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Korovkin type approximation theorem for functions of two variables via statistical summability (C, 1)

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ABSTRACT. The concept of statistical summability (C, 1) has recently been introduced by Moricz (2002). In this paper, we use this notion of summability to prove the Korovkin type approximation theorem for functions of two variables. Finally we construct an example by Bleimann, Butzer and Hahn operators to show that our result is stronger than those of previously proved by other authors for ordinary convergence and statistical convergence.

Keywords: statistical convergence, *A*–statistical convergence, statistical *A*–summability, statistical summability (C, 1), positive linear operator, Korovkin type approximation theorem.

O teorema do tipo aproximação de Korovkin para funções de duas variabilidades pela sumabilidade estatística

RESUMO. O conceito de sumabilidade estatística (C, 1) foi introduzido recentemente por Moricz (2002). Usamos a noção de sumabilidade nesse artigo para provar o teorema de aproximação de Korovkin para funções de duas variabilidades. Construimos um modelo pelos operadores de Bleimann, Butzer and Hahn para mostrar que nossos resultados são mais fortes do que aqueles provados por outros autores para convergência ordinária e convergência statística.

Palavras-chave: convergência estatística, *A*-convergência estatística, *A*-summabilidade estatística, sumabilidade estatística (C, 1), operador linear positivo, teorema do tipo aproximação de Korovkin.

Introduction

The concept of statistical convergence for sequences of real numbers was introduced by Fast (1951) and further studied by many others (FRIDY 1985; MOHIUDDINE; AIYUB, 2012; MOHIUDDINE; ALGHAMDI, 2012; MOHIUDDINE et al., 2010, 2013a).

Let $K \subseteq \mathbb{N}$ and $K_n = \{k \le n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = \lim_n n^{-1} |K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n.

A sequence $x = (x_k)$ of real numbers is said to be statistically convergent to L provided that for every $\varepsilon > 0$ the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \ge \varepsilon\}$ has natural density zero, i.e. for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \le n : |x_k - L| \ge \varepsilon\}| = 0.$$

In this case we write $st\text{-}\lim x = L$. Note that if $x = (x_k)$ is convergent then it is statisically convergent but not conversely. The idea of statistical convergence of double sequences has been introduced by Moricz (2003), Mursaleen and Edely (2003) and further studied by Mohiuddine et al. (2012a, b and d; 2013b), Mursaleen and Mohiuddine (2009).

Let $A = (a_{nk}), n, k \in \mathbb{N}$, be an infinite matrix and $x = (x_k)$ be a sequence. Then the (transformed) sequence, $Ax :=\ (y_n)$, is defined by

$$y_n := \sum_{k=1}^{\infty} a_{nk} x_k,$$

where it is assumed that the series on the right converges for each $n \in \mathbb{N}$. We say that a sequence x is A-summable to the limit ℓ if $y_n \to \ell$ as $n \to \infty$.

A matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known conditions for a matrix to be regular are known as Silverman-Toeplitz conditions (MADDOX, 1970). That is $A = (a_{nk})$ is regular if and only if

$$\|A\| = \sup_n \sum_k |a_{nk}| < \infty, \tag{1}$$

$$\lim_n a_{nk} = 0 \text{ for each } k \in \mathbb{N}, \tag{2}$$

$$\lim_n \sum_k a_{nk} = 1. \tag{3}$$

In Edely and Mursaleen (2009) have given the notion of statistical A-summability for single sequences and statistical A-summability for double sequences has recently been studied in (BELEN et al., 2012).

Let $A = (a_{nk})$ be a nonnegative regular matrix and $x = (x_k)$ be a sequence of real or complex sequences. We say that x is statistically A-summable to L if for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : |y_n - L| \geq \varepsilon\}) = 0.$$

So, if x is statistically A-summable to L then for every $\varepsilon > 0$,

$$\lim_m \frac{1}{m}\left|\{n \leq m : |y_n - L| \geq \varepsilon\}\right| = 0.$$

Note that if a sequence is bounded and A-statistically convergent to L, then it is A-summable to L; hence it is statistically A-summable to L but not conversely [see Edely and Mursaleen (2009)].

If $A = (C,1)$, the Cesàro matrix, then statistical A-summability is reduced to statistical summability (C, 1) due to Moricz (MORICZ, 2002).

For a sequence $x = (x_k)$, let us write $t_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k$. We say that a sequence $x = (x_k)$ is statistically summable (C, 1) if $st\text{-}\lim_{n\to\infty} t_n = L$. In this case we write $L = C_1(st)\text{-}\lim x$.

In the following example we exhibit that a sequence is statistically summable (C, 1) but not statistically convergent. Define a sequence $u = (u_k)$ by

$$u_k = \begin{cases} 1 \text{ if } k = m^2 - m, m^2 - m + 1, \cdots, m^2 - 1; \\ -m \text{ if } k = m^2, m = 2,3,4,\cdots; \\ 0 \text{ otherwise.} \end{cases} \tag{4}$$

Then

$$t_n = \frac{1}{n+1}\sum_{k=0}^{n} u_k$$

$$= \begin{cases} \frac{s+1}{n+1} \text{ if } n = m^2 - m + s; s = 0,1,2,\cdots,m-1; m = 2,3,\cdots; \\ 0 \text{ otherwise.} \end{cases} \tag{5}$$

It is easy to see that $\lim_{n\to\infty} t_n = 0$ and hence $st\text{-}\lim_{n\to\infty} t_n = 0$, i.e. $u = (u_k)$ is statistically summable (C, 1) to 0. On the other hand $st\text{-}\liminf_{k\to\infty} u_k = 0$ and $st\text{-}\limsup_{k\to\infty} u_k = 1$, since the sequence $(m^2)_{m=2}^{\infty}$ is statistically convergent to 0. Hence $u = (u_k)$ is not statistically convergent.

Let $I = [0,\infty)$ and $C(I)$ denote the space of all continuous real valued functions on I. Let $C_B(I) = \{f \in C(I) : f$ is bounded on $I\}$. $C(I)$ and $C_B(I)$ are equipped with norm

$$\| f \|_{C(I)} = \sup_{x\in I} | f(x) | .$$

Let $H_\omega(I)$ denote the space of all real valued functions f on I such that

$$| f(s) - f(x) | \leq \omega(f; | \frac{s}{1+s} - \frac{x}{1+x} |),$$

where:

ω is the modulus of continuity, i.e.

$$\omega(f;\delta) = \sup_{s,x\in I}\{| f(s) - f(x) | : | s - x | \leq \delta\}.$$

It is to be noted that any function $f \in H_\omega(I)$ is continuous and bounded on I.

The following Korovkin type theorem [see Korovkin (1960)] was proved by Çakar and Gadjiev (1999).

Theorem 1: Let (L_n) be a sequence of positive linear operators from $H_\omega(I)$ into $C_B(I)$. Then for all $f \in H_\omega(I)$

$$\lim_{n\to\infty} \| L_n(f;x) - f(x) \|_{C_B(I)} = 0$$

if and only if

$$\lim_{n\to\infty} \| L_n(f_i;x) - g_i \|_{C_B(I)} = 0 (i = 0,1,2),$$

where:

$$g_0(x) = 1, g_1(x) = \frac{x}{1+x}, g_2(x) = (\frac{x}{1+x})^2.$$

Erkus and Duman (2005) have given the st_A-version of the above theorem for functions of two variables. Quite recently, Korovkin type of approximation theorems have been proved in Alotaibi and Mursaleen (2012), Alotaibi et al. (2013), Anastassiou et al. (2011), Belen and Mohiuddine (2013), Braha et al. (2014), Demirci and Karakus (2011, 2013), Dirik and Demirci (2010a and b), Edely et al. (2010), Mohiuddine (2011), Mohiuddine and Alotaibi (2013a and b), Mohiuddine et al. (2012c), Mursaleen and Alotaibi (2011, 2012, 2013), Mursaleen and Ahmad (2013), Mursaleen and Edely (2009), Mursaleen and Kiliçman (2013), Mursaleen et al. (2012) for functions of one and two variables by using almost convergence, statistical convergence, A-statistical convergence, statistical A-summability and weighted statistical convergence of single and double sequences. In this paper, we use the notion of statistical summability (C, 1) to prove a Korovkin type approximation theorem for functions of two variables with the help of test functions $1, \frac{x}{1+x}, \frac{y}{1+y}, (\frac{x}{1+x})^2+(\frac{y}{1+y})^2$.

Main result

Let $I=[0,\infty)$ and $K=I\times I$. We denote by $C_B(K)$ the space of all bounded and continuous real valued functions on K equipped with norm

$$\| f\|_{C_B(K)} := \sup_{(x,y)\in K} |f(x,y)|, f\in C_B(K).$$

Let $H_{\omega^*}(K)$ denote the space of all real valued functions f on K such that

$$|f(s,t)-f(x,y)|\le \omega^*(f;\sqrt{(\frac{s}{1+s}-\frac{x}{1+x})^2+(\frac{t}{1+t}-\frac{y}{1+y})^2})$$

where:

ω^* is the modulus of continuity, i.e.

$$\omega^*(f;\delta)=\sup_{(s,t),(x,y)\in K}\{|f(s,t)-f(x,y)|:\sqrt{(s-x)^2+(t-y)^2}\le\delta\}$$

It is to be noted that any function $f\in H_{\omega^*}(K)$ is bounded and continuous on K, and a necessary and sufficient condition for $f\in H_{\omega^*}(K)$ is that

$$\lim_{\delta\to 0}\omega^*(f;\delta)=0.$$

We prove the following result:

Theorem 2: Let $A=(a_{nk})$ be nonnegative regular summability matrix. Let (T_k) be a sequence of positive linear operators from $H_{\omega^*}(K)$ into $C_B(K)$. Then for all $f\in H_{\omega^*}(K)$

$$C_1(st)-\lim_{k\to\infty}\| T_k(f;x,y)-f(x,y)\|_{C_B(K)}=0 \quad (5)$$

if and only if

$$C_1(st)-\lim_{k\to\infty}\| T_k(1;x,y)-1\|_{C_B(K)}=0 \quad (6)$$

$$C_1(st)-\lim_{k\to\infty}\| T_k(\frac{s}{1+s};x,y)-\frac{x}{1+x}\|_{C_B(K)}=0 \quad (7)$$

$$C_1(st)-\lim_{k\to\infty}\| T_k(\frac{t}{1+t};x,y)-\frac{y}{1+y}\|_{C_B(K)}=0 \quad (8)$$

$$C_1(st)-\lim_{k\to\infty}\| T_k((\frac{s}{1+s})^2+(\frac{t}{1+t})^2;x,y) -((\frac{x}{1+x})^2+(\frac{y}{1+y})^2)\|_{C_B(K)}=0 \quad (9)$$

Proof.: Since each of the functions $f_0(x,y)=1$, $f_1(x,y)=\frac{x}{1+x}$, $f_2(x,y)=\frac{y}{1+y}$, $f_3(x,y)=(\frac{x}{1+x})^2+(\frac{y}{1+y})^2$ belongs to $H_{\omega^*}(K)$, conditions (6)–(9) follow immediately from (5). Let $f\in H_{\omega^*}(K)$ and $(x,y)\in K$ be fixed. Then for $\varepsilon>0$ there exist $\delta_1,\delta_2>0$ such that $|f(s,t)-f(x,y)|<\varepsilon$ holds for all $(s,t)\in K$ satisfying $|\frac{s}{1+s}-\frac{x}{1+x}|<\delta_1$ and $|\frac{t}{1+t}-\frac{y}{1+y}|<\delta_2$. Let

$$K(\delta):=\{(s,t)\in K:\sqrt{(\frac{s}{1+s}-\frac{x}{1+x})^2+(\frac{t}{1+t}-\frac{y}{1+y})^2} <\delta=\min\{\delta_1,\delta_2\}\}$$

Hence

$$|f(s,t)-f(x,y)|=|f(s,t)-f(x,y)|\chi_{K(\delta)}(s,t) +|f(s,t)-f(x,y)|\chi_{K\setminus K(\delta)}(s,t) \le\varepsilon+2N\chi_{K\setminus K(\delta)}(s,t) \quad (10)$$

where:

χ_D denotes the characteristic function of the set D and $N=\| f\|_{C_B(K)}$. Further we get

$$\chi_{K\setminus K(\delta)}(s,t)\le\frac{1}{\delta_1^2}(\frac{s}{1+s}-\frac{x}{1+x})^2+\frac{1}{\delta_2^2}(\frac{t}{1+t}-\frac{y}{1+y})^2 \quad (11)$$

Combining (10) and (11), we get

$$|f(s,t)-f(x,y)|\leq \varepsilon+\frac{2N}{\delta^2}\{(\frac{s}{1+s}-\frac{x}{1+x})^2+(\frac{t}{1+t}-\frac{y}{1+y})^2\} \quad (12)$$

After using the properties of f, a simple calculation gives that

$$|T_k(f;x,y)-f(x,y)|\leq \varepsilon+M\{|T_k(f_0;x,y)-f_0(x,y)| + |T_k(f_1;x,y)-f_1(x,y)| + |T_k(f_2;x,y)-f_2(x,y)|+|T_k(f_3;x,y)-f_3(x,y)|\}$$

where:

$$M:=\varepsilon+N+\frac{4N}{\delta^2}.$$

Now replacing $T_k(.;x,y)$ by $\frac{1}{m+1}\sum_{k=0}^{m}T_k(.;x,y)$ and then by $B_m(.;x,y)$ and taking $\sup_{(x,y)\in K}$, we get

$$\|B_m(f;x,y)-f(x,y)\|_{C(K)}\leq \varepsilon+M(\|B_m(f_0;x,t)-f_0(x,y)\|_{C_B(K)} + \|B_m(f_1;x,y)-f_1(x,y)\|_{C_B(K)}+\|B_m(f_2;x,y)-f_2(x,y)\|_{C_B(K)} + \|B_m(f_3;x,y)-f_3(x,y)\|_{C_B(K)}) \quad (13)$$

For a given $r>0$ choose $\varepsilon>0$ such that $\varepsilon<r$. Define the following sets

$$D:=\{m\leq n: \|B_m(f;x,y)-f(x,y)\|_{C_B(K)}\geq r\},$$

$$D_1:=\{m\leq n: \|B_m(f_0;x,t)-f_0(x,y)\|_{C_B(K)}\geq \frac{r-\varepsilon}{4K}\},$$

$$D_2:=\{m\leq n: \|B_m(f_1;x,t)-f_1(x,y)\|_{C_B(K)}\geq \frac{r-\varepsilon}{4K}\},$$

$$D_3:=\{m\leq n: \|B_m(f_2;x,t)-f_2(x,y)\|_{C_B(K)}\geq \frac{r-\varepsilon}{4K}\},$$

$$D_4:=\{m\leq n: \|B_m(f_3;x,t)-f_3(x,y)\|_{C_B(K)}\geq \frac{r-\varepsilon}{4K}\}.$$

Then from (13), we see that $D\subset D_1\cup D_2\cup D_3\cup D_4$ and therefore $\delta(D)\leq\delta(D_1)+\delta(D_2)+\delta(D_3)+\delta(D_4)$. Hence conditions (6)–(9) imply the condition (5).

This completes the proof of the theorem.

For $m=0$ in the above theorem, we have the following special case which is two variables version of Theorem 1:

Corollary 2: Let $A=(a_{nk})$ be nonnegative regular summability matrix. Let (T_k) be a sequence of positive linear operators from $H_{\omega^*}(K)$ into $C_B(K)$. Then for all $f\in H_{\omega^*}(K)$

$$\lim_{k\to\infty}\|T_k(f;x,y)-f(x,y)\|_{C_B(K)}=0 \quad (14)$$

if and only if

$$\lim_{k\to\infty}\|T_k(1;x,y)-1\|_{C_B(K)}=0 \quad (15)$$

$$\lim_{k\to\infty}\|T_k(\frac{s}{1+s};x,y)-\frac{x}{1+x}\|_{C_B(K)}=0 \quad (16)$$

$$\lim_{k\to\infty}\|T_k(\frac{t}{1+t};x,y)-\frac{y}{1+y}\|_{C_B(K)}=0 \quad (17)$$

$$\lim_{k\to\infty}\|T_k((\frac{s}{1+s})^2+(\frac{t}{1+t})^2;x,y)-((\frac{x}{1+x})^2+(\frac{y}{1+y})^2)\|_{C_B(K)}=0 \quad (18)$$

Statistical rate of convergence

In this section, using the concept of statistically summable (C, 1) we study the rate of convergence of positive linear operators with the help of the modulus of continuity. Let us recall, for $f\in H_{\omega^*}(K)$

$$|f(s,t)-f(x,y)|\leq\omega^*(f;\sqrt{(\frac{s}{1+s}-\frac{x}{1+x})^2+(\frac{t}{1+t}-\frac{y}{1+y})^2}),$$

where:

$$\omega^*(f;\delta)=\sup_{(s,t),(x,y)\in K}\{|f(s,t)-f(x,y)|:\sqrt{(s-x)^2+(t-y)^2}\leq\delta\}$$

We have the following result:

Theorem 3: Let (T_k) be a sequence of positive linear operators from $H_{\omega^*}(K)$ into $C_B(K)$. Assume that

$$C_1(st)-\lim_{k\to\infty}\|T_k(f_0)-f_0\|_{C_B(K)}=0, \quad (19)$$

$$C_1(st)-\lim_{n\to 0}\omega^*(f;\delta_n)=0, \quad (20)$$

where:

$$\delta_n=\sqrt{\|T_k(\psi)\|_{C_B(K)}}\text{ with }\psi=\psi(s,t)=(\frac{s}{1+s}-\frac{x}{1+x})^2+(\frac{t}{1+t}-\frac{y}{1+y})^2$$

Then for all $f\in H_{\omega^*}(K)$

$$C_1(st)-\lim_{k\to\infty}\|T_k(f)-f\|_{C_B(K)}=0.$$

Proof.: Let $f\in H_{\omega^*}(K)$ be fixed and $(x,y)\in K$ be fixed. Using linearity and positivity of the operators T_k for all $n\in\mathbb{N}$, we have

$$|T_k(f;x,y)-f(x,y)|\leq T_k(|f(s,t)-f(x,y)|;x,y) + |f(x,y)||T_k(f_0;x,y)-f_0(x,y)|$$

$$\leq T_k(\omega^*(f;\delta\frac{\sqrt{(\frac{s}{1+s}-\frac{x}{1+x})^2+(\frac{t}{1+t}-\frac{y}{1+y})^2}}{\delta});x,y)$$
$$+\|f\|_{C_B(K)}|T_k(f_0;x,y)-f_0(x,y)|$$
$$\leq T_k((1+[\frac{\sqrt{(\frac{s}{1+s}-\frac{x}{1+x})^2+(\frac{t}{1+t}-\frac{y}{1+y})^2}}{\delta}])\omega^*(f;\delta);x,y)$$
$$+\|f\|_{C_B(K)}|T_k(f_0;x,y)-f_0(x,y)|$$
$$\leq \omega^*(f;\delta)T_k((1+\frac{(\frac{s}{1+s}-\frac{x}{1+x})^2+(\frac{t}{1+t}-\frac{y}{1+y})^2}{\delta^2});x,y)$$
$$+\|f\|_{C_B(K)}|T_k(f_0;x,y)-f_0(x,y)|$$
$$\leq \omega^*(f;\delta)|T_k(f_0;x,y)-f_0(x,y)|$$
$$+\|f\|_{C_B(K)}|T_k(f_0;x,y)-f_0(x,y)|+\omega^*(f;\delta)$$
$$+\frac{\omega^*(f;\delta)}{\delta^2}T_k(((\frac{s}{1+s}-\frac{x}{1+x})^2+(\frac{t}{1+t}-\frac{y}{1+y})^2;x,y)).$$

Hence

$$\|T_k(f)-f\|_{C_B(K)}\leq\|f\|_{C_B(K)}\|T_k(f_0)-f_0\|_{C_B(K)}$$
$$+\omega^*(f;\delta)\|T_k(f_0)-f_0\|_{C_B(K)}$$
$$+\frac{\omega^*(f;\delta)}{\delta^2}\|T_k(\psi)\|_{C_B(K)}+\omega^*(f;\delta).$$

Now if we choose $\delta:=\delta_n:=\sqrt{\|T_k(\psi)\|_{C_B(K)}}$, then

$$\|T_k(f)-f\|_{C_B(K)}\leq\|f\|_{C_B(K)}\|T_k(f_0)-f_0\|_{C_B(K)}$$
$$+\omega^*(f;\delta_n)\|T_k(f_0)-f_0\|_{C_B(K)}+2\omega^*(f;\delta_n).$$

Therefore

$$\|T_k(f)-f\|_{C_B(K)}\leq M\{\|T_k(f_0)-f_0\|_{C_B(K)}+\omega^*(f;\delta_n)$$
$$\|T_k(f_0)-f_0\|_{C_B(K)}+\omega^*(f;\delta_n)\}$$

where:
$M=\max\{2,\|f\|_{C_B(K)}\}$. Now, for a given $r>0$, choose $\varepsilon>0$ such that $\varepsilon>r$. Let us write

$$E:=\{k\leq n:\|T_k(f;x,y)-f(x,y)\|_{C_B(K)}\geq r\},$$
$$E_1:=\{k\leq n:\|T_k(f_0;x,t)-f_0(x,y)\|_{C_B(K)}\geq\frac{r}{3K}\},$$
$$E_2:=\{k\leq n:\omega^*(f;\delta_n)\geq\frac{r}{3K}\},$$
$$E_3:=\{k\leq n:\omega^*(f;\delta_n)\|T_k(f_0;x,t)-f_0(x,y)\|_{C_B(K)}\geq\frac{r}{3K}\}$$

Then $E\subset E_1\cup E_2\cup E_3$ and therefore $\delta(E)\leq\delta(E_1)+\delta(E_2)+\delta(E_3)$. Using conditions (19) and (20) we conclude

$$C_1(st)-\lim_{k\to\infty}\|T_k(f)-f\|_{C_B(K)}=0.$$

This completes the proof of the theorem.

Example and the concluding remark

We show that the following double sequence of positive linear operators satisfies the conditions of Theorem 2 but does not satisfy the conditions of Corollary 2 and Theorem 2 of (ERKUS; DUMAN, 2005).

Example 1: Consider the following Bleimann et al. (1980) (of two variables) operators:

$$B_n(f;x,y):=\frac{1}{(1+x)^n(1+y)^n}\sum_{j=0}^{n}\sum_{k=0}^{n}f(\frac{j}{n-j+1},\frac{k}{n-k+1})\binom{n}{j}\binom{n}{k}x^jy^k \tag{21}$$

where:
$f\in H_\omega(K)$, $K=[0,\infty)\times0,\infty)$ and $n\in\mathbb{N}$.

Since

$$(1+x)^n=\sum_{j=0}^{n}\binom{m}{j}x^j,$$

it is easy to see that

$$B_n(f_0;x,y)\to1=f_0(x,y).$$

Also by simple calculation, we obtain

$$B_n(f_1;x,y)=\frac{n}{n+1}(\frac{x}{1+x})\to\frac{x}{1+x}=f_1(x,y),\text{ and}$$
$$B_n(f_2;x,y)=\frac{n}{n+1}(\frac{y}{1+y})\to\frac{y}{1+y}=f_2(x,y).$$

Finally, we get

$$B_n(f_3;x,y)=\frac{n(n-1)}{(n+1)^2}(\frac{x}{1+x})^2+\frac{n}{(n+1)^2}(\frac{x}{1+x})$$
$$+\frac{n(n-1)}{(n+1)^2}(\frac{y}{1+y})^2$$

$$+\frac{n}{(n+1)^2}(\frac{y}{1+y})$$

$$\to (\frac{x}{1+x})^2 + (\frac{y}{1+y})^2 = f_3(x,y).$$

Now, take $A=C(1,1)$ and define $u=(u_n)$ by (4). Let the operator $L_n : H_\omega(K) \to C_B(K)$ be defined by

$$L_n(f;x,y) = (1+|u_n|)B_n(f;x,y).$$

It is easy to see that the sequence (L_n) satisfies the conditions (6), (7), (8) and (9). Hence by Theorem 2, we have

$$C_1(st)-\lim_{n\to\infty} \| L_n(f;x) - f(x) \|_{C_B(K)} = st-\lim_{m\to\infty}$$

$$\| \frac{1}{m}\sum_{n=1}^{m} L_n(f;x) - f(x) \|_{C_B(K)} = 0.$$

On the other hand, the sequence (L_n) does not satisfy the conditions of Corollary 2 and Theorem 2 of (EDELY et al., 2010), since (u_n) as well as (L_n) is neither convergent nor statistically (nor *A*–statistically) convergent. That is, Corollary 2 and Theorem 2 of (ERKUS; DUMAN, 2005) do not work for our operators L_n. Hence our Theorem 2 is stronger than Corollary 2 and Theorem 2 of (ERKUS; DUMAN, 2005).

Conclusion

Korovkin type approximation theorems have recently been proved for different types of summability methods, e.g. staitistcal convergence, A-statistical convergence, Statistical A-summability etc. In this paper, we have proved such approximation theorem for functions of two variables with the help of test functions $1, \frac{x}{1+x}, \frac{y}{1+y}, (\frac{x}{1+x})^2 + (\frac{y}{1+y})^2$ by using the notion of statistical summability (C, 1). Through an example, we have also justified that our result was stronger than those of previously proved for ordinary convergence and statistical convergence.

References

ALOTAIBI, A.; MURSALEEN, M. A-Statistical summability of fourier series and Walsh-Fourier series. **Applied Mathematics and Information Sciences**, v. 6, n. 3, p. 535-538, 2012.

ALOTAIBI, A.; MURSALEEN, M.; MOHIUDDINE, S. A. Statistical approximation for periodic functions of two variables. **Journal of Function Spaces and Applications**, v. 2013, p. 1-5, 2013. (Article ID 491768).

ANASTASSIOU, G. A.; MURSALEEN, M.; MOHIUDDINE, S. A. Some approximation theorems for functions of two variables through almost convergence of double sequences. **Journal of Computational Analysis and Applications**, v. 13, n. 1, p. 37-46, 2011.

BELEN, C.; MOHIUDDINE, S. A. Generalized weighted statistical convergence and application. **Applied Mathematics and Computation**, v. 219, n. 18, p. 9821-9826, 2013.

BELEN, C.; MURSALEEN, M.; YILDIRIM, M. Statistical A-summability of double sequences and a Korovkin type approximation theorem. **Bulletin of the Korean Mathematical Society**, v. 49, n. 4, p. 851-861, 2012.

BLEIMANN, G.; BUTZER, P. L.; HAHN, L. A Bernstein type operator approximating continuous functions on semiaxis. **Indagationes Mathematicae**, v. 42, n. 3, p. 255-262, 1980.

BRAHA, N. L.; SRIVASTAVA, H. M.; MOHIUDDINE, S. A. A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean. **Applied Mathematics and Computation**, v. 228, p. 162-169, 2014.

ÇAKAR, Ö.; GADJIEV, A. D. On uniform approximation by Bleimann, Butzer and Hahn on all positive semiaxis. **Transactions of Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences**, v. 19, n. 5, p. 21-26, 1999.

DEMIRCI, K.; KARAKUS, S. Korovkin-type approximation theorem for double sequences of positive linear operators via statistical A-summability. **Results in Mathematics**, v. 63, n. 1-2, p. 1-13, 2013.

DEMIRCI, K.; KARAKUS, S. Statistical A-summability of positive linear operators. **Mathematical and Computer Modelling**, v. 53, n. 1-2, p. 189-195, 2011.

DIRIK, F.; DEMIRCI, K. Korovkin type approximation theorem for functions of two variables in statistical sense. **Turkish Journal of Mathematics**, v. 34, n. 1, p. 73-83, 2010a.

DIRIK, F.; DEMIRCI, K. A Korovkin type approximation theorem for double sequences of positive linesr operators of two variables in A-statistical sense. **Bulletin of the Korean Mathematical Society**, v. 47, n. 4, p. 825-837, 2010b.

EDELY, O. H. H.; MOHIUDDINE, S. A.; NOMAN, A. K. Korovkin type approximation theorems obtained through generalized statistical convergence. **Applied Mathematics Letters**, v. 23, n. 11, p. 1382-1387, 2010.

EDELY, O. H. H.; MURSALEEN, M. On statistical A-summability. **Mathematical and Computer Modelling**, v. 49, n. 3-4, p. 672-680, 2009.

ERKUS, E.; DUMAN, O. A-Statistical extension of the Korovkin type approximation theorem. **Proceedings of**

the Indian Academy of Sciences (Mathematics Sciences), v. 115, n. 4, p. 499-508, 2005.

FAST, H. Sur la convergence statistique. **Colloquium Mathematicum**, v. 2, p. 241-244, 1951.

FRIDY, J. A. On statistical convergence. **Analysis**, v. 5, n. 4, p. 301-313, 1985.

KOROVKIN, P. P. **Linear operators and approximation theory**. Delhi: Hindustan Publishing Co., 1960.

MADDOX, I. J. **Elements of functional analysis**. Cambridge: University Press of Cambridge, 1970.

MOHIUDDINE, S. A. An application of almost convergence in approximation theorems. **Applied Mathematics Letters**, v. 24, n. 11, p. 1856-1860, 2011.

MOHIUDDINE, S. A.; AIYUB, M. Lacunary statistical convergence in random 2-normed spaces. **Applied Mathematics and Information Sciences**, v. 6, n. 3, p. 581-585, 2012.

MOHIUDDINE, S. A.; ALGHAMDI, M. A. Statistical summability through a lacunary sequence in locally solid Riesz spaces. **Journal of Inequalities and Applications**, v. 2012, p. 1-9, 2012. (Article 225).

MOHIUDDINE, S. A.; ALOTAIBI, A. Korovkin second theorem via statistical summability (C, 1). **Journal of Inequalities and Applications**, v. 2013, p. 1-9, 2013a. (Article 149).

MOHIUDDINE, S. A.; ALOTAIBI, A. Statistical convergence and approximation theorems for functions of two variables. **Journal of Computational Analysis and Applications**, v. 15, n. 2, p. 218-223, 2013b.

MOHIUDDINE, S. A.; ALOTAIBI, A.; ALSULAMI, S. M. Ideal convergence of double sequences in random 2-normed spaces. **Advances in Difference Equations**, v. 2012, p. 1-8, 2012a (Article 149).

MOHIUDDINE, S. A.; ALOTAIBI, A.; MURSALEEN, M. Statistical convergence of double sequences in locally solid Riesz spaces. **Abstract and Applied Analysis**, v. 2012, p. 1-9, 2012b. (Article ID 719729).

MOHIUDDINE, S. A.; ALOTAIBI, A.; MURSALEEN, M. Statistical summability (C, 1) and a Korovkin type approximation theorem. **Journal of Inequalities and Applications**, v. 2012, p. 1-8, 2012c. (Article 172).

MOHIUDDINE, S. A.; ŞEVLI, H.; CANCAN, M. Statistical convergence of double sequences in fuzzy normed spaces. **Filomat**, v. 26, n. 4, p. 673-681, 2012d.

MOHIUDDINE, S. A.; ALOTAIBI, A.; MURSALEEN, M. Statistical convergence through de la Vallée-Poussin mean in locally solid Riesz spaces. **Advances in Difference Equations**, v. 2013, p. 1-10, 2013a. (Article 66).

MOHIUDDINE, S. A.; HAZARIKA, B.; ALOTAIBI, A. Double lacunary density and some inclusion results in locally solid Riesz spaces. **Abstract and Applied Analysis**, v. 2013, p. 1-8, 2013b. (Article ID 507962).

MOHIUDDINE, S. A.; ŞEVLI, H.; CANCAN, M. Statistical convergence in fuzzy 2-normed space. **Journal of Computational Analysis and Applications**, v. 12, n. 4, p. 787-798, 2010.

MORICZ, F. Statistical convergence of multiple sequences. **Archiv der Mathematik**, v. 81, n. 1, p. 82-89, 2003.

MORICZ, F. Tauberian conditions under which statistical corvergence follows fron statistical summability (C, 1). **Journal of Mathematical Analysis and Applications**, v. 275, n. 1, p. 277-287, 2002.

MURSALEEN, M.; AHMAD, R. Korovkin type approximation theorem through statistical lacunary summability. **Iranian Journal of Science and Technology Transaction A**, v. 37, n. 2, p. 99-102, 2013.

MURSALEEN, M.; ALOTAIBI, A. Korovkin type approximation theorem for functions of two variables through statistical A-summability. **Advances in Difference Equations**, v. 2012, p. 1-10, 2012. (Article 65).

MURSALEEN, M.; ALOTAIBI, A. Korovkin type approximation theorem for statistical A-summability of double sequences. **Journal of Computational Analysis and Applications**, v. 15, n. 6, p. 1036-1045, 2013.

MURSALEEN, M.; ALOTAIBI, A. Statistical summability and approximation by de la Vallée-Poussin mean. **Applied Mathematics Letters**, v. 24, n. 3, p. 320-324, 2011.

MURSALEEN, M.; EDELY, O. H. H. On the invariant mean and statistical convergence. **Applied Mathematics Letters**, v. 22, n. 11, p. 1700-1704, 2009.

MURSALEEN, M.; EDELY, O. H. H. Statistical convergence of double sequences. **Journal of Mathematical Analysis and Applications**, v. 288, n. 1, p. 223-231, 2003.

MURSALEEN, M.; KARAKAYA, V.; ERTURK, M.; GURSOY, F. Weighted statistical convergence and its application to Korovkin type approximation theorem. **Applied Mathematics and Computation**, v. 218, n. 18, p. 9132-9137, 2012.

MURSALEEN, M.; KILIÇMAN, A. Korovkin second theorem via B-statistical A-summability. **Abstract and Applied Analysis**, v. 2013, p. 1-6, 2013. (Article ID 598963).

MURSALEEN, M.; MOHIUDDINE, S. A. Statistical convergence of double sequences in intuitionistic fuzzy normed spaces. **Chaos, Solitons and Fractals**, v. 41, n. 5, p. 2414-2421, 2009.

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