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On α- τ-disconnectedness and α- τ-connectedness in topological spaces

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ABSTRACT. The aims of this paper is to introduce new approach of separate sets, disconnected sets and connected sets called α- τ-separate sets, α- τ-disconnected sets and α- τ-connected sets of topological spaces with the help of α-open and α-closed sets. On the basis of new introduce approach, some relationship of α- τ-disconnected and α- τ-connected set with α- τ-separate sets have been investigated thoroughly.

Keywords: α-open set, α-closed set, α-closure, α- τ-separate sets, α- τ-disconnected sets and α- τ-connected sets.

Sobre a α- τ-des-conectividade e a α- τ-conectividade em espaços topológicos

RESUMO. Introduz-se nesse ensaio uma nova abordagem de conjuntos separados, conjuntos desconectados e conjuntos conectados chamados α- τ-conjuntos separados, α- τ-conjuntos desconectados e α- τ-conjuntos conectados de espaços topológicos, com o auxílio de conjuntos α- τ-aberto e α- τ-fechado. Baseados nessas abordagens, os relacionamentos de α- τ-conjunto desconectado e α- τ-conjunto conectado com α- τ-conjuntos separados foram analisados.

Palavras-chave: α-conjunto aberto, α-conjunto fechado, α-fechamento, α- τ-conjuntos separados, α- τ-conjuntos desconectados, α- τ-conjuntos conectados.

Introduction

There are several natural approaches that can take to rigorously the concepts of connectedness for a topological spaces. Two most common approaches are connected and path connected and these concepts are applicable Intermediate Mean Value Theorem and use to help distinguish topological spaces. These concepts play a significant role in application in geographic information system studied by Egenhofer and Franzosa (1991), topological modelling studied by Clementini et al. (1994) and motion planning in robotics studied by Farber et al. (2003). The generalization of open and closed set as like $\alpha-$open and $\alpha-$closed sets was introduced by Njastad (1965) which is nearly to open and closed set respectively. These notion are plays significant role in general topology. In this paper, the new approaches of separate sets, disconnected sets and connected sets called $\alpha-\tau-$separate set, $\alpha-\tau-$disconnected sets and $\alpha-\tau-$connected set with the help of $\alpha-$open and $\alpha-$closed set are firstly introduced. Further, some relationship concerning $\alpha-\tau-$disconnected and $\alpha-\tau-$connected sets with $\alpha-\tau-$separate sets are also investigated.

Throughout this paper (X,τ) and (X,τ_α) will always be topological spaces. For a subset A of topological space X, $Int(A), Cl(A), Cl_\alpha(A)$ and $Int_\alpha(A)$ denote the interior, closure, $\alpha-$closure and $\alpha-$interior of A respectively and G_α is the $\alpha-$open set for topology τ_α on X.

Preliminaries

We shall requires the following definitions and results.

Definition 2.1. Levine (1963), defined a subset A of (X,τ) is semi-open if $A\subset Cl(Int(A))$ and its complement is called semi-closed set.

The family $SO(X,\tau)$ of semi-open sets is not a topology on X.

Definition 2.2. Mashhour et al. (1982), defined a subset A of (X,τ) is called pre-open locally dense or nearly open if $A\subset Int(Cl(A))$ and its complement is called pre-closed set.

Theorem 2.3. According to Mashhour et al. (1982), the family $PO(X,\tau)$ of pre-open sets is not a topology on X.

Definition 2.4. Maheshwari and Jain (1982), defined a subset A of (X,τ) is called $\alpha-$open if

$A \subset Int(Cl(Int(A)))$. The family τ_α of $\alpha-$ open sets of (X,τ) is a topology on X which is finer than τ and the complement of an $\alpha-$ open set is called an $\alpha-$ closed set.

Theorem 2.5. According to Njastad (1965), for a subset *A* of (X,τ), thee following are equivalent:

$A \subset \tau_\alpha$;

A is semi open and pre-open;

$A \cap B \in SO(X,\tau)$, for all *B* in $So(X,\tau)$;

$A = O/N$, where $O \in \tau_\alpha$ and *N* is nowhere dense;

There exists *U* contained in τ_α such that $U \subset A \subset scl(U) = Int(Cl(A))$.

Theorem 2.6. The intersection of semi-open (resp. pre-open) set and an $\alpha-$open set is semi-open (resp. pre-open).

Theorem 2.7. According to Njastad (1965), $SO(X,\tau) = SO(X,\tau_\alpha)$ and $PO(X,\tau) = PO(X,\tau_\alpha)$.

Theorem 2.8. According to Njastad (1965), the $\alpha-$open sets *A* and *B* are disjoint if and only if $Int(Cl(Int(A)))$ and $Int(Cl(Int(B)))$ are disjoint.

Definition 2.9. A point *x* in *X* is called an $\alpha-$interior point of a set *A* in *X* if there exists $A \supset G_\alpha \in \tau_\alpha$ such that $y \in G_\alpha$ with $y \neq x$, i.e. *x* in *X* is called an $\alpha-$interior point of a set *A* in *X* if foe every $G_\alpha \in \tau_\alpha$ with $x \in G_\alpha$ and $G_\alpha \subset A$. Collection of all $\alpha-$interior points of A is called $\alpha-$interior of A which is denoted by $Int_\alpha(A)$. Alternatively, we can define as $Int_\alpha(A)$ by $Int_\alpha(A) = \bigcup\{G^\alpha \in \tau_\alpha : G_\alpha \subset A\}$.

Main Results

Definition 3.1. Let (X,τ) and (X,τ_α) be topological spaces. Then the subsets *A* and *B* of (X,τ) are said to be $\alpha-\tau-$separate sets if and only if

A and *B* are non-empty set.

$A \cap Cl_\alpha(B)$ and $B \cap Cl_\alpha(A)$ are non-empty.

Remark 3.2 If *A* and *B* are $\alpha-\tau-$separate sets, then both of them are also disjoint sets.

Definition 3.3. Let (X,τ) and (X,τ_α) be topological spaces. Then the subsets *A* of *X* is said $\alpha-\tau-$disconnected, if there exists G_α and H_α in τ_α such that

$A \cap G_\alpha$ and $A \cap H_\alpha \neq \phi$.

$(A \cap G_\alpha) \cap (A \cap H_\alpha) = \phi$.

$(A \cap G_\alpha) \cup (A \cap H_\alpha) = A$.

(X,τ_α) is said to be $\alpha-\tau-$disconnected if there exists non-empty G_α and H_α in τ_α such that $G_\alpha \cap H_\alpha \neq \phi$ and $G_\alpha \cup H_\alpha \neq X$.

Definition 3.4. Let (X,τ) be a topological space and *A* be non-empty subset of *X*. let G_α be arbitrary in τ_α, then collection $\tau_\alpha^A = \{G_\alpha \cap A : G_\alpha \in \tau_\alpha\}$ is a topology on *A*, called the subspace or relative topology of topology τ_α.

Theorem 3.5. If (X,τ) a disconnected space and (X,τ_α) is a topological space, then (X,τ_α) is $\alpha-\tau-$disconnected.

Proof. As (X,τ) disconnected and τ_α is finer than, hence by Theorem 3.1 (X,τ) is disconnected.

Theorem 3.6. Let (X,τ) and (X,τ_α) are spaces, then (X,τ) is $\alpha-\tau-$disconnected if and only if there exists non-empty proper subset of *X* which is both $\alpha-$open and $\alpha-$closed.

Proof. Necessity: Let (X,τ_α) be $\alpha-\tau$-disconnected. Then, by Definition 3.3 there exist non-empty sets G_α and H_α in τ_α such that $G_\alpha \bigcap H_\alpha$ is non-empty and $G_\alpha \bigcup H_\alpha = X$. Since $G_\alpha \bigcap H_\alpha = \phi$ and H_α is open in τ_α show that $G_\alpha = X - H_\alpha$, but it is α-closed. Hence G_α is non-empty proper subset of X which is α-closed as well α-open.

Sufficiency: Suppose A is non-empty proper subset of X such that it is α-open as well α-closed. Now A is nonempty α-closed show that $X-A$ is non-empty and α-open. Suppose $B = X - A$, then $A \bigcup B = X$ and $A \bigcap B = \phi$. Thus A and B are non-empty disjoint α-open as well as α-closed subset of X such that $A \bigcup B = X$. Consequently X is $\alpha-\tau$-disconnected.

Theorem 3.7. Every (X,τ_α) discrete space is $\alpha-\tau$-disconnected if the space contains more than one element.

Proof. Let (X,τ_α) be discrete space such that $X = \{a,b\}$ contains more than one element. But τ is discrete topology so $\tau = \{\phi, X, \{a\}, \{b\}\}$ and family of all α-open sets is $\tau_\alpha = \{\phi, X, \{a\}, \{b\}\}$.

All α -closed sets are $\phi, X, \{a\}, \{b\}$. Since $\{a\}$ is non-empty proper subset of X which is both α -open and α -closed in X. Finally, we can say that (X, τ_α) is $\alpha - \tau$-disconnected by Theorem 3.6.

Theorem 3.8. A topological space (X, τ_α) is $\alpha - \tau$ -connected if and only if one non-empty subset which is both α -open and α -closed in X is X itself.

Proof. Necessity: Assume that (X, τ_α) is $\alpha - \tau$ -connected topological space. So, our assumption show that (X, τ_α) is not α -disconnected i.e. there does not exist a pair of non-empty disjoin α -open and α -closed A and B such that, $A \bigcup B = X$. This shows that there exist non-empty subsets (other than X) which are both and α -open and α -closed in X .

Sufficiency: Suppose that (X, τ_α) is topological space such that the only non-empty subsets of X which is α -open as well as α -closed in X is X itself. By hypothesis, there does not exist a partition of the space X . Hence X is not $\alpha - \tau$ -disconnected i.e. $\alpha - \tau$ connected.

Theorem 3.9. Let A be a non-empty subset of topological space (X, τ). Let τ_α^A be the relative topology on A , then A is $\alpha - \tau$ -disconnected if and only if A is $\alpha - \tau_\alpha^A$ -disconnected.

Proof. Necessity: Let A be a $\alpha - \tau$ -disconnected and let $G_\alpha \bigcup H_\alpha$ be a $\alpha - \tau$ -disconnection on A. By Definition 3.3, there exists non-empty G_α and H_α in τ_α such that

$$A \bigcap G_\alpha, A \bigcap H_\alpha \neq \phi;$$
$$(A \bigcap G_\alpha) \bigcap (A \bigcap H_\alpha) = \phi;$$
$$(A \bigcap G_\alpha) \bigcup (A \bigcap H_\alpha) = A.$$

Now by the definition of relative topology, if G_α and H_α in τ_α, then there exist G_α^1 and H_α^1 in τ_α^A such that $G_\alpha^1 = A \bigcap G_\alpha$ and $H_\alpha^1 = A \bigcap H_\alpha$.

Now by (1) G_α^1 and H_α^1 are non-empty. Hence $A \bigcap G_\alpha^1$ and $A \bigcap H_\alpha^1$ are non-empty.

Similarly by (2) and (3), we can say that $(A \bigcap G_\alpha^1) \bigcap (A \bigcap H_\alpha^1) = \phi$ and $(A \bigcap G_\alpha^1) \bigcup (A \bigcap H_\alpha^1) = A$ respectively. Consequently A is $\alpha - \tau_\alpha^A$ -disconnected.

Sufficiency: Suppose that A is $\alpha - \tau_\alpha^A$ -disconnected and $M_\alpha^A \bigcap N_\alpha^A$ is a $\alpha - \tau_\alpha^A$ -disconnection on A . By definition, we can say that

$$M_\alpha^A, N_\alpha^A \neq \phi;$$
$$M_\alpha^A, N_\alpha^A \in \tau_\alpha^A;$$
$$(A \bigcap M\alpha^A) \bigcap (A \bigcap N_\alpha^A) = \phi;$$
$$(A \cap M_\alpha^A) \cup (A \cap N_\alpha^A) = A.$$

Now (2) $\Rightarrow$ there exists $M_\alpha^1, N_\alpha^1 \in \tau_\alpha$ such that $M_\alpha^A = A \cap M_\alpha^1, N_\alpha^A = A \cap N_\alpha^1.$ But by (1) we can say that $A \cap M_\alpha^1, A \cap N_\alpha^1 \neq \phi.$ Now $(A \cap M_\alpha^A) = A \cap (A \cap M_\alpha^1) = (A \cap A) \cap M_\alpha^1 = A \cap M_\alpha^1.$ Similarly we can say that $A \cap N_\alpha^A = A \cap N_\alpha^1.$

Now (3) $\Rightarrow (A \cap M_\alpha^1) \cap (A \cap N_\alpha^1) = \phi.$

Similarly (4) $\Rightarrow (A \cap M_\alpha^1) \cup (A \cap N_\alpha^1) = A$.

Finally, we can say that A is $\alpha - \tau -$ disconnected.

Theorem 3.10. The union of two non-empty $\alpha - \tau$ -separate subsets of topological space (X, τ_α) is $\alpha - \tau$ -disconnected.

Proof. Let A and B be $\alpha - \tau -$ separate subsets of (X, τ_α) . Then by definition 3.1, we can say that A and B non-empty.

$$A \cap Cl_\alpha(B), B \cap Cl_\alpha(A) = \phi;$$
$$A \cap B = \phi.$$

Let $X - Cl_\alpha(A) = G_\alpha$ and $X - Cl_\alpha(B) = H_\alpha$. Then $Cl_\alpha(A)$ and $Cl_\alpha(B)$ are non-empty α -closed subsets of X which shows that G_α and H_α are non-empty α -open subsets of X.

Since,

$$G_\alpha \cup H_\alpha = (X - Cl_\alpha(A)) \cup (X - Cl_\alpha(B)) = X - Cl_\alpha(A) \cap Cl_\alpha(B)$$

we have

$$(A \cup B) \cap G_\alpha = (A \cup B) \cap (X - Cl_\alpha(B))$$
$$= [A \cap (X - Cl_\alpha(B))] \cup [B \cap (X - Cl_\alpha(B))]$$
$$= \phi \cup B$$
$$(A \cup B) \cap G_\alpha = B.$$

Similarly, $(A \cup B) \cap H_\alpha = A.$ Now (1) shows that $(A \cup B) \cap G_\alpha, (A \cup B) \cap H_\alpha \neq \phi$. Additionally, (3) shows that

$[(A \cup B) \cap H_\alpha] \cap [(A \cup B) \cap G_\alpha] = \phi$ and

$[(A \cup B) \cap H_\alpha] \cup [(A \cup B) \cap G_\alpha] = A \cup B$

Finally, we can say that there exists G_α and H_α in τ_α such that

$(A \cup B) \cap G_\alpha, (A \cup B) \cap H_\alpha \neq \phi$

$[(A \cup B) \cap H_\alpha] \cap [(A \cup B) \cap G_\alpha] = \phi$ and

$[(A \cup B) \cap H_\alpha] \cup [(A \cup B) \cap G_\alpha] = A \cup B$.

So, $G_\alpha \cup H_\alpha$ is a $\alpha - \tau$-disconnection of $A \cup B$. Hence $A \cup B$ is $\alpha - \tau$-disconnected.

Theorem 3.11. Let (X, τ) and (X, τ_α) be topological spaces and A be a subset of X and let $G_\alpha \cup H_\alpha$ be a $\alpha - \tau$-disconnection of A. Then $A \cap G_\alpha$ and $A \cap H_\alpha$ are $\alpha - \tau$-separate subsets of topological space (X, τ_α).

Proof. Let $G_\alpha \cup H_\alpha$ be a given $\alpha - \tau$-disconnection of subset A of (X, τ_α). To prove $A \cap G_\alpha$ and $A \cap H_\alpha$ are $\alpha - \tau$-separate subsets, we must show that

$A \cap G_\alpha$ and $A \cap H_\alpha$ are non-empty;

$[Cl_\alpha(A \cap G_\alpha)] \cap (A \cap H_\alpha) = \phi$ and

$[Cl_\alpha(A \cap H_\alpha)] \cap (A \cap G_\alpha) = \phi$;

Now by our assumption and definition, we can say that there exist $G_\alpha, H_\alpha \in \tau_\alpha$ such that

$(A \cap G_\alpha)$ and $(A \cap H_\alpha)$ is non-empty;

$(A \cap G_\alpha) \cap (A \cap H_\alpha) = \phi$;

$(A \cap G_\alpha) \cup (A \cap H_\alpha) = A$.

Evidently, $(4) \Rightarrow (1)$.

To prove (2) suppose it is not possible i.e. $Cl_\alpha(A \cap G_\alpha) \cap (A \cap H_\alpha) \neq \phi$.

Then, there exists $x \in Cl_\alpha(A \cap G_\alpha) \cap (A \cap H_\alpha)$ which implies that $x \in Cl_\alpha(A \cap G_\alpha)$ and $x \in A, x \in H_\alpha$, that is $(A \cap G_\alpha) \cap H_\alpha \neq \phi$. Therefore $(A \cap G_\alpha) \cap (A \cap H_\alpha) \neq \phi$. But it is contrary to (5). Finally our assumption that is wrong.

Theorem 3.12. A subset Y of a topological space X is $\alpha - \tau$-disconnected if and only if it is union of two $\alpha - \tau$-separate sets.

Proof. Necessity: Suppose $Y = A \cup B$, where A and B are $\alpha - \tau$-separate sets of X.

By theorem 3.10, $A \cup B$ is $\alpha - \tau$-disconnected. Hence, Y is $\alpha - \tau$-disconnected.

Sufficiency: Let Y is $\alpha - \tau$-disconnected. To prove that there exists two $\alpha - \tau$-separate subsets of A, B in X such that $Y = A \cup B$. By assumption, Y is $\alpha - \tau$-disconnected show that there exists a $\alpha - \tau$-disconnection $G_\alpha \cup H_\alpha$ of Y. Therefore by Definition 3.3, we can say that there exists G_α H_α in τ_α such that

$Y \cap G_\alpha$ and $Y \cap H_\alpha$ are non-empty;

$(Y \cap G_\alpha) \cap (Y \cap H_\alpha) = \phi$;

$(Y \cap G_\alpha) \cup (Y \cap H_\alpha) = Y$.

Since $(Y \cap G_\alpha)$ and $(Y \cap H_\alpha)$ are separated sets, if we write $A = (Y \cap G_\alpha)$ and $B = (Y \cap H_\alpha)$, then by (3) $Y = A \cup B$. Finally, we can say that there exist two $\alpha - \tau$-separate sets A and B in X such that $Y = A \cup B$.

Theorem 3.13. If Y is an $\alpha - \tau$-connected subset of topological space X such that $Y \subset A \cup B$, where A and B is $\alpha - \tau$-connected, then $Y \subset A$ and $Y \subset B$.

Proof. Since the inclusion $Y \subset A \cup B$ holds by the hypothesis we have $(A \cup B) \cap Y = Y$ which yields that $Y = (Y \cap A) \cup (Y \cap B)$. Now we want to prove that $(Y \cap A), (Y \cap B) = \phi$. Suppose, $(Y \cap A), (Y \cap B) \neq \phi$. Now,

$(Y \cap A) \cap Cl_\alpha(Y \cap B) \subset (Y \cap A) \cap (Cl_\alpha(Y) \cap Cl_\alpha(B)) = (Y \cap Cl_\alpha(Y)) \cap (A \cap Cl_\alpha(B)) = (Y \cap (A \cap Cl_\alpha(B)) = Y \cap \phi = \phi$

i.e., $(Y \cap A) \cap Cl_\alpha(Y \cap B)) = \phi$.

Similarly, we can prove that $Cl_\alpha(Y \cap A) \cap (Y \cap B) = \phi$. Hence, from the above result we can say that Y is a union of two $\alpha - \tau$-separate sets $(Y \cap A)$ and $(Y \cap B)$. Consequently, Y is $\alpha - \tau$-disconnected. But this contradicts the fact that Y is $\alpha - \tau$-connected. Hence we can say that $(Y \cap A)$, $(Y \cap B) = \phi$. Now if

$(Y \cap A) = \phi$, then $Y = \phi \cup (Y \cap B) = (Y \cap B)$ which gives that $Y \subset B$. Similarly, we can prove that $Y \subset A$ if $(Y \cap A) = \phi$.

Conclusion

We have introduced new approach of separate sets, disconnected sets and connected sets called $\alpha - \tau -$separate sets, $\alpha - \tau -$disconnected sets and $\alpha - \tau -$connected sets of topological spaces with the help of $\alpha -$ open and $\alpha -$ closed sets and investigated their properties. The results of this paper will help to study various weak and strong form of connectedness and disconnectedness in topological spaces and it can be also applied for the study of topology in robotics, topological modeling and geographical information system.

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