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New quaternary codes obtained from a combinatorial optimization algorithm

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ABSTRACT. New codes over Z_4 (the integers mod 4) found by use of a combinatorial optimization algorithm are tabulated. Also, from one of these codes, a new Z_4-linear code better than any previously known binary code in terms of number of codewords and minimum distance is presented.

Key words: Z_4-linear code, combinatorial optimization algorithm, Brazil.

RESUMO. Novos códigos quaternários obtidos de um algoritmo de otimização combinatorial. Novos códigos sobre Z_4 (o anel dos integrais módulo 4) encontrados através da utilização de um algoritmo de otimização combinatorial são tabulados. Também apresentamos, entre estes códigos, um novo código Z_4- linear melhor do que qualquer código binário previamente conhecido em termos do número de palavras-código e distância.

Palavras-chave: códigos Z_4- lineares, algoritmo de otimização combinatorial, códigos quaternários, Brasil.

In this paper we tabulate new codes over Z_4 (the integers mod 4), which we call quaternary codes, by using a combinatorial optimization algorithm (Said and Palazzo, 1993). These new codes improve in terms of achieving larger minimum distances when compared with the minimum distances of previously known codes having the same code parameters (length and dimension) (Kschischang and Pasupathy, 1992). The main consequence of our results is the presentation of good Z_4-linear codes from these quaternary codes, via Gray map from Z_4 to Z_2 (Hammons Jr. *et al.,* 1994; Gerônimo, 1997).

The results shown in Table 2 indicate that the corresponding binary codes (obtained from the quaternary codes through Gray map) achieve the upper bounds on several minimum distances, as shown in Brouwer and Verhoeff's table (Brouwer and Verhoeff, 1993). The only binary codes in which the minimum distances of the corresponding Z_4-linear codes improve the upper bound shown in (Brouwer and Verhoeff, 1993) are the Nordstron-Robinson code (16,8,6), (Hammons *et al.,* 1994; Forney *et al.,* 1993), and the code (7,3,6). In Calderbank (1995), a (7,3,6)-quaternary cyclic code is presented; however, the binary image of this code is one which has a better distance than the linear binary code with the same parameters.

In Table 3 we have included those codes which have not achieved the upper bound on the minimum distance as shown in Brouwer and Verhoeff's table. However, they considerably improve the lower bounds shown in Kschischang and Pasupathy's table (Kschischang and Pasupathy, 1992).

Preliminaries

By a quaternary code of length n, we shall mean an additive subgroup of Z_4^n. A *mapping* is a function
$\phi: Z_4^n \text{-->} Z_2^{2n}$ defined by
$\phi(c)=(\beta(c),\gamma(c))=$
$=(\beta(c_1), \beta(c_2),..., \beta(c_n), \gamma(c_1), \gamma(c_2),..., \gamma\beta((c_n)),$
$c=(c_1,c_2,..., c_n)$ em Z_4^n
where β Z_4-->Z_2 and β Z_4-->Z_2 are defined as in Table 1.

Table 1. Gray mapping

c	β(c)	γ(c)
0	0	0
1	0	1
2	1	1
3	1	0

In the search for the best linear group codes we have used the following key properties of the mapping ϕ, namely,

1. The mapping ϕ is an isometry from (Z_4^n,d_L) to (Z_2^{2n},d_H);
2. The mapping ϕ is a matched labeling, i.e., $d_H(\phi(x), \phi(y)) = d_H(\phi(x-y), \phi(0)) = d_L(x-y,0) = w_L(x-y)$.

Definition: (Hammons Jr. *et al.,* 1994) A binary code C in Z_2^{2n} of even length is Z_4-linear if its coordinates can be arranged in such a way that C is the image of a quaternary code under the mapping ϕ.

The Z_4-linear codes are geometrically uniform, (Forney, 1991), for Z_4 is a transitive subgroup of the symmetry group of Z_2^2 (Forney Jr. *et al.,* 1993), and ϕ is a matched mapping from Z_4 onto Z_2^2, in Loeliger's sense (Loeliger, 1991) (the only modification to be introduced in Loeliger's definition is that the Euclidean distance must be replaced by the Hamming distance).

Quaternary Code Search via Combinatorial Optimization Algorithm

The algorithm employed in the search for new quaternary codes is based on the combinatorial optimization problem inherent in the design of good unit-memory convolutional codes (UMC). Fortunately, efficient methods to solve this problem were proposed in Said and Palazzo (1993), and in Palazzo (1995). These design methods are based on mathematical optimization models with the purpose of providing a systematic way of finding these codes. In order to use standard optimization techniques, desirable properties such as convexity and linearity, usually found in these models, are looked for. In particular, the UMC codes design problem is decomposed into two subproblems which are formulated as standard optimization problems with integer variables. The solution of each one of them comes from the use of the corresponding local search algorithms, which were shown to be quite efficient heuristic algorithms.

The decomposition of the UMC problem is carried out as follows:

1. Column Types Selection Problem}: Find the sets of generator matrix columns that form distinct (2n,k) block codes with minimum distance greater than or equal to the maximal d_{free} possible;
2. Column Types Matching Problem: Use these sets of columns to find the pairwise combination of the columns in G'_0 and G'_1 that forms a UMC with d_{free} equal to the maximal d_{free} value.

However, in the search for the best quaternary linear codes shown in the tables we have made extensive use of the Column Types Selection Problem method of decomposition. This decomposition method is the one which has inherited the well-known knapsack problem.

Table of Quaternary Codes

From the combinatorial optimization algorithm proposed by Said and Palazzo (1993), a great number of quaternary linear group codes have been obtained.

The algorithm uses the following facts:

- $d_L(x,y)=w_L(x-y)$, i.e., it is sufficient to find the Lee weight of the codewords to obtain the minimum Lee distance;
- The length of the binary codes is twice the length of the corresponding quaternary codes;
- The number of codewords of the binary codes is 2^{2k}, when k is the number of generators.

Example 1: Our objective here is to find the best (14,6)-binary code which is Z_4-linear. From Brouwer and Verhoeff's table, (Brouwer and Verhoeff, 1993), the best known binary code with these parameters has an upperbound on the minimum Hamming distance equal to 5. The corresponding Z_4-linear code has parameter (7,3). Therefore, if a better code is to be found its minimum Lee distance should be 6. Hence, a (7,3,6) code. With these parameters, the best quaternary code found by the combinatorial optimization algorithm has minimum Lee distance 6. The generator matrix and the weight enumerator function are respectively

$$\begin{vmatrix} 1 & 3 & 0 & 3 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 1 & 1 & 0 \\ 0 & 3 & 2 & 0 & 3 & 1 & 1 \end{vmatrix}$$

And W_{Lee}(C)$=1+42x^6+7x^8+14x^{10}$.

Taking this example into consideration, the Lee weight enumerator function is read as follows: there is one codeword of weight 0; there are 42 codewords of weight 6; 7 codewords of weight 8, and 14 codewords of weight 10. The notation used in the tables that follow is 0(1) 6(42) 8(7) 10(14).

Example 2: The (16,8,6) Nordstron-Robinson code, (Hammons *et al.,* 1994; Forney *et al.,* 1993). The corresponding quaternary code has parameters (8,4), and minimum Lee distance equal to 6. This code was obtained by the combinatorial optimization algorithm. Its generator matrix is given by

$$\begin{vmatrix} 0 & 3 & 0 & 0 & 3 & 2 & 3 \\ 3 & 2 & 0 & 1 & 0 & 0 & 3 \\ 3 & 3 & 0 & 2 & 2 & 1 & 2 \\ 1 & 0 & 3 & 0 & 1 & 2 & 0 \end{vmatrix}$$

The weight enumerator function is $W_{Lee}(C)=1+112x^6+30x^8+112x^{10}+x^{16}$.

This code can also be obtained from a cyclic code with generator polynomial given by

$g(x)=x^3+3x^2+2x+3$ and a parity-check digit. The corresponding quaternary code has parameters n=8, k=4, $d_{Hamming}$=6.

The next Tables are organized as follows:

- n - length of the 4-ary group code;
- k - number of generators of the 4-ary group code;
- $LB\text{-}UB_{BV}$ - lower and upper bounds on the minimum Hamming distance of the binary linear codes from Brouwer and Verhoeff's Table, (Brouwer and Verhoeff, 1993);
- LB_{KP} - lower bound on the minimum distance of the quaternary linear codes from Kschischang and Pasupathy's Table, (Kschischang and Pasupathy, 1992);
- d_{Lee} - minimum Lee distance corresponding to the Hamming distance associated with the binary code through the mapping ϕ;
- G - generator matrix of the quaternary code;
- w_{Lee} - spectrum of Lee weight of the quaternary code corresponding to the Hamming weight spectrum of the binary code.

Table 2. 4-ary linear group codes

N	K	$LB\text{-}UB_{BV}$	D_{LEE}	G	W_{LEE}
5	2	4	4	21033	0(1) 4(2) 5(8)
				32313	6(4) 8(1)
5	3	3	3	33013	0(1) 3(8) 4(18)
				11322	5(16) 6(8) 7(8)
				03323	8(5)
6	3	4	4	013120	0(1) 4(6)
				223133	5(24) 6(16)
				130123	8(9) 9(8)
6	4	3	3	220233	0(1) 3(16) 4(39)
				312220	5(48) 6(48) 7(48)
				120323	8(39) 9(16) 12(1)
				113320	
7	3	5	6	1303210	0(1) 6(42)
				0231110	8(7) 10(14)
				0320311	
7	4	4	4	3233100	0(1) 4(14) 5(56)
				0113222	6(49) 7(16) 8(49)
				0301301	9(56) 10(14) 14(1)
				2302211	
8	2	8	8	10012132	0(1) 8(11) 10(4)
				32112203	
8	5	4	4	11311121	0(1) 4(47) 5(72)
				13103030	6(98) 7(192) 8(207)
				30022321	9(176) 10(124) 11(64)
				32133021	12(33) 13(8) 14(2)
				33122222	
9	3	8	8	210010231	0(1) 8(46) 12(16)
				033212122	16(1)
				333013333	
9	4	6	6	312313003	0(1) 6(23) 7(56)
				231011101	8(45) 9(16) 10(34)
				101002021	11(56) 12(18) 14(7)
				000212033	
9	5	4	4	310112323	0(1) 4(3) 5(40)
				132121310	6(104) 7(128) 8(113)
				312031231	9(176) 10(232) 11(128)
				121001323	12(41) 13(40) 14(16)
				333031300	16(2)
9	6	4	4	200111232	0(1) 4(78) 5(144)
				202332111	6(228) 7(528) 8(708)
				003311020	9(736) 10(696) 11(480)
				303112320	12(298) 13(144) 14(36)
				310323230	15(16) 16(3)

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				121033233	
9	7	2	2	130300331 033032211 210313020 033200323 013320022 122313300 323331322	0(1) 2(3) 3(60) 4(210) 5(504) 6(1134) 7(2052) 8(2748) 9(2960) 10(2748) 11(2052) 12(1134) 13(504) 14(210) 15(60) 16(3) 18(1)
10	2	10	10	2113113301 3003132233	0(1) 10(12) 12(2) 16(1)
10	3	8	8	2231210233 1201032110 3322323201	0(1) 8(7) 9(24) 10(16) 12(6) 13(8) 16(2)
10	4	8	8	3031030202 1013333110 2011103020 0331022003	0(1) 8(130) 12(120) 16(5)
10	5	6	6	3021010203 2321022233 3233203033 1312030303 0310330031	0(1) 6(90) 8(255) 10(332) 12(255) 14(90) 20(1)
10	6	4	4	0023301031 2130330023 1003312121 1023221033 1130123230 0120332332	0(1) 4(5) 5(64) 6(240) 7(320) 8(250) 9(640) 10(1056) 11(640) 12(250) 13(320) 14(240) 15(64) 16(5) 20(1)
10	7	4	4	2023102010 0231021311 1303110333 3002301321 2222013202 3020032011 2132332200	0(1) 4(125) 5(256) 6(480) 7(1280) 8(2050) 9(2560) 10(2880) 11(2560) 12(2050) 13(1280) 14(480) 15(256) 16(125) 20(1)
10	8	2	2	1211211231 3110311220 1331230002 0012011023 0033021203 2023233133 2002002333 3003132223	0(1) 2(5) 3(82) 4(324) 5(928) 6(2388) 7(4936) 8(7902) 9(10368) 10(11542) 11(10620) 12(7860) 13(4768) 14(2436) 15(1000) 16(297) 17(64) 18(13) 19(2)
11	3	9	9	22030013033 20302101123 11333033313	0(1) 9(10) 10(15) 11(16) 12(14) 13(2) 14(1) 15(4) 20(1)
11	4	8	8	23231030101 00112113103 00013233231 30023300203	0(1) 8(26) 9(68) 10(40) 12(32) 13(56) 14(24) 16(5) 17(4)
11	6	6	6	11332312000 01010120313 03322230210 32023102133 10033312120 12221023120	0(1) 6(184) 8(594) 10(1248) 12(1304) 14(600) 16(149) 18(16)
12	2	12	12	123213003232 210310123210	0(1) 12(9) 14(6)
12	4	8	8	210331133232 122203302201 303002123130 313232320033	0(1) 8(1) 9(32) 10(52) 11(40) 12(34) 13(24) 14(20) 15(24) 16(20) 17(8)
12	5	8	8	130221313233 030120030203 132020333320 001010313333 120333211222	0(1) 8(128) 10(218) 12(324) 14(244) 16(91) 18(18)
13	2	13	13	0130133022132 3132002101323	0(1) 13(8) 14(4) 16(3)
13	3	12	12	3023233230001 0020130312132 0121303321200	0(1) 12(46) 16(15) 20(2)
13	4	10	10	3322320031203 0131333110200 1303020322303	0(1) 10(56) 12(87) 14(52) 16(45) 18(12) 20(3)

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				3001212032231	
13	5	8	8	3312121020012	0(1) 8(10) 9(72)
				3110310210101	10(148) 11(104) 12(60)
				2102333333231	13(144) 14(200) 15(144)
				0313303232222	16(53) 17(40) 18(36)
				3212331310120	19(8) 20(4)
14	6	8	8	20233323212003	0(1) 8(50) 9(166)
				30023331113103	10(218) 11(280) 12(428)
				32132133032202	13(566) 14(630) 15(576)
				00331221210232	16(493) 17(338) 18(166)
				20212333030330	19(104) 20(52) 21(18)
				31233200310321	22(10)
15	2	16	16	212101032301332	0(1) 16(15)
				100321102211211	
15	3	14	14	223130021310233	0(1) 14(15) 15(32)
				131333133313331	16(15) 30(1)
				303213323202330	
15	4	12	12	333130123231001	0(1) 12(60) 14(86)
				211312013333131	16(52) 18(32) 20(22)
				103221313223032	22(2) 24(1)
				012300113230303	
15	6	9-10	9	223303300022103	0(1) 9(74) 10(171) 11(208)
				231011232212031	12(302) 13(434) 14(521) 15(600)
				012111301021212	16(593) 17(462) 18(333) 19(208)
				312303323130323	20(94) 21(54) 22(31)
				021103210122230	23(8) 24(2)
				203300332323102	
16	2	16	16	1032201131201321	0(1) 16(3) 17(8)
				1231113002211102	18(4)
16	6	10	10	0101023303131033	0(1) 10(138) 12(499)
				3133133122133320	14(824) 16(1114) 18(950)
				1321230220321013	20(445) 22(100) 24(21)
				3023300300022011	26(4)
				0310212203102021	
				2101233013003030	
17	3	16	16	01021120123322110	0(1) 16(14) 17(32)
				33012102013223031	18(16) 32(1)
				00123033102331322	
17	4	14	14	22033332323011333	0(1) 14(28) 15(76)
				13220121020121320	16(53) 18(16) 19(40)
				02233303332212221	20(8) 22(20) 23(12)
				22332233333333001	24(2)
17	5	12-13	12	03230123122131203	0(1) 12(40) 13(96)
				32132101213010302	14(100) 15(96) 16(117)
				02121302123302332	17(144) 18(120) 19(80)
				31321332212331102	20(88) 21(80) 22(36)
				32222231100213113	23(16) 24(10)
18	3	16	16	223130022310033312	0(1) 16(2) 17(16)
				131333132013331333	18(24) 19(16) 20(4)
				303213323102030233	32(1)
18	5	13-14	13	033133330100200332	0(1) 13(40) 14(89) 15(118)
				123233202011123120	16(128) 17(108) 18(92) 19(100)
				201301231302311201	20(110) 21(88) 22(73) 23(54)
				300113122130300301	24(17) 25(4) 26(2)
				230122231033323121	
18	6	12	12	212202002013302132	0(1) 12(191) 14(510)
				323331112112111211	16(804) 18(1008) 20(967)
				003023302200101331	22(454) 24(115) 26(44)
				302231330023321310	28(2)
				330121030203303321	
				312021103202030202	
19	2	20	20	1031320211100332122	0(1) 20(14) 24(1)
				1133032123211301302	
19	3	18	18	1213221003330202113	0(1) 18(30) 20(30)
				0112330310320220333	22(2) 32(1)
				1033213022011313202	
19	4	16	16	0023101323201123002	0(1) 16(68) 18(82)
				3232223321321013212	20(50) 22(24) 24(17)
				0300302332222011303	26(14)
				2330313320330212003	
19	5	14-15	14	3023112212013113103	0(1) 14(40) 15(144)
				2101211120001100002	16(103) 18(120) 19(232)

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				33230203323330333110	20(110) 22(88) 23(128)
				3330103322201032222	24(40) 26(8) 27(8)
				3320300033203100112	28(2)
19	6	12-14	12	2312102023113200301	0(1) 12(20) 13(128) 14(206)
				2122200223223103130	15(234) 16(291) 17(372) 18(524)
				1010203230202303023	19(546) 20(458) 21(428) 22(368)
				2121212213010011232	23(238) 24(124) 25(92) 26(52)
				2310311112001003110	27(6) 28(2) 29(4)
				1233033033213022230	30(2)
20	4	16-17	16	11101333333030033322	0(1) 16(4) 17(40) 18(61)
				21103310122331331212	19(36) 20(18) 21(20) 22(26)
				22133333333303201003	23(20) 24(5) 25(4)
				21032330220122130330	26(9) 27(8) 28(4)
20	6	13-14	13	02220030101011110303	0(1) 13(46) 14(134) 15(174)
				12323030121112102100	16(213) 17(350) 18(426) 19(444)
				32100300033332103111	20(468) 21(466) 22(494)
				13321003320213330330	23(368) 24(194) 25(154)
				32012331221023011013	26(94) 27(36) 28(20)
				32303221021030033103	29(8) 30(4) 31(2)
21	2	22	22	121020101121211333210	0(1) 22(12) 24(3)
				003131310221322031312	
21	3	20	20	321131313113012310101	0(1) 20(32) 22(24)
				211300320323102231133	24(6) 32(1)
				022123310330011221320	
21	5	16	16	313231102000302122020	0(1) 16(101) 18(218)
				223011003300101323113	20(208) 22(206) 24(170)
				113330200103322000031	26(86) 28(32) 30(2)
				112023332310323010330	
				310230322133330203002	
23	2	24	24	03211213233121003230203	0(1) 24(11) 26(4)
				11312112220023210012333	
23	3	22	22	00223021323301132332003	0(1) 22(14) 23(32)
				03032130013123113322331	24(14) 30(2) 32(1)
				12323303333013323213033	
23	4	20	20	10210021013323022303300	0(1) 20(74) 22(80)
				23100303021032333230323	24(44) 26(24) 28(18)
				00131223302133331112012	30(8) 32(7)
				20331332232332222133302	
23	5	18-19	18	30232203130211321220201	0(1) 18(124) 20(195)
				33222001030130332033301	22(212) 24(195) 26(172)
				31030221001021011332211	28(81) 30(36) 32(8)
				11122330012033031333223	
				20021321331203310323201	
23	6	16-18	16	30011220211103032313333	0(1) 16(125) 18(412)
				30231211222302003023002	20(622) 22(896) 24(856)
				00130223030312322211133	26(692) 28(362) 30(104)
				13103021123301201011231	32(18) 34(8)
				31132310102233003310123	
				30301133133132130210301	
25	2	26	26	2332030203113020231133231	0(1) 26(12) 28(2)
				0122332331200331100333122	32(1)
25	3	24	24	2323310112310022300031121	0(1) 24(12) 25(32)
				3010220020012112331333233	26(16) 32(3)
				2132333121031013223300002	
25	5	20-21	20	1000310120321301132100313	0(1) 20(150) 22(158)
				2111313133011320013202330	24(239) 26(166) 28(179)
				2301000310311123131330121	30(82) 32(36) 34(10)
				0213222232103032003302033	36(3)
				1313201332123330202223000	
26	4	24	24	30011101333232203310330323	0(1) 24(172) 28(32) 32(51)
				21322130331230322301222311	
				01213023132303313323023211	
				32231300303203330303032202	
27	2	28	28	133303003221221130320102323	0(1) 28(9) 30(6)
				230033122103231101201321021	
27	3	26	26	301132211000311202123231110	0(1) 26(12) 27(32)
				330003123012011210232313123	28(12) 30(4) 32(3)
				003201012321312331322333303	
27	4	24	24	030120220323220011323213013	0(1) 24(103) 26(16)
				300133030110333331130222230	28(86) 30(16) 32(27)
				011203201020021312221211303	36(2) 40(5)
				300133202123130232001301022	
28	4	24-25	24	21321110033320033110333302131	0(1) 24(7) 25(56) 26(46)
				1230301203133312300101102102	27(12) 28(40) 29(36) 30(13)
				1020221123320213021231332103	31(4) 32(14) 33(14) 36(2)

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				1211303332200032023120003130	37(4) 38(5) 41(2)
29	3	28	28	10123020133201013133233302332 11131330020030232100313131222 03322033121320331212033033303	0(1) 28(12) 29(32) 30(12) 32(3) 34(4)
30	2	32	32	132013303213203302021012132233 233303321201113010200133221212	0(1) 32(15)
30	4	26-28	26	131233320001103220201322103300 202203022031003020313110331021 003123132131201001312222010300 333133023330311312030201303332	0(1) 26(10) 27(36) 28(59) 29(52) 30(8) 31(16) 32(26) 33(8) 34(12) 35(8) 36(9) 37(4) 40(1) 42(2) 43(4)
31	2	32	32	0032211110303312022333323102312 1102113301122012201032200233333	0(1) 32(3) 33(8) 34(4)
32	3	32	32	21300230212321333122003110012031 21022332103022321323232103322113 30203132132010220120123311030112	0(1) 32(59) 40(4)

Table 3. 4-ary linear group codes

N	K	LB_{KP}	D_{LEE}	G	W_{LEE}
7	2	5	6	1322013 2011333	0(1) 6(2) 7(8) 8(3) 10(2)
11	2	8	10	33222313032 11330012332	0(1) 10(1) 11(8) 12(3) 14(3)
11	5	6	6	23010220312 10210222301 21100100230 33123112033 33013323023	0(1) 6(4) 7(56) 8(146) 9(104) 10(60) 11(176) 12(216) 13(144) 14(60) 15(24) 16(21) 17(8) 18(4)
14	5	8	9	31233030132300 01211323330233 10111002222013 30013203233300 12321202113330	0(1) 9(20) 10(81) 11(116) 12(108) 13(104) 14(127) 15(168) 16(131) 17(68) 18(47) 19(36) 20(16) 22(1)
15	5	8	10	130320021123202 020020111300323 111032333120230 032130131310311 120201130120312	0(1) 10(18) 11(96) 12(130) 13(64) 14(118) 15(192) 16(103) 17(64) 18(110) 19(96) 20(22) 22(10)
16	4	11	12	0312332203110022 1203330310311121 3102023102013102 0010233333033131	0(1) 12(1) 13(36)14(61) 15(40) 16(21) 17(20) 18(26) 19(20)20(9) 21(8) 22(9) 23(4)
16	5	8	11	1002032300211201 3231323213100200 0333203102300333 3331132310322033 2032120322232130	0(1) 11(30) 12(89) 13(116) 14(112) 15(102) 16(114) 17(136) 18(120) 19(90) 20(51) 21(36) 22(24) 23(2) 24(1)
20	5	12	15	32121102030031300130 30100211213311313302 12001333001013302023 30003212310032323123 33332133202000032003	0(1) 15(52) 16(118) 17(100) 18(76) 19(112) 20(141) 21(104) 22(64) 23(84) 24(85) 25(52) 26(20) 27(8) 28(7)
22	3	16	20	0113313333233010101332 1010330013222210233211 3313212020320333330311	0(1) 20(2) 21(16) 22(24) 23(16) 24(2) 28(2) 32(1)
22	5	13	16	3123022003122232221332 0033233003230203121110 2201331312032203210110 1121301010232310120301 1003302330311031103231	0(1) 16(7) 17(68) 18(102) 19(94) 20(92) 21(98) 22(122) 23(108) 24(94) 25(72) 26(58) 27(54) 28(28) 29(18) 30(6) 32(2
22	6	12	15	3213221320031330112202 3111002033323231022022 0133010333010021313230 1132103130220231323001 0203200233300313133001 1122113310303032200321	0(1) 15(56) 16(154) 17(180) 18(262) 19(344) 20(373) 21(440) 22(457) 23(444) 24(401) 25(356) 26(273) 27(176) 28(95) 29(48) 30(27) 31(4) 34(5)
24	5	15	18	110123101303311300123221 103001113322213200012011 000330011111321323201331 030213332121021002332200 101200210012201313233201	0(1) 18(8) 19(54) 20(136) 21(114) 22(56) 23(106) 24(132) 25(94) 26(40) 27(74) 28(104) 29(46) 30(24) 31(22) 32(11) 33(2)
24	6	14	16	221310130123131323320321 223013331330333202221012 032112123202230022230330	0(1) 16(3) 17(72) 18(168) 19(232) 20(239) 21(286) 22(384) 23(424) 24(474) 25(410) 26(360)

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				303303102131032203331010	27(380) 28(270) 29(178)
				210233333333123223032113	30(112) 31(48) 32(34)
				122233300020302213032223	33(14) 35(4) 36(3)
25	4	16	22	131231012020320110331 2002	0(1) 22(24) 23(96)
				020101322103100221130 2133	24(52) 26(32) 30(8)
				003120232110233113000 3322	31(32) 32(11)
				220312311033300032033 1302	
25	6	14	17	2221211021102123033121311	0(1) 17(8) 18(82) 19(182)
				2303112001020231103310031	20(213) 21(210) 22(324) 23(418)
				0102211103101222330202313	24(409) 25(426) 26(399) 27(382)
				1102100101330220233222212	28(331) 29(278) 30(207) 31(102)
				3001211213232000211312122	32(66) 33(38) 34(11)
				1333200332122231012231133	35(4) 36(4) 38(1) \end{tabular}
26	6	14	18	10213011122003133032022200	0(1) 18(16) 19(82) 20(200)
				32120232121133233220120122	21(216) 22(237) 23(306) 24(329)
				33311121201021113320011020	25(406) 26(443) 27(446) 28(389)
				02002221032123300310003101	29(334) 30(277) 31(166) 32(98)
				01223013122330013222103211	33(66) 34(49) 35(24)
				03333032020023322323231101	36(7) 37(2) 38(2)
27	5	17	21	012230210000312130020233311	0(1) 21(18) 22(67) 23(118)
				203203200201310021130331301	24(105) 25(98) 26(96) 27(84)
				132112002111132011213211323	28(86) 29(82) 30(75) 31(54)
				022311132310210012020030030	32(60) 33(38) 34(18)
				333333121321033122202323023	35(16) 36(4) 37(4)
27	6	15	19	111002032030120130132010210	0(1) 19(16) 20(92) 21(208)
				222030213002030332033001313	22(203) 23(242) 24(335) 25(336)
				120202121330101323021032121	26(403) 27(418) 28(388) 29(392)
				303313210330003211301320200	30(333) 31(246) 32(196) 33(144)
				021011001123321200012302131	34(81) 35(38) 36(11)
				332203031322310221300022222	37(8) 38(4) 44(1)
28	5	18	22	2021023331313203113300233123	0(1) 22(18) 23(86) 24(99) 25(102)
				0203312203230003023032312121	26(110) 27(80) 28(89) 29(88) 30(85)
				3210020010121313120203132030	31(72) 32(57) 33(46) 34(40)
				0133300312003330220131312103	35(32) 36(9) 37(4)
				1212312333020002001301011313	38(3) 39(2) 40(1)
28	6	16	20	2103213031232231023023033133	0(1) 20(48) 22(368)
				3232113231011203002032031023	24(545) 26(664) 28(821)
				3030032211111003313000103211	30(722) 32(576) 34(244)
				3212320213022110330321311200	36(83) 38(18) 40(6)
				2031232321121233231221202200	
				1103300200321332333320021111	
29	6	17	21	00031132131120121202320001121	0(1) 21(30) 22(110) 23(192)
				00202212012330001110232102312	24(241) 25(254) 26(267) 27(376)
				30323223331231001311302310132	28(381) 29(352) 30(435) 31(384)
				00310221001323020203003332131	32(302) 33(272) 34(193) 35(120)
				20223102332223212013013320103	36(95) 37(50) 38(19)
				23032323303233301032013332102	39(16) 40(4) 41(2)
30	6	18	22	312110020312320103233302221311	0(1) 22(33) 23(122) 24(190)
				332330313131211331213330322211	25(218) 26(282) 27(332) 28(318)
				222102013122110203223303303120	29(374) 30(358) 31(328) 32(398)
				030222003010232321313030330200	33(354) 34(274) 35(212) 36(137)
				223010321222301020233222130112	37(66) 38(45) 39(30) 40(11)
				222331003233023020003322033123	41(12) 44(1)
31	3	23	30	1133200330221102203203120011132	0(1) 30(12) 31(32)
				1103223221322300321131222110323	32(15) 38(4)
				3230322021202113102100323313330	
31	5	20	25	2231122030131110300111213003313	0(1) 25(48) 26(90) 27(78)
				1233012021002033301123001221231	28(91) 29(88) 30(89) 31(102)
				3021113310323321310232100022031	32(89) 33(90) 34(52) 35(50)
				3000022130002213011110320131133	36(63) 37(36) 38(25)
				2303000033221213311100113320122	39(10) 40(12) 41(10)
31	6	18	21	0023332221302232001303003100031	0(1) 21(4) 22(13) 23(36)
				1313201130233011010212121222210	24(73) 25(176) 26(296) 27(320)
				0303123303303332130103232221231	28(283) 29(314) 30(346) 31(348)
				0200123222011132121120323332230	32(390) 33(364) 34(298) 35(264)
				2203303020222031131313223221302	36(229) 37(152) 38(93) 39(56)
				3020322032103301122202332030332	40(16) 41(12) 42(10) 45(2)
32	4	22	28	11101012301333333211300313213333	0(1) 28(4) 29(30) 30(69)
				01322032233132030302301011001232	31(60) 32(22) 33(16) 34(8)
				23310223321312332223232212331220	36(2) 37(16) 38(18) 39(4)
				12232123323010211203303223223200	44(2) 45(2) 46(1) 48(1)
32	5	20	25	02102033321013122121222220032113	0(1) 25(4) 26(20) 27(78)
				01100323110032202130233312330123	28(126) 29(116) 30(96) 31(74)
				02111010030231230030201132101133	32(81) 33(72) 34(60) 35(82)
				32300022333221013011002221001033	36(58) 37(52) 38(40) 39(22)

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				23033202213301123303330333332302	40(22) 41(12) 42(8)
32	6	18	22	00013301300031233220120210301220	0(1) 22(2) 23(34) 24(55)
				21323030021023330123223002222222	25(28) 26(146) 27(430) 28(271)
				10102103001123222220312312301332	29(76) 30(388) 31(572) 32(381)
				20211010100113300102020010013032	33(92) 34(332) 35(572) 36(250)
				31100130133101123321032232120212	37(52) 38(138) 39(178) 40(59)
				03302330330322002313310001032221	41(8) 42(18) 43(6) 44(7)

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