation defined by

$$ {}_0^c D_t^\alpha u(r,t)=u_{rr}(r,t)-q(r)u(r,t), 0<r<\pi, t>0, \quad (5)$$

$$\begin{aligned} &u_r(0,t)-hu(0,t)=0,\\ &u_r(\pi,t)+Hu(\pi,t)=0, \quad (6)\\ &u(r,0)=f(r).\end{aligned}$$

where:

$q\in L_1(0,\pi), f\in L_2(0,\pi),$

and

${}_0^C D_t^\alpha u(t), 0<\alpha<1$

is the Caputo fractional derivative Tuan (2011).

Here the solution u depends on the lateral boundary conditions. One may rewrite main four cases for the direct problem, i.e.

$$\begin{aligned} u_r(0,t)-hu(0,t)&=0,\ h\neq\infty,\\ u(0,t)&=0,\ h=\infty,\\ u_r(\pi,t)+Hu(\pi,t)&=0,\ H\neq\infty,\\ u(\pi,t)&=0,\ H=\infty. \end{aligned} \tag{7}$$

If $u(0,t)=u(\pi,t)=0$, it means
$h=H=\infty$,

where:

h and H have been fixed, $\varphi(r,\lambda)$ is a solution of the Sturm-Liouville problem at $r=0$

$$\begin{aligned} &-\varphi''(r,\lambda)+q(r)\varphi(r,\lambda)=\lambda\varphi(r,\lambda), \quad 0<r<\pi\\ &\varphi(0,\lambda)=1, \qquad \varphi'(0,\lambda)=h, \qquad \text{if} \quad h\neq\infty,\\ &\varphi(0,\lambda)=0 \qquad \varphi'(0,\lambda)=1 \qquad \text{if} \quad h=\infty, \end{aligned} \tag{8}$$

λ_n and $\varphi(r,\lambda_n)$

are the eigenvalues and eigenfunctions of the boundary value problem (8) with the boundary condition

$$\begin{aligned} &\varphi'(\pi,\lambda)+H\varphi(\pi,\lambda)=0, \qquad \text{if} \quad H\neq\infty\\ &\varphi(\pi,\lambda)=0, \qquad \text{if} \quad H=\infty. \end{aligned} \tag{9}$$

In all cases, α_n shows the $L_2(0,\pi)$ norm of $\varphi(r,\lambda_n)$. The normalized eigenfunctions

$$\psi_n(r)=\frac{1}{\alpha_n}\varphi(r,\lambda_n), \quad \|\psi_n\|_2=1,$$

where:

$$\alpha_n=\|\varphi(r,\lambda_n)\|_2.$$

Therefore, the generalized Fourier series of any initial condition $f\in L_2(0,\pi)$ is obtained by

$$f(r)=\sum_{n\geq 0}c_n\varphi(r,\lambda_n)$$

where:

$$c_n=\frac{\langle f,\psi_n\rangle}{\alpha_n}=\frac{\langle f,\varphi(r,\lambda_n)\rangle}{\alpha_n^2}. \tag{10}$$

By virtue of the seperation of variables method one can find the solution in the following form

$$u(r,t)=\sum_{n\geq 0}T_n(t)\varphi(r,\lambda_n),$$

where:

$T_n(t)$ is a solution of the fractional differential equation founded below:

$${}_0^C D_t^{\alpha}T_n(t)=-\lambda T_n(t) \qquad\qquad T_n(0)=c_n.$$

The solution can be written as follows:

$$T_n(t)=c_nE_\alpha\left(-\lambda_n t^\alpha\right).$$

Take into account of the separation of variables method the solution $u^f=u$ of (6) in the series form

$$u(r,t)=\sum_{n\geq 0}c_nE_\alpha\left(-\lambda_n t^\alpha\right)\varphi(r,\lambda_n) \tag{11}$$

$$u_r(r,t)=\sum_{n\geq 0}c_nE_\alpha\left(-\lambda_n t^\alpha\right)\varphi'(r,\lambda_n). \tag{12}$$

Lemma: For each fixed $t>0$ the series (11) and (12) converge uniformly on $[0,\pi]$.

Proof: Let $h\neq\infty$. The asymptotic formulas, Levitan and Gasymov (1964), Levitan and Sargsjan (1975), Levitan (1987), for the solution $\varphi(r,\lambda_n)$

$$\varphi(r,\lambda)=\cos\sqrt{\lambda}r+O\left(\frac{1}{\sqrt{\lambda}}\right) \tag{13}$$

by differentiating the last equation, we get

$$\varphi'(r,\lambda)=-\sqrt{\lambda}\sin\sqrt{\lambda}r+h\cos\sqrt{\lambda}r+O\left(\frac{1}{\sqrt{\lambda}}\right) \tag{14}$$

and the eigenvalues

$$\sqrt{\lambda_n}=n+\frac{c}{n}+O\left(\frac{1}{n}\right), \qquad n\to\infty \tag{15}$$

where:

$$c=\tfrac{1}{\pi}\left(h+H+\tfrac{1}{2}\int_0^\pi q(\tau)d\tau\right).$$

Which yield

$$\varphi(r,\lambda_n)=O(1), \qquad \varphi'(r,\lambda_n)=O(n), \qquad r\in[0,\pi]. \tag{16}$$

Let $h=\infty$. Then the asymptotic formulas, Levitan and Gasymov (1964), Levitan and Sargsjan (1975), Levitan (1987), for the solution $\varphi(r,\lambda_n)$ and its derivative have the following form

$$\varphi(r,\lambda)=\frac{\sin\sqrt{\lambda}r}{\sqrt{\lambda}}+O\left(\frac{1}{\lambda}\right)$$
$$\varphi'(r,\lambda)=\cos\sqrt{\lambda}r+O\left(\frac{1}{\sqrt{\lambda}}\right). \quad (17)$$

For the eigenvalues we have, Levitan and Gasymov (1964), Levitan and Sargsjan (1975), Levitan (1987),

$$\varphi(r,\lambda)=\frac{\sin\sqrt{\lambda}r}{\sqrt{\lambda}}+O\left(\frac{1}{\lambda}\right)$$
$$\varphi'(r,\lambda)=\cos\sqrt{\lambda}r+O\left(\frac{1}{\sqrt{\lambda}}\right). \quad (18)$$

Consequently,

$$\varphi(r,\lambda_n)=O\left(\frac{1}{n}\right),\quad \varphi'(r,\lambda_n)=O(1),\quad r\in[0,\pi]. \quad (19)$$

The Mittag-Leffler function is bounded Kilbas et al. (2006),

$$|E_\alpha(z)|<\frac{C}{1+|z|},\qquad \frac{\alpha\pi}{2}<\mu\le|\arg(z)|\le\pi, \quad (20)$$

therefore,

$$\left|E_\alpha(-\lambda_n t^\alpha)\right|<\frac{C}{1+\lambda_n|t|^\alpha},\quad 0\le|\arg(t)|\le\mu\le\min\left\{\pi,\frac{\pi}{\alpha}-\frac{\pi}{2}\right\}. \quad (21)$$

For $t>0$, we get

$$E_\alpha(-\lambda_n t^\alpha)=O\left(\frac{1}{n^2}\right). \quad (22)$$

We have now, by the Cauchy-Schwartz inequality,

$$\left|\sum_{n\ge N}c_nE_\alpha(-\lambda_nt^\alpha)\varphi(r,\lambda_n)\right|^2\le\sum_{n\ge N}|c_n|^2\sum_{n\ge N}\left|E_\alpha(-\lambda_nt^\alpha)\varphi(r,\lambda_n)\right|^2$$
$$\le C\sum_{n\ge N}|c_n|^2\sum_{n\ge N}\frac{1}{n^4}\le\ell^2 \quad (23)$$

Let N choose large enough and $r\in[0,\pi]$. Hence, the series (11) converge uniformly on $[0,\pi]$. Similarly, for the series (12) we have

$$\left|\sum_{n\ge N}c_nE_\alpha(-\lambda_nt^\alpha)\varphi'(r,\lambda_n)\right|^2\le\sum_{n\ge N}|c_n|^2\sum_{n\ge N}\left|E_\alpha(-\lambda_nt^\alpha)\varphi'(r,\lambda_n)\right|^2$$
$$\le C\sum_{n\ge N}|c_n|^2\sum_{n\ge N}\frac{1}{n^2}\le\ell^2 \quad (24)$$

if N is chosen large enough and $r\in[0,\pi]$. Consequently, the series (12) converge uniformly on $[0,\pi]$.

The uniform convergence of the series (11) and (12) on $[0,\pi]$ allows us to represent the readings at the boundary points $r=0$ and $r=\pi$ as series of Mittag-Leffler functions. We consider Mittag-Leffler function series expansions which can be summarized in the following cases Tuan (2011).

$$\begin{aligned}&\text{For } h=\infty \text{ and } r=0, && u_r(0,t)=\textstyle\sum_{n\ge0}c_nE_\alpha(-\lambda_nt^\alpha)\\ &\text{For } h\ne\infty \text{ and } r=0, && u(0,t)=\textstyle\sum_{n\ge0}c_nE_\alpha(-\lambda_nt^\alpha)\\ &\text{For } H=\infty \text{ and } r=\pi, && u_r(\pi,t)=\textstyle\sum_{n\ge0}c_nE_\alpha(-\lambda_nt^\alpha)\varphi'(\pi,\lambda_n)\\ &\text{For } H\ne\infty \text{ and } r=\pi, && u(r,t)=\textstyle\sum_{n\ge0}c_nE_\alpha(-\lambda_nt^\alpha)\varphi(\pi,\lambda_n).\end{aligned} \quad (25)$$

Results

Let $\lambda_0<\lambda_1<\lambda_2...\to\infty$ be the eigenvalues of the problem (5) and

$$0<r_1^n<...<r_j^n<\pi,\ \ j=1,2,...,n-1$$

where:

be nodal points of n-th eigenfunction. It is shown that the set of all nodal points $\{r_j^n\}$ is dense in $[0,\pi]$. In fact, judicious choice of one nodal point r_j^n for each y_n, $n>1$ also forms a dense set in $[0,\pi]$. Using (11) and (13), and we may write

$$u(r,t)=\sum_{n\ge0}c_nE_\alpha(-\lambda_nt^\alpha)\left(\cos\sqrt{\lambda}r+O\left(\frac{1}{\sqrt{\lambda}}\right)\right)$$

and then we obtained

$$\left|u(r,t)-\sum_{n\ge0}c_nE_\alpha(-\lambda_nt^\alpha)\left(\cos\sqrt{\lambda}r+O\left(\frac{1}{\sqrt{\lambda}}\right)\right)\right|<M.$$

Approximately, we rewrite last equation as follows:

$$\left|u(r,t)-\sum_{n\ge0}c_nE_\alpha(-\lambda_nt^\alpha)\left(\cos\sqrt{\lambda}r\right)\right|<M.$$

By virtue of (23), $u(r,t)$ vanishes in the intervals whose end points are solutions of the following inequality

$$\sum_{n\geq 0} c_n E_\alpha\left(-\lambda_n t^\alpha\right)\left(\cos\sqrt{\lambda} r\right) < M$$

$$\cos\sqrt{\lambda} r < \frac{M}{\sum_{n\geq 0} c_n E_\alpha\left(-\lambda_n t^\alpha\right)} \leq \ell^2.$$

If we consider,

$$c_n = \frac{\langle f, \varphi(r,\lambda_n)\rangle}{\alpha_n^2}$$

and equations (21) and (22) then c_n and $E_\alpha\left(-\lambda_n t^\alpha\right)$ are bounded, respectively.

$$\cos\sqrt{\lambda} r \leq \ell^2$$
$$\sqrt{\lambda} r = \arccos \ell^2$$

the expanding $\arccos \ell^2$

$$\sqrt{\lambda} r = \left(j - \frac{1}{2}\right)\pi + \frac{\ell^4 + 1}{2\ell^2} + O(1), \ \ (j = 1, 2, \dots, n-1)$$
$$r = \frac{\left(j - \frac{1}{2}\right)\pi}{\sqrt{\lambda}} + \frac{\ell^4 + 1}{2\sqrt{\lambda}\ell^2} + O\left(\frac{1}{\sqrt{\lambda}}\right), \ (j = 1, 2, \dots, n-1, \ n = 1, 2, \dots)$$
$$r_j^n = \frac{\left(j - \frac{1}{2}\right)\pi}{n} + \frac{\ell^4 + 1}{2\ell^2 n} + O\left(\frac{1}{n}\right).$$

The nodal length is

$$I_j^n = r_{j+1}^n - r_j^n$$
$$I_j^n = \left(\frac{j + 1 - \frac{1}{2}}{n}\right)\pi - \left(\frac{j - \frac{1}{2}}{n}\right)\pi + O\left(\frac{1}{n}\right)$$
$$I_j^n = \frac{\pi}{n} + O\left(\frac{1}{n}\right).$$

Theorem: Suppose that q is integrable at fractional diffusion equation. Then h and $q - \int_0^\pi q$ are uniquely determined by any dense set of nodal points.

Proof: For $h \neq \infty$, consider the second problem with $\widetilde{h}$ and $\widetilde{q}$. Furthermore, let the nodal points r_j^n, r_j^n satisfy $r_j^n = r_j^n$ and form a dense set in $[0,\pi]$.

Considering solutions of $(5)-(6)$ as u_n for (h,q) and $\widetilde{u}_n$ for $\left(\widetilde{h},\widetilde{q}\right)$

$$\left[u_n'\widetilde{u}_n - u_n\widetilde{u}_n'\right]_r = \left[\left|\sum_{n\geq 0} c_n E_\alpha\left(-\lambda_n t^\alpha\right)\left(\varphi_n'\widetilde{\varphi}_n - \varphi_n\widetilde{\varphi}_n'\right)\right|\right]_r \quad (26)$$
$$= \left[q - \widetilde{q} + \widetilde{\lambda}_n - \lambda_n\right]\varphi_n\widetilde{\varphi}_n.$$

Recalling $r_j^n = r_j^n$ then integrating both sides of (26) from 0 to r_j^n and using the boundary conditions

$$\sum_{n\geq 0} c_n E_\alpha\left(-\lambda_n t^\alpha\right)\left(h - \widetilde{h}\right)\varphi_n(0)\widetilde{\varphi}_n(0) = \int_0^{r_j^n}\left[q - \widetilde{q} + \widetilde{\lambda}_n - \lambda_n\right]\varphi_n\widetilde{\varphi}_n dr \quad (27)$$

where:
$\widetilde{\lambda}_n = \lambda_n$ are uniformly bounded in n and the $\varphi_n\widetilde{\varphi}_n$ are uniformly bounded in n and $r \in [0,\pi]$.

We now select a subsequence of nodes from the dense set. If the subsequence tends to zero, then the right side of (27) is equal to zero. Then we obtain the following equations:

$$\sum_{n\geq 0} c_n E_\alpha\left(-\lambda_n t^\alpha\right)\left(h - \widetilde{h}\right)\varphi_n(0)\widetilde{\varphi}_n(0) = 0,$$

$$\sum_{n\geq 0} c_n E_\alpha\left(-\lambda_n t^\alpha\right)\varphi_n(0)\widetilde{\varphi}_n(0) \neq 0,$$

hence we get $h = \widetilde{h}$.

For $H \neq \infty$, similarly, to get that $H = \widetilde{H}$, integrating both sides of (26) from r_j^n to π and select a subsequence that tends to γ

$$\sum_{n\geq 0} c_n E_\alpha\left(-\lambda_n t^\alpha\right)\left(H - \widetilde{H}\right)\varphi_n(\pi)\widetilde{\varphi}_n(\pi) = \int_{r_j^n}^{\pi}\left[q - \widetilde{q} + \widetilde{\lambda}_n - \lambda_n\right]\varphi_n\widetilde{\varphi}_n dr \quad (28)$$

according to this results, we can say $H = \widetilde{H}$. Since $h = \widetilde{h}$, $H = \widetilde{H}$.

Now we take a sequence r_j^n accumulating at an arbitrary $r \in [0,\pi]$ and using the above method,

$$\int_0^{r_j^n}\left[u_n'\widetilde{u}_n - u_n\widetilde{u}_n'\right]_r dr = \int_0^{r_j^n}\left[\left|\sum_{n\geq 0} c_n E_\alpha\left(-\lambda_n t^\alpha\right)\left(\varphi_n'\widetilde{\varphi}_n - \varphi_n\widetilde{\varphi}_n'\right)\right|\right]_r dr$$
$$= \int_0^{r_j^n}\left[q - \widetilde{q} + \widetilde{\lambda}_n - \lambda_n\right]\varphi_n\widetilde{\varphi}_n dr$$
$$= \left[u_n'\left(r_j^n\right)\widetilde{u}_n\left(r_j^n\right) - u_n\left(r_j^n\right)\widetilde{u}_n'\left(r_j^n\right)\right]$$
$$= \sum_{n\geq 0} c_n E_\alpha\left(-\lambda_n t^\alpha\right)\left(\varphi_n'\left(r_j^n\right)\widetilde{\varphi}_n\left(r_j^n\right) - \varphi_n\left(r_j^n\right)\widetilde{\varphi}_n'\left(r_j^n\right)\right)$$
$$= \int_0^{r_j^n}\left[q - \widetilde{q} + \widetilde{\lambda}_n - \lambda_n\right]\varphi_n\widetilde{\varphi}_n dr$$

$$0 = \int_0^{r_j^n}\left[q - \widetilde{q} + \widetilde{\lambda}_n - \lambda_n\right]\varphi_n\widetilde{\varphi}_n dr,$$

using the asymptotic formulas of λ_n and $\widetilde{\lambda}_n$,

$$\int_0^{r_j^n}\left[q-\widetilde{q}+\frac{1}{\pi}\left(\widetilde{h}+\widetilde{H}+\frac{1}{2}\int_0^{\pi}\widetilde{q}(\tau)d\tau-h-H-\frac{1}{2}\int_0^{\pi}q(\tau)d\tau\right)\right]\varphi_n\widetilde{\varphi}_n ds=0$$

considering $H=\widetilde{H}$, and $h=\widetilde{h}$ and taking a sequence r_j^n accumulating at an arbitrary $r\in(0,\pi)$. Thus

$$\int_0^{r}\left(q-\widetilde{q}-\int_0^{\pi}(\widetilde{q}(\tau)-q(\tau))d\tau\right)\varphi_n\widetilde{\varphi}_n ds=0$$

for all r. It is clear that $q-\int_0^{\pi}q(\tau)d\tau$ is uniquely determined by a dense set of nodes.

Corollary: For the fractional inverse nodal problem, the potential q is uniquely determined by a dense set of nodes and the constant

$$c=\tfrac{1}{\pi}\left(h+H+\tfrac{1}{2}\int_0^{\pi}q(\tau)d\tau\right).$$

Proof: Suppose that $c=\widetilde{c}$. Since $h=\widetilde{h}$ and $H=\widetilde{H}$ it follows that $\int_0^{\pi}q=\int_0^{\pi}\widetilde{q}$. Thus the proof is obvious for $q=\widetilde{q}$ almost everywhere on $(0,\pi)$.

Conclusion

In this paper, we have extended the scope of the inverse nodal problem by proving the uniqueness theorem for the fractional diffusion equation. We obtain the uniqueness of the potential function for the diffusion equation by using nodal data and fractional calculus.

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Received on May 17, 2012.
Accepted on June 22, 2012.