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# An improvement for an algorithm for finding a minimum feedback arc set for planar graphs

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**ABSTRACT.** Given a directed graph $G$, a *covering* is a subset $B$ of arcs which meets all directed cuts of $G$. Equivalently, the contraction of the elements of $B$ makes $G$ strongly connected. An $O(n^5)$ primal-dual algorithm is presented by Frank (1981) for finding a minimum covering of a directed graph. For a planar graph, the dual problem is to find a minimum set of arcs whose removal makes $G$ acyclic. The dual problem may be solved with Frank's algorithm. Further, some improvements that may be used to make such algorithm faster in practical cases are prescuted.

**Key words:** feedback arc set, kernel, covering directed cuts.

**RESUMO. Um algorítmo melhorado para encontrar FAS para grafos planares.** Dado um grafo orientado $G$, uma cobertura é um subconjunto $B$ de arestas que interceptam todos os cortes de $G$. De maneira equivalente, a contração das arestas de $B$ tornam o grafo $G$ fortemente conexo. Um algoritmo primal-dual de complexidade $O(n^5)$ é apresentado por Frank (1981), este algoritmo encontra uma cobertura mínima do grafo orientado. No caso de um grafo planar, o problema dual será encontrar um conjunto mínimo de arestas cuja remoção torna $G$ acíclico. Neste trabalho será mostrado como utilizar o algoritmo de Frank para resolver o problema dual. Será também apresentado uma melhoria que torna o algoritmo de Frank mais eficiente em casos práticos.

**Palavras-chave:** FAZ, núcleos, cortes de coberturas orientadas.

**History.** A problem in the topology of directed graphs (digraphs) that has attracted some interest is the following: to determine, for an arbitrary digraph, a minimum set of arcs which, if removed, leaves the resultant digraph free of directed loops (dicircuits). The problem was originally suggested by Runyon[1], who observed that the analysis of sequential switching circuits with feedback paths would be simplified by the knowledge of such a set. Increased interest in this problem was in a large measure due to Moore, who had encouraged attempts to find a solution. It was at first expected that a simple and efficient algorithm might be found, perhaps even an algorithm such that the number of operations required would increase linearly with the number of nodes in the graph. However, the problem turned out to be more difficult and suggested algorithms fell far short of that goal.

This problem lies within in a series of unsolved problems called NP-complete as proved by Karp (1972). Previous investigations into this problem have produced algorithms which, in general, yield a sub optimum solution (Younger, 1963; Diaz *et al.* 1972) or require severe restrictions on the source graph (Ramamoorthy, 1967), or are practical only for small graphs, as circuits (Lempel *et al.,* 1966; Divieti and Granselli, 1968; Guardabassi, 1971). Finally it was solved for planar directed graphs (Lucchesi and Younger, 1967; Frank, 1981; Karzanov, 1979) and for reducible flow graphs (Ramachandran, 1988). Some heuristics were developed to find some sub optimum solutions (Eades *et al.* 1989, 1993, 1995). Another variant of the problem is the Feedback Vertex Set which has also several polynomially solvable cases including: the cyclically reducible graphs (Wang *et al.,*

[1] Seshu and Reed in the book of Ford and Fulkerson (1962) included this among a list of research problems given in the appendix.

1987), a variant[2] of Smith and Walford (1975), the chordal graphs (Corneil and Fonlupt, 1988; Yannakakis and Gavril 1987), the interval graphs (Marathe, Rangan and Ravi, 1992; Lu and Tang, 1997), the permutation graphs (Brandstädt and Kratsch, 1985; Brandstädt, 1993; Liang 1994). For a survey on the Feedback Arc (Vertex) Set Problem the reader may refer to Festa, Pardalos and Resende (1999). We point out that the authors in this survey are not aware of the algorithm of Frank (1981) improved here.

## The problem

**The feedback arc set (FAS).** Given a directed graph $D=(V,E)$, find a minimum cardinality subset $E' \subseteq E$ such that $E'$ contains at least one arc from every directed cycle in $D$.

## Finding a FAS for a planar praph

**Basic definitions.**

Let $G = (V,E)$ be a *directed graph*, or *digraph*, with the vertex set $V$ and arc set $E$. The arcs are ordered pairs $(u,v)$ where $u$ is the *tail* and $v$ is the *head*.

A digraph is said to be *planar* if it can be drawn in a plane without crossings or overlappings. In other words the digraph *admits* a *planar drawing*. Note that a graph may admit several planar drawings. A face in a planar drawing is a continuous subset of the surface surrounded by a subset of arcs of $G$.

A *potential* $p$ is an integer value function on $V$.

A *path* $P$ is a subset $\{v_1, (v_1,v_2),v_2,...,v_{k-1}, (v_{k-1},v_k),v_k\}$ of $G$ where $v_l \neq v_j$ if $l>j$ and $(v_j,v_{j+1})$ belongs to $E$.

A *directed circuit*, or *dicircuit*, is a path $P$ from $v_1$ to $v_k$ in $G$ plus the arc $(v_k,v_1)$ (note that the arc $(v_k,v_1)$ must be in $E$).

A set $X$ is said to be a *kernel* if there is no arc leaving $X$. The non-empty set $D(X)$ of arcs entering $X$ is called *directed cut*, or *dicut*. It is very easy to show that a digraph $G$ is not strongly connected if and only if there are no kernels in $G$ but $V$.

We say that a digraph $D$ is a *dual* of a digraph $G$ if (i) $G$ admits a planar drawing such that each vertex of $G$ is associated to a face in $D$ and vice-versa. (ii) for each arc $e$ in $G$ there is an arc $e'$ in $D$ such that $e'$ has its tail in the face (vertex of $D$) at right of $e$ and head in the face at left of $e$. In our convention we assume that the associated set of arcs of a clockwise dicircuit $C$ in the planar drawing of $G$ is a dicut $D(X)$ in $D$ where $X$ is the set of vertices associated with the inside faces of the region bounded by the dicircuit $C$. Exceptionally, an arc $e$ of $G$ may not separate two faces in the planar drawing, namely, when $e$ is the only incident arc with a set of vertices $X$. In this case the corresponding arc $e'$ in $D$ forms a self-loop on the vertex associated with the outside face of the set $X$. Another problem exists when more than one arc connects the same pair of vertices. It is very easy to handle these problems by deleting the self-loops from $D$ (these arcs will never be in a covering) and all multiple arcs connecting the same oriented pair of arcs (only one of these multiple arcs can be a minimum covering).

It is an easy exercise to show that circuits in a planar drawing of a digraph are related to dicuts in the corresponding dual digraph. For more detailed description of dual graphs and its properties see Gibbons (1980).

A kernel $X$ is said to be *strict* with respect to a subset $B \subseteq E$ if for each arc of $B$ entering $X$ there is one weak component in $V\backslash X$, i.e. this arc leaves a component of $V\backslash X$ and enters $X$. This kernel is said to be *1-strict* if it is strict with only one arc of $B$ entering $X$. Let $R(x)$ be the intersection of all strict kernels containing a fixed vertex $x$. In section on improvements it is shown how $R(x)$ can be efficiently calculated.

A *covering* is a subset $B$ of arcs which meets all dicuts of $G$. We call the arc set $B$ blue arcs and the arc set $V\backslash B$ white arcs. Obviously, $B$ is a covering if and only if by contracting its elements $G$ becomes strongly connected.

If a digraph $G$ is planar then we can find its minimum FAS by finding the associated arc set of a minimum covering for its dual.

Let $B$ be a minimum covering of a digraph $D$ which is a dual digraph of a digraph G. Let $F$ be the set of the arcs of $G$ related to the arcs in $B$. The set of arcs $F$ forms a minimum FAS in $G$. Therefore, the subgraph $G\backslash F$ is acyclic, conversely the contraction of the edges in $B$ from $D$ yields a planar digraph $D'$. It is an easy exercise to show that $G\backslash F$ is a dual digraph of $D'$. Thus, we will concentrate our attention on the dual problem which is to find a minimum cost covering for a digraph.

In our work we depend heavily on the following characterization by Lucchesi and Younger as follows.

**Theorem 1.** [Lucchesi-Younger (1978)]. *The minimum cardinality of a covering is equal to the maximum number of (arc) disjoint dicuts.*

In other words, we only need find a covering $B$ and a family $\mathcal{K}$ of kernels such that:

- Every blue arc is in exactly one dicut $D(X)$, $X$ in $\mathcal{K}$.
- Every white arc is in at most one dicut $D(X)$, $X$ in $\mathcal{K}$.

[2] The exact algorithm is exponential, but polynomial in some cases.

- There is exactly one blue arc entering $X$ for each $X$ in $\mathcal{K}$.

In the next section we will see how to produce the family $\mathcal{K}$ by potentials.

**Covering and potentials.**

A *potential p* is an integer value function on $V$.

For each arc $(x,y)$ of $E$ let a *differential* be a function $d(x,y)= 1-p(y)+p(x)$. It was shown by Frank in 1981 that the family $\mathcal{K}$ and the covering $B$ can be easily produced if you find a covering $B$ and a potential $p$ for which the following optimality criteria hold:

a) For every blue arc $(x,y)$, $d(x,y)\leq 0$.
b) For every white arc $(x,y)$, $d(x,y)\geq 0$.
c) For every $y$ in $R(x)$, $p(y)\geq p(x)$.

The core of Frank's algorithm is the following.

**Frank's algorithm.**

**Input:** A covering B, a potential p, and a blue arc (a,b) such that (b) and (c) of the optimality criteria hold but (a,b) violates (a).

**Output:** A covering $B'$ and a potential $p'$ such that (b) and (c) of the optimality criteria hold again, $(a,b)$ does not violate (a) and every arc which violates (a) with respect to $B'$ and $p'$ also violates (a) with respect to $B$ and $p$.

Given that such an algorithm is available and repeating it successively until there is no blue arc violating (a), at the beginning the covering $B$ may be the arc set of a spanning tree and $p\equiv 0$. Then after no more than $|B|\leq n-1$ applications of this algorithm where $n=|V|$, we will get a covering $B$ and a potential $p$ which satisfy all three optimality criteria.

**Improving a covering-potential pair.**

To describe the core procedure properly, we define an auxiliary graph $H=(V,A)$ as follows. Let $A$ be the following three - not necessarily disjoint - parts $A_B$, $A_W$ and $A_R$. You may note that $H$ depends on $G$, $B$ and $p$, and $H$ may contain multiple arcs.

$$A=\begin{cases} A_B=\{(x,y):(x,y)\in B, d(x,y)\geq 0\} \\ A_W=\{(y,x):(y,x)\in E\setminus B, d(x,y)\leq 0\} \\ A_R=\{(x,y): y\in R(x), p(y)=p(x)\} \end{cases}$$

We will try to find a *ba*-path from $b$ to $a$ in $H$ for an arc $(a,b)$ which does not satisfy (a) of the optimality criteria. We have two cases.

**Case 1. Improving the potential.** There is no path from $b$ to $a$ in $H$.

In this case let $T$ be the vertex set $\{x{:}x$ can be reached from $b$ in $H\}$. If $a$ is not in $T$ (there is not such path change $p$ as follows.

$$p'(x)=\begin{cases} p(x) & if \quad x\notin T \\ p(x)+\delta & if \quad x\in T \end{cases}$$

Where

$$\delta=\min\begin{cases} \delta_e=d(a,b) \\ \delta_B=\min\{-d(x,y):(x,y)\in B,(x,y)\, leaves \quad T\} \\ \delta_W=\min\{d(x,y):((x,y)\in E\setminus B,(x,y)\, enters \quad T\} \\ \delta_R \min\{p(y)-p(x): x\in T, y\in R(x)\setminus T\} \end{cases}$$

From the definition of $\delta$ we have for the new $d'(x,y)= 1-p'(y)+p'(x)$ that:

$$\delta=\begin{cases} d(x,y)-\delta & if\,(x,y) \quad leaves \quad T \\ d(x,y)+\delta & if\,(x,y) \quad enters \quad T \\ d(x,y) & otherwise \end{cases}$$

We have now that after at most $n-1$ iterations, either $\delta=\delta_e$ or $a\in T$ is achieved (case 2). If we need such iterations, we observe that the definition of $\delta$ assures that $H'$ contains at least one arc leaving $T$ (which is in $A_B$, $A_W$ or $A_R$ according as $\delta$ is equal to $\delta_B$, $\delta_W$ or $\delta_R$. Consequently the set $T'$ of vertices which can be reached by a directed path from $b$ in $H'$ properly includes $T$. (Note that, in this case, the new auxiliary graph arises simply from the old one by joining some new arcs leaving $T$ and deleting some old ones entering $T$.)

**Case 2. Improving the covering.** There is a path from $b$ to $a$ in $H$.

Let $F^\star=\{(xy):(yx)\in F\}$.

In this case let $U$ be the shortest path from $a$ to $b$ in $H$. Since $(a,b)\in A_B$, $U$ and $(a,b)$ form a directed circuit $C$ in $H$. Let $C_B$ and $C_W$ be the set of blue and white arcs of $C$ respectively. Let $B'=B\setminus C_B\cup C_W{}^\star$. Now this change was between a set of vertices which satisfy the optimality criteria.

A complete proof of correctness of the above algorithm may be found in Frank (1981).

## Complexity

Let $n$ and $m$ be $|V|$ and $|E|$ respectively.

Let $B$ be the covering $\{e_i=(x_i,y_i){:}i=1,2,...,l\}$. Let $G_i$ denote the graph obtained by adding to $G$ a set $(B\setminus e_i)^\star$ of new arcs (i. e. the new arcs are the reversed elements of $(B\setminus e_i)$). Let $R_i(x)=\{y$: $y$ can be reached from $x$ in $G_i\}$. Therefore, using the well-known labeling technique (Ford and Fulkerson, 1962), $R_i(x)$

can be produced in at most $O(n)$ steps (since $G_i$ is planar).

Let $R(x) = \bigcap_{i=1,2,\ldots,l} R_i(x)$.

By the last paragraph we see that we can construct $R(x)$ in $O(n^2)$ steps for fixed $x$. So determining all $R(x)$'s requires at most $O(n^3)$ steps. Therefore, this part has complexity $O(n^3)$.

The other part of the algorithm finds a path from $b$ to $a$ in $H$. To this end we may apply again the labeling technique. In case 1 the set $T$ is just the set of vertices having received a label during the labeling algorithm. In case 2 the path $U$ produced by the labeling algorithm is automatically free of red arcs which does not increase the number of steps.

The labeling algorithm uses at most $O(n)$ steps for a planar graph. Moreover, if $\delta > \delta_e$ occurs in the course of the algorithm and we apply it again with the modified potential, then the labels calculated previously may be used ($T \subseteq T'$, you may note that, in this case the new auxiliary graph arises simply from the old one by joining some new arcs leaving $T$ and deleting some old ones entering $T$).

Therefore, the core of the algorithm needs at most $O(n^3)$ steps. Since it is applied at most $(n-1)$ times, the complexity of the whole algorithm developed here is $O(n^4)$.

## Improvements

To improve the time complexity of this algorithm we took three actions:

1. Find $R(b)$ for a vertex $b$ of one arc $(a,b)$ in $B$ which does not satisfy our optimality criteria. We call the arc $(a,b)$ *good blue arc* or *bad blue arc* f $a \in R(b)$ or $a \notin R(b)$ respectively.
2. Change the color of all good blue arcs to white.
3. Run the algorithm for all bad blue arcs.

The proof of correctness of this improved algorithm follows from the fact that Frank's algorithm works regardless the input order of the violating arcs. Therefore, without loss of generality, we should input such arcs in a specified order (good blue arcs first). The step 2 follows from the fact that the shortest path from $b$ to $a$ in the auxiliary graph is just the red arc $(b,a)$ and the removing of a good blue arc does not change the potential, consequently, any other good blue arc is still violating our optimalty criteria.

We ran both algorithms in Turbo Pascal 5.5 on a 386 SX (slow) machine obtaining the results shown in figures 1 and 2. It is clear that our algorithm is faster than Frank's algorithm and it suggests that ours is running in time $O(n^3)$ rather than $O(n^4)$.

Although our improvements in this algorithm seems to be only better by constant factor, in practical cases it showed to be better by a linear factor.

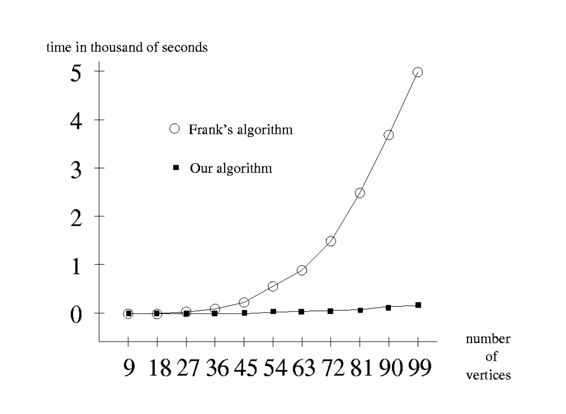


**Figure 1.** The performance of both algorithms

| $\|B\|$ | $\|V\|$ | $\|E\|$ | $\frac{\|E\|}{\|V\|}$ | $a_1$ | $a_2$ | $\frac{\|a_2\|}{\|a_1\|}$ |
|---|---|---|---|---|---|---|
| 2 | 9 | 18 | 2.00 | 0.71 | 0.93 | 1.3 |
| 5 | 18 | 42 | 2.33 | 1.65 | 7.36 | 4.5 |
| 8 | 27 | 66 | 2.44 | 3.90 | 32.52 | 8.3 |
| 11 | 36 | 90 | 2.50 | 9.88 | 97.38 | 9.9 |
| 14 | 45 | 114 | 2.53 | 14.83 | 234.37 | 15.8 |
| 17 | 54 | 138 | 2.55 | 30.34 | 481.69 | 15.9 |
| 20 | 63 | 162 | 2.57 | 48.44 | 893.39 | 18.4 |
| 23 | 72 | 186 | 2.58 | 71.08 | 1522.53 | 21.4 |
| 26 | 81 | 210 | 2.59 | 100.57 | 2444.07 | 24.3 |
| 29 | 90 | 234 | 2.60 | 167.09 | 3716.59 | 22.24 |
| 32 | 99 | 258 | 2.60 | 182.35 | 5458.23 | 29.93 |

**Figure 2.** $a_1$=improved Frank's algorithm, $a_2$=Frank's algorithm, time in seconds.

## The improved algorithm

**Step 0**

**0.1** (Start) The covering $B$ is the arc set of a spanning tree and $p \equiv 0$.

**0.2** Determine $R(b)$ for all blue arc $(a,b)$ which does not satisfy ***(a)*** of the optimality criteria.

**0.3** Change the color of each good blue arc to white.

**Step 1**

**1.1** Determine $R(x)$ for all $x \in V$.

**1.2.** If every blue arc satisfies ***(a)*** of the optimality criteria: ***Halt***. The current covering B is optimal.

**1.3** Select a blue arc $e=(a,b)$ violating ***(a)*** of the optimality criteria.

**1.4** Construct the auxiliary graph $H$ and try to find a path from $b$ to $a$ in $H$ by the labeling technique Ford and Fulkerson (1962). In the new auxiliary graph $H'$

the arc set spanned by $T$ is the same as in $H$. Moreover, the definition of $\delta$ assures that $H'$ contains at least one arc leaving $T$ (which is in $A_B$, $A_W$ or $A_R$) according as $\delta$ is equal to $\delta_B$, $\delta_W$ or $\delta_R$). Consequently the set $T'$ of vertices which can be reached by a directed path from $b$ in $H'$ properly includes $T$. (Note that, in this case the new auxiliary graph arises simply from the old one by joining some new arcs leaving $T$ and deleting some old ones entering $T$. Thus, if $\delta < \delta_e$ occurs (it went in this step from 2.2) we may use the labels calculated but not yet removed previously. If this path $U$ exists then go to step 3.

**Step 2** (potential change). Let $T$ be the set of labeled vertices. Calculate $\delta$ and let $p(x) := p(x) + \delta$ for $x \in T$.
**2.1** If $\delta = \delta_e$ remove all labels and go to 1.2
**2.2** Go to 1.4.

**Step 3** (covering change) Let $C$ be $U+e$, $C_B$ and $C_W$ be the set of blue and white arcs of $C$, respectively, and $B$: be $B \backslash C_B \cup C_W{}^\star$. Return to 1.1.

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