



New characterization of $b-m_2$ developable surfaces

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ABSTRACT. In this paper, $b-m_2$ developable surfaces of biharmonic b -slant helices in the special three-dimensional ϕ -Ricci symmetric para-Sasakian manifold P is studied. Explicit parametric equations of $b-m_2$ developable surfaces of biharmonic b -slant helices in the special three-dimensional ϕ -Ricci symmetric para-Sasakian manifold P are characterized.

Keywords: bienergy, biharmonic curve, para-Sasakian manifold, Bishop frame, developable surface.

Novas caracterizações das superfícies desenvolvimentáveis de $b - m_2$

RESUMO. Analisam-se as superfícies desenvolvimentáveis $b - m_2$ de hélices bi-harmonicas com inclinação b em ϕ -tri-dimensional especial de múltiplo P para-Sasakiano simétrico. São caracterizados as equações paramétricas das superfícies desenvolvimentáveis $b - m_2$ de hélices bi-harmonicas com inclinação b em ϕ -tri-dimensional especial de múltiplo P para-Sasakiano simétrico.

Palavras-chave: bi-energia; curva bi-harmônica; múltiplo para-Sasakiano; superfície desenvolvimentável; esquema de Bishop

Introduction

In differential geometry (CADDEO; MONTALDO, 2001), (DIMITRIC, 1992), (LOUBEAU; ONICIUC, 2007), (O'NEILL, 1983) that under the assumption of sufficient differentiability, a developable surface is either a plane, conical surface, cylindrical surface or tangent surface of a curve or a composition of these types. Thus a developable surface is a ruled surface, where all points of the same generator line share a common tangent plane. The rulings are principal curvature lines with vanishing normal curvature and the Gaussian curvature vanishes at all surface points. Therefore developable surfaces are also called single-curved surfaces, as opposed to double-curved surfaces (CARMO, 1976).

On the other hand, a smooth map $\phi: N \rightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |T(\phi)|^2 dv_h,$$

where:

$T(\phi) := \text{tr} \nabla^\phi d\phi$ is the tension field of ϕ .

The Euler-Lagrange equation of the bienergy is given by $T_2(\phi) = 0$. Here the section $T_2(\phi)$ is defined

by $T_2(\phi) = -\Delta_\phi T(\phi) + \text{tr} R(T(\phi), d\phi)d\phi$, and called the bitension field of ϕ (ARSLAN et al., 2005), (EELLS; LEMAIRE, 1978), (EELLS; SAMPSON, 1964). Non-harmonic biharmonic maps are called proper biharmonic maps.

New methods for constructing a canal surface surrounding a biharmonic curve in the Lorentzian Heisenberg group Heis^3 were given, (KORPINAR; TURHAN, 2011; 2012), (TURHAN; KORPINAR, 2011). Also, in (KORPINAR; TURHAN, 2010), (TURHAN; KORPINAR, 2010) they characterized biharmonic curves in terms of their curvature and torsion. Also, by using timelike biharmonic curves, they give explicit parametrizations of canal surfaces in the Lorentzian Heisenberg group Heis^3 .

In this paper, we study $b-m_2$ developable surfaces of biharmonic b -slant helices in the special three-dimensional ϕ -Ricci symmetric para-Sasakian manifold P . We characterize the biharmonic curves in terms of their curvature and torsion in the special three-dimensional ϕ -Ricci symmetric para-Sasakian manifold P . Finally, we find out explicit parametric equations of $b-m_2$ developable surfaces of biharmonic b -slant helices in the special three-dimensional ϕ -Ricci symmetric para-Sasakian manifold P . Additionally, we illustrate our results in Figure 1, 2 and 3.

Preliminaries

An n -dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ, ξ, η, g) , where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi = 0, \eta(\xi) = 1, g(X, \xi) = \eta(X), \quad (1)$$

$$\phi^2(X) = X - \eta(X)\xi, \quad (2)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (3)$$

for any vector fields X, Y on M (BLAIR, 1975).

In addition, if (ϕ, ξ, η, g) , satisfy the equations

$$d\eta = 0, \nabla_X \xi = \phi X, \quad (4)$$

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, X, Y \in \chi(M) \quad (5)$$

then M is called a para-Sasakian manifold or, briefly a P -Sasakian manifold.

Definition 1: A para-Sasakian manifold M is said to be locally ϕ -symmetric if

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

where:

for all vector fields X, Y, Z, W orthogonal to ξ .

This notion was introduced by (SATO, 1976) for a Sasakian manifold.

Definition 2: A para-Sasakian manifold M is said to be ϕ -symmetric if

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

where:

for all vector fields X, Y, Z, W on M (BLAIR, 1976).

Definition 3: A para-Sasakian manifold M is said to be ϕ -Ricci symmetric if the Ricci operator satisfies

$$\phi^2((\nabla_X Q)(Y)) = 0,$$

where:

for all vector fields X and Y on M and $S(X, Y) = g(QX, Y)$, (TAKASHI, 1977).

If X, Y are orthogonal to ξ , then the manifold is said to be locally ϕ -Ricci symmetric, (SATO, 1976).

The three-dimensional manifold

$$P = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$$

where:

(x, y, z) are the standard coordinates in \mathbb{R}^3 . We choose the vector fields

$$e_1 = e^x \frac{\partial}{\partial y}, e_2 = e^x \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right), e_3 = -\frac{\partial}{\partial x} \quad (6)$$

are linearly independent at each point of P .

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_3) \text{ for any } Z \in \chi(P).$$

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = e_2, \phi(e_2) = e_1, \phi(e_3) = 0.$$

Then using the linearity of and g we have

$$\eta(e_3) = 1,$$

$$\phi^2(Z) = Z - \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

where:

for any $Z, W \in \chi(P)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost para-contact metric structure on P .

Let ∇ be the Levi-Civita connection with respect to g . Then, we have

$$[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = e_2.$$

Taking $e_3 = \xi$, and using the Koszul's formula, we obtain

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = -e_3, \nabla_{e_2} e_3 = e_2, \quad (7)$$

$$\nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = 0.$$

Moreover we put

$$R_{ijk} = R(e_i, e_j)e_k, R_{ijkl} = R(e_i, e_j, e_k, e_l),$$

where:

the indices i, j, k and l take the values 1, 2 and 3.

$$R_{122} = -e_1, R_{133} = -e_1, R_{233} = -e_2,$$

and

$$R_{1212} = R_{1313} = R_{2323} = 1. \quad (8)$$

Biharmonic b-slant helices in the special three-dimensional ϕ -ricci symmetric para-sasakian manifold P

Let us consider biharmonicity of curves in the special three-dimensional ϕ -Ricci Symmetric para-Sasakian manifold P. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame field along γ . Then, the Frenet frame satisfies the following Frenet--Serret equations:

$$\begin{aligned} \nabla_t \mathbf{t} &= \kappa \mathbf{n}, \\ \nabla_t \mathbf{n} &= -\kappa \mathbf{t} + \tau \mathbf{b}, \\ \nabla_t \mathbf{b} &= -\tau \mathbf{n}, \end{aligned} \quad (9)$$

where:

$\kappa = |\mathbf{T}(\gamma)| = |\nabla_t \mathbf{t}|$ is the curvature of γ and τ its torsion and

$$\begin{aligned} g(\mathbf{t}, \mathbf{t}) &= 1, g(\mathbf{n}, \mathbf{n}) = 1, g(\mathbf{b}, \mathbf{b}) = 1, \\ g(\mathbf{t}, \mathbf{n}) &= g(\mathbf{t}, \mathbf{b}) = g(\mathbf{n}, \mathbf{b}) = 0. \end{aligned}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative, (BISHOP, 1975). The Bishop frame is expressed as

$$\begin{aligned} \nabla_t \mathbf{t} &= k_1 \mathbf{m}_1 + k_2 \mathbf{m}_2, \\ \nabla_t \mathbf{m}_1 &= -k_1 \mathbf{t}, \\ \nabla_t \mathbf{m}_2 &= -k_2 \mathbf{t}, \end{aligned} \quad (10)$$

where:

$$\begin{aligned} g(\mathbf{t}, \mathbf{t}) &= 1, g(\mathbf{m}_1, \mathbf{m}_1) = 1, g(\mathbf{m}_2, \mathbf{m}_2) = 1, \\ g(\mathbf{t}, \mathbf{m}_1) &= g(\mathbf{t}, \mathbf{m}_2) = g(\mathbf{m}_1, \mathbf{m}_2) = 0. \end{aligned}$$

Here, we shall call the set $\{\mathbf{t}, \mathbf{m}_1, \mathbf{m}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\zeta(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \zeta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s) \cos \zeta(s), \\ k_2 &= \kappa(s) \sin \zeta(s). \end{aligned}$$

The relation matrix may be expressed as

$$\begin{aligned} \mathbf{t} &= \mathbf{t}, \\ \mathbf{n} &= \cos \zeta(s) \mathbf{m}_1 + \sin \zeta(s) \mathbf{m}_2, \\ \mathbf{b} &= -\sin \zeta(s) \mathbf{m}_1 + \cos \zeta(s) \mathbf{m}_2. \end{aligned}$$

On the other hand, using above equation we have

$$\begin{aligned} \mathbf{t} &= \mathbf{t}, \\ \mathbf{m}_1 &= \cos \zeta(s) \mathbf{n} - \sin \zeta(s) \mathbf{b}, \\ \mathbf{m}_2 &= \sin \zeta(s) \mathbf{n} + \cos \zeta(s) \mathbf{b}. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t^1 \mathbf{e}_1 + t^2 \mathbf{e}_2 + t^3 \mathbf{e}_3, \\ \mathbf{m}_1 &= m_1^1 \mathbf{e}_1 + m_1^2 \mathbf{e}_2 + m_1^3 \mathbf{e}_3, \end{aligned} \quad (11)$$

$$\mathbf{m}_2 = m_2^1 \mathbf{e}_1 + m_2^2 \mathbf{e}_2 + m_2^3 \mathbf{e}_3.$$

Theorem 1: $\gamma: I \rightarrow \mathbf{P}$ is a biharmonic curve with Bishop frame if and only if

$$k_1^2 + k_2^2 = \text{constant} \neq 0,$$

Definition 4: A regular curve $\gamma: I \rightarrow \mathbf{P}$ is called a slant helix provided the unit vector \mathbf{m}_1 of the curve γ has constant angle E with unit vector field u along γ , that is

$$g(\mathbf{m}_1(s), u) = \cos E \text{ for all } s \in I. \quad (12)$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on Bishop curvatures.

To separate a slant helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as b-slant helix.

Lemma 1: Let $\gamma: I \rightarrow \mathbf{P}$ be a unit speed non-geodesic biharmonic b-slant helix. Then, the parametric equations of γ are

$$x(s) = \sin E s + A_1,$$

$$y(s) = \frac{e^{\sin E s + A_1}}{A_0^2 + \sin^2 E} (\sin E - A_0) \cos E \cos[A_0 s + A] \quad (13)$$

$$+ \frac{e^{\sin E s + A_1}}{A_0^2 + \sin^2 E} (\sin E + A_0) \cos E \sin[A_0 s + A] + A_2,$$

$$z(s) = \frac{e^{\sin E s + A_1}}{A_0^2 + \sin^2 E} A_0 \cos E \cos[A_0 s + A] - \frac{e^{\sin E s + A_1}}{A_0^2 + \sin^2 E} \sin E \cos E \sin[A_0 s + A] + A_3,$$

where:

A, A_1, A_2, A_3 are constants of integration and

$$A_0 = \left(\frac{k_1^2 + k_2^2 - \cos^2 E}{\cos^2 E} \right)^{\frac{1}{2}}.$$

Now, we illustrate our result in Figure 1.

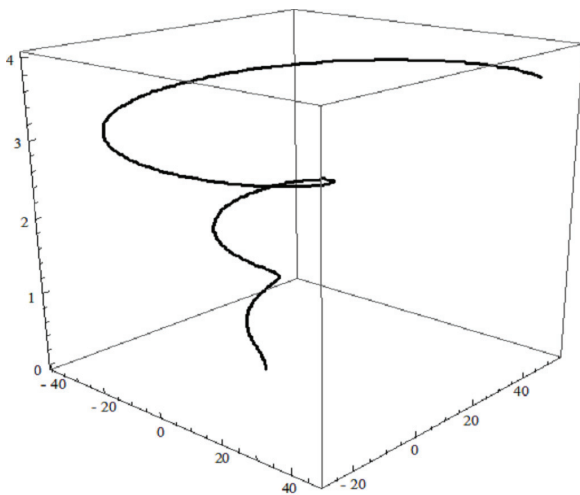


Figure 1. For $A = A_1 = A_2 = A_3 = 0$.

b- \mathbf{m}_2 developable surfaces of biharmonic b-slant helices in the special three-dimensional ϕ -ricci symmetric para-sasakian manifold P

The $b\text{-}\mathbf{m}_2$ developable of γ is a ruled surface

$$R(s, u) = \gamma(s) + u \mathbf{m}_2(s). \quad (14)$$

Firstly, we need following Lemma.

Lemma 2: Let $\gamma: I \rightarrow P$ be a unit speed non-geodesic biharmonic b-slant helix. Then, the parametric equations of γ are

$$\begin{aligned} \gamma(s) = & \left[\frac{1}{A_0^2 + \sin^2 E} (\sin E - A_0) \cos E \cos[A_0 s + A] \right. \\ & + \frac{1}{A_0^2 + \sin^2 E} (\sin E + A_0) \cos E \sin[A_0 s + A] \\ & + e^{-\sin E s - A_1} A_2 \\ & + \frac{1}{A_0^2 + \sin^2 E} A_0 \cos E \cos[A_0 s + A] \\ & - \frac{1}{A_0^2 + \sin^2 E} \sin E \cos E \sin[A_0 s + A] + e^{-\sin E s - A_1} A_3 \mathbf{e}_1 \\ & - \left[\frac{1}{A_0^2 + \sin^2 E} A_0 \cos E \cos[A_0 s + A] \right. \\ & - \frac{1}{A_0^2 + \sin^2 E} \sin E \cos E \sin[A_0 s + A] + e^{-\sin E s - A_1} A_3 \mathbf{e}_2 \\ & \left. \left. - [\sin E s + A_1] \mathbf{e}_3, \right. \right. \end{aligned} \quad (15)$$

where:

A, A_1, A_2, A_3 are constants of integration and

$$A_0 = \left(\frac{k_1^2 + k_2^2 - \cos^2 E}{\cos^2 E} \right)^{\frac{1}{2}}.$$

Proof. Substituting (4) to (13), we have (15). Thus, the proof is completed.

Theorem 2. Let R be $b\text{-}\mathbf{m}_2$ developable of a unit speed non-geodesic biharmonic b-slant helix in P . Then, the vector equation of $b\text{-}\mathbf{m}_2$ developable is given by

$$\begin{aligned} R(s, u) = & \left[\frac{1}{A_0^2 + \sin^2 E} (\sin E - A_0) \cos E \cos[A_0 s + A] \right. \\ & + \frac{1}{A_0^2 + \sin^2 E} (\sin E + A_0) \cos E \sin[A_0 s + A] + e^{-\sin E s - A_1} A_2 \\ & + \frac{1}{A_0^2 + \sin^2 E} A_0 \cos E \cos[A_0 s + A] + u \sin[A_0 s + A] \\ & - \frac{1}{A_0^2 + \sin^2 E} \sin E \cos E \sin[A_0 s + A] + e^{-\sin E s - A_1} A_3 \mathbf{e}_1 \\ & - \left[\frac{1}{A_0^2 + \sin^2 E} A_0 \cos E \cos[A_0 s + A] - u \cos[A_0 s + A] \right. \\ & - \frac{1}{A_0^2 + \sin^2 E} \sin E \cos E \sin[A_0 s + A] + e^{-\sin E s - A_1} A_3 \mathbf{e}_2 \\ & \left. \left. - [\sin E s + A_1] \mathbf{e}_3, \right. \right. \end{aligned} \quad (16)$$

where:

A, A_1, A_2, A_3 are constants of integration and

$$A_0 = \left(\frac{k_1^2 + k_2^2 - \cos^2 E}{\cos^2 E} \right)^{\frac{1}{2}}.$$

Proof. By the Bishop (1975) formula, we have the following equation

$$\mathbf{m}_2 = \sin[A_0s + A]\mathbf{e}_1 - \cos[A_0s + A]\mathbf{e}_2, \quad (17)$$

where:

A is constants of integration and

$$A_0 = \left(\frac{k_1^2 + k_2^2 - \cos^2 E}{\cos^2 E} \right)^{\frac{1}{2}}.$$

Consequently, the equation of R can be found from (14), (17). This concludes the proof of Theorem.

We can prove the following interesting main result.

Theorem 3. Let R be b-m₂ developable of a unit speed non-geodesic biharmonic b-slant helix in P. Then the parametric equations of b-m₂ developable are given by

$$\begin{aligned} x_R(s, u) &= \sin Es + A_1, \\ y_R(s, u) &= e^{\sin Es + A_1} \left[\frac{1}{A_0^2 + \sin^2 E} (\sin E - A_0) \cos E \cos[A_0s + A] \right. \\ &\quad + \frac{1}{A_0^2 + \sin^2 E} (\sin E + A_0) \cos E \sin[A_0s + A] + e^{-\sin Es - A_1} A_2 \\ &\quad + \frac{1}{A_0^2 + \sin^2 E} A_0 \cos E \cos[A_0s + A] + u \sin[A_0s + A] \\ &\quad \left. - \frac{1}{A_0^2 + \sin^2 E} \sin E \cos E \sin[A_0s + A] + e^{-\sin Es - A_1} A_3 \right] \quad (18) \\ z_R(s, u) &= e^{\sin Es + A_1} \left[\frac{1}{A_0^2 + \sin^2 E} A_0 \cos E \cos[A_0s + A] - u \cos[A_0s + A] \right. \\ &\quad - \frac{1}{A_0^2 + \sin^2 E} \sin E \cos E \sin[A_0s + A] + e^{-\sin Es - A_1} A_3, \\ &\quad \left. - \frac{1}{A_0^2 + \sin^2 E} \sin E \cos E \sin[A_0s + A] + e^{-\sin Es - A_1} A_3 \right], \end{aligned}$$

where:

A, A₁, A₂, A₃ are constants of integration.

Proof. We assume that γ is a unit speed b-slant helix.

Substituting (4) to (16), we have (18). Thus, the proof is completed.

Thus, we proved the following:

Corollary 1: Let R be b-m₂ developable of a unit speed non-geodesic biharmonic b-slant helix in P. Then, normal of b-m₂ developable of γ is

$$\mathbf{n}_R = \sin E \cos[\tan Es + A]\mathbf{e}_1 + \sin E \sin[\tan Es + A]\mathbf{e}_2 + \cos E \mathbf{e}_3,$$

where:

A is constants of integration.

Proof. It is obvious from Theorem 15 Hence the proof is completed.

Finally, the obtained parametric equations are illustrated in Figure 2 and 3:

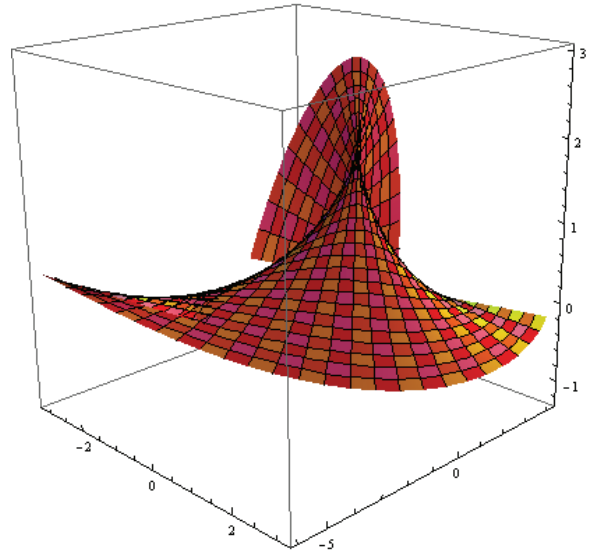


Figure 2. For $A = -A_1 = A_2 = A_3 = 1$.

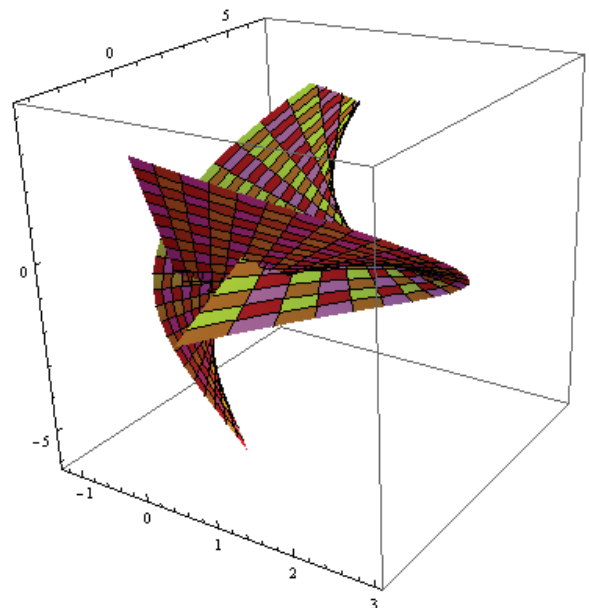


Figure 3. For $-A = A_1 = -A_2 = A_3 = -1$.

Conclusion

In the last decade there has been a growing interest in the theory of biharmonic maps which can

be divided in two main research directions. On the one side, constructing the examples and classification results have become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations.

In this paper, $b\text{-}\mathbf{m}_2$ developable surfaces of biharmonic b -slant helices in the special three-dimensional ϕ -Ricci symmetric para-Sasakian manifold P is studied. Explicit parametric equations of $b\text{-}\mathbf{m}_2$ developable surfaces of biharmonic b -slant helices in the special three-dimensional ϕ -Ricci symmetric para-Sasakian manifold P are characterized.

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