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MATEMATICS

On the new approach for the energy of elastica

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ABSTRACT. In this work, we firstly describe conditions for being elastica in Minkowski space E^4_1. Then we investigate the energy of the elastic curves and exploit its relationship with the energy of Bishop vectors belong to that elastic curves E^4_1. Finally, we characterize non-elastic curves in E^4_1 and compute their energy to see the distinction between energies for the curves of elastic and non-elastic case in Minkowski space E^4_1.Mathematics Subject Classifications: 53C41, 53A10.

Keywords: energy; Minkowski space; elastic curves; parallel vector fields.

Sobre a nova abordagem para a energia da elástica

RESUMO. Neste trabalho, descrevemos em primeiro lugar as condições de elástica no espaço de Minkowski E^4_1. Em seguida, investigamos a energia das curvas elásticas e exploramos sua relação com a energia do Triedo de Bishop pertencendo a essas curvas elásticas E^4_1. Finalmente, caracterizamos curvas não elásticas em E^4_1 e calculamos sua energia para ver a disticção entre as energias para as curvas do caso elástico e não elastico no espaço de Minkowski E^4_1.

Palavras-chave: energia; espaço de Minkowski; curvas elásticas; campos vetoriais paralelos.

Introduction

Materials having the feature of a deformable structure such as cloth, flexible metals, rubber, paper are the main subject and research field for the elasticity theory. However, elastica can be considered from a variety of the different perspective that enlights a broad range of physical and mathematical studies. Studies concerned about the elastica firstly focus on the research of mechanical equilibrium, the study of variational problems, and the solution of the elliptic integral.

One of the earliest approach on elastica yields prolific consequences on the equilibrium of moments which constitute elementary principle of statics. Further, it is seen that elastica gives a natural solution for the variational problem which deals with the minimizing of bending energy of the elastic curve. Later, the equivalence between the motion of the simple pendulum and elastica's fundamental differential equation was investigated. Recently, numerical computation implemented on the elastica used to develop mathematical spline theory (Love, 2013).

Potential elastic energy takes place when materials are stretched, compressed or deformed in any way. That is, these deformed bodies store potential energy when there exists a force on them. This potential energy is exerted to bring the deformed body back to its neutral position prior to deformation (Terzopoulost, Platt, Barr, & Fleischert, 1987). We carry studies on the potential energy of elastic curves into a Minkowski space E^4_1.

Minkowski spacetime or Minkowski space can be thought a combination of time dimension and Euclidean space into a four-dimensional manifold. This added time dimension makes a significant difference between Minkowski and Euclidean space, namely, we do not have coordinate dependence in Minkowski space as opposed to Euclidean space. This new space structure helps to understand better ofsome mathematical and also physical phenomena.

In this space, mass-energy equivalence states relationship between mass and energy. Special relativity attempts to estimate this equivalence by the formula $E = mc^2$, where c is the light's speed in a vacuum (Einstein, 1905). Thus we may have abetter understanding of mass-energy and motion-energy concepts if we compute the energy of particles in that space. For this purpose, (Körpinar & Demirkol, 2017) characterized the energy of a particle in Minkowski space E^4_1 for alternative parallel frame created firstly by R. L. Bishop (Bishop, 1975). The advantage and necessity of this frame is that it has no vanishing curvature, which solves a serious problem working on usual Bishop frame since zero curvature prevents to define normal and binormal vectors. (Altin, 2011) computed energy of Frenet vector fields for given non-lightlike curves. (Körpinar, 2014) discussed timelike biharmonic particle's energy in Heisenberg spacetime.

The manuscript of the paper as the following:

In this study, we approach the concept of the potential energy of the elastic materials from a different point of view. We firstly determine differential equations satisfied by non-rigid deformable curves in order to model the behaviorof elastic curves in 4-dimensional Minkowski space E_1^4. Then, we compute the energy of the elastic curves using a variational method in Bishop vectors according to different cases in E^4_1. The method we use for computing the energy of Bishop vector fields in this study is that considering a vector field as a map from manifold M to the Riemannian manifold (TM, p_s), where TM is tangent bundle of a Riemannian manifold and p_s is a Sasaki metric induced from TM naturally. Then, we construct a new equivalence including theenergy of elastic curves, theenergy of Bishop vectors and well-established formula known as bending energy functional fordifferent type of curves in E^4_1. Finally, we define non-elastic curves to characterize their structure which makes them different from elastic curves. Then, we discover a connection between the energies of elastic and non-elastic curves from point of geometrical view in E^4_1.

Material and methods

Minkowski space E_1^4 corresponds to four dimensional Euclidean space with the induced Lorentzian metric defined as Equation 1:

$$\langle p,u\rangle = -p_1u_1 + \sum p_iu_i \,|_{i=2,3,4} \tag{1}$$

where: $p,u \in R^4$ For an arbitrary curve $\boldsymbol{\alpha}: I \subset R \rightarrow E^4_1$ $\boldsymbol{\alpha} \in E^4_1$ is called a lightlike, timelike or spacelike curve if velocity vector of the curve satisfies $\langle \boldsymbol{\alpha}'(t), \boldsymbol{\alpha}'(t)\rangle = 0$, $\langle \boldsymbol{\alpha}'(t), \boldsymbol{\alpha}'(t)\rangle < 0$, $\langle \boldsymbol{\alpha}'(t), \boldsymbol{\alpha}'(t)\rangle > 0$ for each $t \in I$, respectively. Furthermore $\boldsymbol{\alpha}$ is named unit speed curve if $\|\boldsymbol{\alpha}'(t)\| = 1$. In this study, we only consider non-lightlike unit speed curves and use a pseudo orthonormal frame {T, $\mathbf{E_1}$, $\mathbf{E_2}$, $\mathbf{E_3}$} which is attained by Lorentzian rotation on Bishop frame.

Case 1: If $\boldsymbol{\alpha}: I \subset R \rightarrow E_1^4$ unit speed curve is timelike then $\mathbf{T}$ is timelike and parallel frame vectors $\mathbf{E_1}$, $\mathbf{E_2}$, $\mathbf{E_3}$ are spacelike. Thus, we have Equation 2:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{E}_1 + k_2\mathbf{E}_2 + k_3\mathbf{E}_3, \\ \nabla_{\mathbf{T}}\mathbf{E}_1 &= k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{E}_2 &= k_2\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{E}_3 &= k_3\mathbf{T}, \end{aligned} \tag{2}$$

where $\kappa = \sqrt{k_1^2 + k_2^2 + k_3^2}$ is defined as curvature and k_1, k_2 and k_3 denote principal curvatures of the curve $\boldsymbol{\alpha}$ according to parallel frame (Erdoğdu, 2015).

If $\boldsymbol{\alpha}: I \subset R \rightarrow E_1^4$ unit speed curve is spacelike, then $\mathbf{T}$ is spacelike. Therefore we have following Bishop equations with respect to the parallel frame vectors $\mathbf{E_1}$, $\mathbf{E_2}$, $\mathbf{E_3}$.

Case2: Let$\mathbf{T}$, $\mathbf{E_2}$, $\mathbf{E_3}$ are spacelike and $\mathbf{E_1}$ is a timelike for a unit speed curve $\boldsymbol{\alpha}$. Then we have Equation 3:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{E}_1 + k_2\mathbf{E}_2 + k_3\mathbf{E}_3, \\ \nabla_{\mathbf{T}}\mathbf{E}_1 &= k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{E}_2 &= -k_2\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{E}_3 &= -k_3\mathbf{T}, \end{aligned} \tag{3}$$

where $\kappa = \sqrt{\left|-k_1^2 + k_2^2 + k_3^2\right|}$ is defined as curvature and k_1, k_2 and k_3 denote principal curvatures of the curve $\boldsymbol{\alpha}$ according to parallel frame (Erdoğdu, 2015).

Case 3: Let $\mathbf{T}$, $\mathbf{E_1}$, $\mathbf{E_3}$ are spacelike and $\mathbf{E_2}$is a timelike for a unit speed curve$\boldsymbol{\alpha}$. Then we have Equation 4:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{E}_1 + k_2\mathbf{E}_2 + k_3\mathbf{E}_3, \\ \nabla_{\mathbf{T}}\mathbf{E}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{E}_2 &= k_2\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{E}_3 &= -k_3\mathbf{T}, \end{aligned} \tag{4}$$

where $\kappa = \sqrt{\left|k_1^2 - k_2^2 + k_3^2\right|}$ is defined as curvature and k_1, k_2 and k_3 denote principal curvatures of the curve $\boldsymbol{\alpha}$ according to parallel frame (Erdoğdu, 2015).

Case 4: Let $\mathbf{T}$, $\mathbf{E_1}$, $\mathbf{E_2}$ are spacelike and $\mathbf{E_3}$ is a timelike for a unit speed curve$\boldsymbol{\alpha}$. Then we have Equation 5:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{E}_1 + k_2\mathbf{E}_2 + k_3\mathbf{E}_3, \\ \nabla_{\mathbf{T}}\mathbf{E}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{E}_2 &= -k_2\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{E}_3 &= k_3\mathbf{T}, \end{aligned} \tag{5}$$

where $\kappa = \sqrt{\left|k_1^2 + k_2^2 - k_3^2\right|}$ is defined as curvature and k_1, k_2 and k_3 denote principal curvatures of the curve $\boldsymbol{\alpha}$ according to parallel frame (Erdoğdu, 2015).

Results and discussion

Energy on the Bishop vector field

We first give the fundamental definitions and propositions which are used to compute the energy of the vector field.

Definition3.1: For two Riemannian manifolds (M, p) and $(N, \widetilde{h})$ the energy of a differentiable map $f : (M, \rho) \to (N, \widetilde{h})$ can be defined as:

$$\varepsilon nergy(f) = \frac{1}{2}\int_M \sum_{a=1}^{n} \widetilde{h}\left(df(e_a), df(e_a)\right)v, \tag{6}$$

where $\{e_a\}$ is a local basis of the tangent space and v is the canonical volume form in M (Wood, 1997).

Definition 3.2: Let $Q : T(T^1M) \to T^1M$ be the connection map. Then following two conditions hold:i) $\omega \circ Q = \omega \circ d\omega$ and $\omega \circ Q = \omega \circ \widetilde{\omega}$, where $\widetilde{\omega} : T(T^1M) \to T^1M$ is the tangent bundle projection;ii)for $\rho \in T_xM$ and a section $\xi : M \to T^1M$; we have:

$$Q(d\xi(\rho)) = \nabla_\rho \xi, \tag{7}$$

where ∇ is the Levi-Civita covariant derivative (Wood, 1997).

Definition 3.3: For $\varsigma_1, \varsigma_2 \in T_\xi(T^1M)$, we define:

$$\rho_S(\varsigma_1, \varsigma_2) = \rho(d\omega(\varsigma_1), d\omega(\varsigma_2)) + \rho(Q(\varsigma_1), Q(\varsigma_2)). \tag{8}$$

This yields a Riemannian metric on TM. As known ρ_S is called the Sasaki metric that also makes the projection $\omega : T^1M \to M$ a Riemannian submersion (Wood, 1997).

Energy on the elastic curves

The research on the curvature-based energy for space curves began with Bernoulli and Euler's studies on elastic thin beams and rods. This type of energy is both essential in the mechanical context and also significant in computer vision, image processing and computer vision besides mathematical and physical importance.

Let $\boldsymbol{\alpha} \in E_1^4$ be a regular curve defined on any fixed interval $[y_1, y_2]$so thatwe have Equation 9:

$$\boldsymbol{\alpha} : [y_1, y_2] \to \mathrm{E}_1^4 \quad v = \left\|\boldsymbol{\alpha}'(t)\right\| = \frac{ds}{dt} \neq 0. \tag{9}$$

As an advantage of studying Minkowski space with parallel frame vectors, curvature of the curve $\boldsymbol{\alpha}$ is not vanish. Thus, elastica is defined for the curve $\boldsymbol{\alpha}$ in E_1^4 over the each point on a fixed interval $[y_1, y_2]$as a minimizer of the bending energy as in the Equation 10:

$$G = \frac{1}{2}\int_{y_1}^{y_2} \left\|\boldsymbol{\alpha}''\right\|^2 dt \tag{10}$$

with some boundary conditions (Guven, Valencia, &Vazquez-Montejo, 2014).

For any two points $p_1, p_2 \in R^4$ and any two non-zero vectors p_1, $p_2$space of smooth curves is defined as Equation 11:

$$\varphi = \left\{\boldsymbol{\alpha} : \boldsymbol{\alpha}(y_i) = p_i, \boldsymbol{\alpha}'(y_i) = p_i'\right\} \tag{11}$$

It is also defined the smooth curves of unit speed as a subspace of φ as the following way in the Equation 12:

$$\varphi_a = \left\{\boldsymbol{\alpha} \in \varphi : \left\|\boldsymbol{\alpha}'\right\| = 1\right\} \tag{12}$$

Then $G^\pi : \phi \to R$ can be defined by Equation 13:

$$G^\pi(\boldsymbol{\alpha}) = \frac{1}{2}\int_\alpha \left\|\boldsymbol{\alpha}''\right\|^2 + \Gamma(t)\left(\left\|\boldsymbol{\alpha}'\right\|^2 - 1\right)dt, \tag{13}$$

where $\Gamma(t)$ is a pointwise multiplier. A stationary point of G^π is the minimum of G on φ_a for some $\Gamma(t)$ according to multiplier principle of Lagrange.

Let $\boldsymbol{\alpha}$ be an extremum of G^π and V be a vector field along $\boldsymbol{\alpha}$, which is a curve's infinitesimal variation, then we get Equation 14 (Singer, 2007).

$$\partial G^\pi(V) = \frac{\partial}{\partial \Upsilon} G^\pi(\boldsymbol{\alpha} + \Upsilon V)\,|_{\Upsilon=0} = 0. \tag{14}$$

We obtain significant differences both on the conditions that have to be satisfied by elastica and on the energy of elastic curves by using Lorentzian metric for different type of curves in E_1^4.

Case 1: Let $\boldsymbol{\alpha} \in E_1^4$ be a unit speed timelike curve defined on any fixed interval $[y_1, y_2]$so that:

$$\boldsymbol{\alpha} : [y_1, y_2] \to E_1^4 \quad v = \left\|\boldsymbol{\alpha}'(t)\right\| = \frac{ds}{dt} \neq 0. \tag{15}$$

By using the pseudo orthonormal frame given by (Equation 2) we already computed the energy of tangent vector $\mathbf{T}$ and parallel frame vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ for timelike curve $\boldsymbol{\alpha} \in E_1^4$, (Körpinar & Demirkol, 2017). This study is helpful to see a relation between the energy of Bishop vectors and bending energy functional which is defined in theEquation 16:

$$G = \frac{1}{2}\int_{\boldsymbol{\alpha}} \left\|\nabla_{\mathbf{T}}\mathbf{T}\right\|^2 ds = \frac{1}{2}\int_{\boldsymbol{\alpha}} \kappa^2 ds, \tag{16}$$

where $\kappa = \sqrt{k_1^2 + k_2^2 + k_3^2}$ is defined as curvature and k_1, k_2 and k_3 denote principal curvatures of the curve $\boldsymbol{\alpha}$ according to parallel frame.

Let V be a vector field along $\boldsymbol{\alpha}$ such that it is a curve's infinitesimal variation. By using equations (Equation 13) and (Equation 14) we get Equation 17 and 18:

$$0 = \frac{1}{2}\frac{\partial}{\partial \Upsilon}\int_{y_1}^{y_2} \left\|(\boldsymbol{\alpha} + \Upsilon V)''\right\|^2 + \Gamma\left(\left\|(\boldsymbol{\alpha} + \Upsilon V)'\right\|^2 - 1\right) dt \tag{17}$$

$$= \int_{y_1}^{y_2} \left\langle \boldsymbol{\alpha}'', V'' \right\rangle dt - \int_{y_1}^{y_2} \Gamma \left\langle \boldsymbol{\alpha}', V' \right\rangle dt. \tag{18}$$

Applying integration by parts, we obtain Equation 19:

$$0 = \left\langle \boldsymbol{\alpha}'', V' \right\rangle - \left\langle V, \Gamma\boldsymbol{\alpha}' + \boldsymbol{\alpha}''' \right\rangle + \int_{y_1}^{y_2} \left\langle V, E(\boldsymbol{\alpha}) \right\rangle dt, \tag{19}$$

where $E(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'''' + \left(\Gamma\boldsymbol{\alpha}'\right)'$. Being elastica implies that we have Equation 20:

$$E(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'''' + \left(\Gamma\boldsymbol{\alpha}'\right)' \equiv 0 \tag{20}$$

for some function $\Gamma(t)$. Thanks to Noether's Theorem we know that from Equation 21:

$$J = \boldsymbol{\alpha}''' + \Gamma\boldsymbol{\alpha}' \tag{21}$$

is a constant vector field. For a parametrized curve $\boldsymbol{\alpha}$ with the arc-length s, we have Equation 22 and 23 if we consider the (Equation 2):

$$\boldsymbol{\alpha}' = \mathbf{T}, \quad \boldsymbol{\alpha}'' = k_1\mathbf{E}_1 + k_2\mathbf{E}_2 + k_3\mathbf{E}_3, \tag{22}$$

$$\boldsymbol{\alpha}''' = k_1'\mathbf{E}_1 + k_2'\mathbf{E}_2 + k_3'\mathbf{E}_3 + \kappa^2\mathbf{T}. \tag{23}$$

Thus we get Equation 24:

$$\mathrm{J} = k_1'\mathbf{E}_1 + k_2'\mathbf{E}_2 + k_3'\mathbf{E}_3 + \left(\kappa^2 + \Gamma\right)\mathbf{T}. \tag{24}$$

By the fact that J is a constant vector field we find $\mathrm{J}_s = 0$. From this, we have following Equation 25, 26, 27 and 28:

$$k_1'' + \left(k_1^2 + k_2^2 + k_3^2 + \Gamma\right)k_1 = 0, \tag{25}$$

$$k_2'' + \left(k_1^2 + k_2^2 + k_3^2 + \Gamma\right)k_2 = 0, \tag{26}$$

$$k_3'' + \left(k_1^2 + k_2^2 + k_3^2 + \Gamma\right)k_3 = 0, \tag{27}$$

$$3\left(k_1k_1' + k_2k_2' + k_3k_3' + \frac{\Gamma'}{3}\right) = 0, \tag{28}$$

and if we solve them we will get $\Gamma(s) = \frac{-3}{2}\kappa^2 + \frac{\Omega}{2}$, for some constant Ω. Finally we get a vector field J along the curve and some other restrictions as stated in the following Equation 29, 30, 31 and 32, respectively.

$$\mathrm{J} = \frac{\Omega - \kappa^2}{2}\mathbf{T} + k_1'\mathbf{E}_1 + k_2'\mathbf{E}_2 + k_3'\mathbf{E}_3, \tag{29}$$

$$0 = k_1'' - \frac{k_1}{2}\left(k_1^2 + k_2^2 + k_3^2 - \Omega\right), \tag{30}$$

$$0 = k_2'' - \frac{k_2}{2}\left(k_1^2 + k_2^2 + k_3^2 - \Omega\right), \tag{31}$$

$$0 = k_3'' - \frac{k_3}{2}\left(k_1^2 + k_2^2 + k_3^2 - \Omega\right) \tag{32}$$

If we assume that we have Equation 33:

$$k_1^2 + k_2^2 + k_3^2 - \Omega = \sin s, \tag{33}$$

then we can solve the differential equation system and get the following plot for the sample solution family (Figure 1).

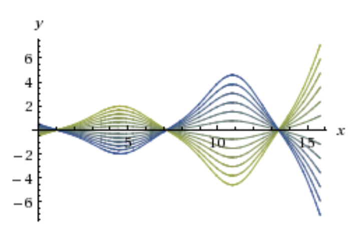

Figure 1. Sample solution family.

Theorem 3.4: Constant vector field's energy by using Sasaki metric is stated byEquation 34:

$$\varepsilon nergy_1(J) = \frac{-s}{2}. \tag{34}$$

Proof: From Equation 6 and 7 we obtain Equation 35:

$$\varepsilon nergy_1(J) = \frac{1}{2}\int_0^s \rho_S\left(dJ(\mathbf{T}), dJ(\mathbf{T})\right) ds. \tag{35}$$

Using the Equation 8, we obtain the Equation 36:

$$\rho_S\left(dJ(\mathbf{T}),dJ(\mathbf{T})\right)=\rho(d\omega(J(\mathbf{T})),d\omega(J(\mathbf{T})))+\rho(Q(J(\mathbf{T})),Q(J(\mathbf{T}))). \quad (36)$$

Since J is a section, we get Equation 37:

$$d(\omega)\circ d(J)=d(\omega\circ J)=d(id_C)=id_{TC}. \quad (37)$$

We also know that from Equation 38:

$$Q(J(\mathbf{T}))=\nabla_{\mathbf{T}}J=0. \quad (38)$$

Thus, we find from the former statement to the Equation 39:

$$\rho_S\left(dJ(\mathbf{T}),dJ(\mathbf{T})\right)=\rho\left(\mathbf{T},\mathbf{T}\right)+\rho\left(\nabla_{\mathbf{T}}J,\nabla_{\mathbf{T}}J\right) \\ =-1. \quad (39)$$

So we can easily obtain Equation 40 as in the following form:

$$\varepsilon nergy_1\left(J\right)=\frac{-s}{2}. \quad (40)$$

This completes the proof.

Corollary 3.5: For a unit speed timelike curve $\alpha\in E_1^4$, we have following relationgiven in the Equation 41:

$$\varepsilon nergy_1\left(J\right)+G=\varepsilon nergy\left(\mathbf{T}\right) \quad (41)$$

Case 2: Unit speed spacelike curve $\boldsymbol{\alpha}\in E_1^4$ with the characterization of spacelike vectors $\mathbf{T}$, $\mathbf{E}_2$, $\mathbf{E}_3$ and timelike vector $\mathbf{E}_1$ on any fixed interval $[y_1, y_2]$ is defined in the Equation 42:

$$\boldsymbol{\alpha}:\left[y_1,y_2\right]\rightarrow E_1^4 \quad v=\left\|\boldsymbol{\alpha}'\left(t\right)\right\|=\frac{ds}{dt}\neq 0. \quad (42)$$

By using the pseudo orthonormal frame given by (Equation 3) we already computed the energy of spacelike vectors $\mathbf{T}$, $\mathbf{E}_2$, $\mathbf{E}_3$ and timelike vector $\mathbf{E}_1$, (Körpinar & Demirkol, 2017). This study is helpful to see a relation between the energy of Bishop vectors and bending energy functional which is defined in theEquation 43:

$$G=\frac{1}{2}\int_{\boldsymbol{\alpha}}\left\|\nabla_{\mathbf{T}}\mathbf{T}\right\|^2 ds=\frac{1}{2}\int_{\boldsymbol{\alpha}}\kappa^2 ds, \quad (43)$$

where $\kappa=\sqrt{\left|-k_1^2+k_2^2+k_3^2\right|}$ is defined as curvature and k_1, k_2 and k_3 denote principal curvatures of the curve $\boldsymbol{\alpha}$ according to parallel frame.

Let V be a vector field along $\boldsymbol{\alpha}$ such that it is a curve's infinitesimal variation. By using Equation 13 and 14 we get Equation 44 and 45:

$$0=\frac{1}{2}\frac{\partial}{\partial\Upsilon}\int_{y_1}^{y_2}\left\|\left(\boldsymbol{\alpha}+\Upsilon V\right)''\right\|^2+\Gamma\left(\left\|\left(\boldsymbol{\alpha}+\Upsilon V\right)'\right\|^2-1\right)dt \quad (44)$$

$$=\int_{y_1}^{y_2}\left\langle\boldsymbol{\alpha}'',V''\right\rangle dt+\int_{y_1}^{y_2}\Gamma\left\langle\boldsymbol{\alpha}',V'\right\rangle dt. \quad (45)$$

Applying integration by parts we obtain Equation 46:

$$0=\left\langle\boldsymbol{\alpha}'',V'\right\rangle+\left\langle V,\Gamma\boldsymbol{\alpha}'-\boldsymbol{\alpha}'''\right\rangle+\int_{y_1}^{y_2}\left\langle V,E\left(\boldsymbol{\alpha}\right)\right\rangle dt, \quad (46)$$

where $E\left(\boldsymbol{\alpha}\right)=\boldsymbol{\alpha}''''-\left(\Gamma\boldsymbol{\alpha}'\right)'$. So being elastica implies that we have Equation 47:

$$E\left(\boldsymbol{\alpha}\right)=\boldsymbol{\alpha}''''-\left(\Gamma\boldsymbol{\alpha}'\right)'\equiv 0 \quad (47)$$

for some function $\Gamma(t)$. Thanks to Noether's Theorem we know that Equation 48 satisfies that:

$$J=\boldsymbol{\alpha}'''-\Gamma\boldsymbol{\alpha}' \quad (48)$$

is a constant vector field. For a parametrized curve $\boldsymbol{\alpha}$ with the arc-length s, we have Equation 49 and 50 from the (Equation 3):

$$\boldsymbol{\alpha}'=\mathbf{T},\quad \boldsymbol{\alpha}''=k_1\mathbf{E}_1+k_2\mathbf{E}_2+k_3\mathbf{E}_3, \quad (49)$$

$$\boldsymbol{\alpha}'''=k_1'\mathbf{E}_1+k_2'\mathbf{E}_2+k_3'\mathbf{E}_3+\kappa^2\mathbf{T}. \quad (50)$$

Thus we get Equation 51:

$$J=k_1'\mathbf{E}_1+k_2'\mathbf{E}_2+k_3'\mathbf{E}_3+\left(\kappa^2-\Gamma\right)\mathbf{T}. \quad (51)$$

By the fact that J is a constant vector field we find $J_s = 0$. From this, we have following Equation 52, 53, 54 and 55:

$$k_1''+\left(-k_1^2+k_2^2+k_3^2-\Gamma\right)k_1=0, \quad (52)$$

$$k_2''+\left(-k_1^2+k_2^2+k_3^2-\Gamma\right)k_2=0, \quad (53)$$

$$k_3''+\left(-k_1^2+k_2^2+k_3^2-\Gamma\right)k_3=0, \quad (54)$$

$$3\left(k_1k_1'-k_2k_2'-k_3k_3'-\frac{\Gamma'}{3}\right)=0, \quad (55)$$

and if we solve it we will get $\Gamma(s)=\frac{3}{2}\kappa^2+\frac{\Omega}{2}$, for some constant Ω. Finally we get a vector field J along the curve and some other restrictions as stated in the following Equation 56, 57, 58 and 59, respectively.

$$J=\frac{-\Omega-\kappa^2}{2}\mathbf{T}+k_1'\mathbf{E}_1+k_2'\mathbf{E}_2+k_3'\mathbf{E}_3, \tag{56}$$

$$0=k_1''-\frac{k_1}{2}\left(-k_1^2+k_2^2+k_3^2+\Omega\right), \tag{57}$$

$$0=k_2''-\frac{k_2}{2}\left(-k_1^2+k_2^2+k_3^2+\Omega\right), \tag{58}$$

$$0=k_3''-\frac{k_3}{2}\left(-k_1^2+k_2^2+k_3^2+\Omega\right). \tag{59}$$

If we assume that we have Equation 60:

$$-k_1^2+k_2^2+k_3^2+\Omega=\cos s, \tag{60}$$

then we can solve the differential equation system and get the following plot for the sample solution family (Figure 2).

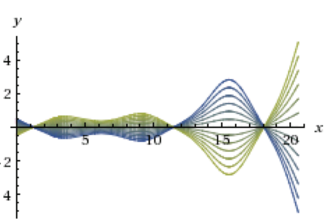

Figure 2. Sample solution family.

Theorem 3.6: Constant vector field's energy by using Sasaki metric is stated by Equation 61:

$$\varepsilon nergy_2(J)=\frac{s}{2}. \tag{61}$$

Corollary 3.7: For a unit speed spacelike curve with the given Bishop characters we have the following Equation 62:

$$\varepsilon nergy_2(J)+G=\varepsilon nergy(\mathbf{T}) \tag{62}$$

Case 3: Let α be a unit speed vector with the Bishop characterization of spacelike vectors **T, E$_1$, E$_3$** and timelike vector **E$_2$**. For a vector field V which is an infinitesimal variation of the curve α, we have constant vector field J and some restrictions as the following Equation 63, 64, 65 and 66:

$$J=\frac{-\Omega-\kappa^2}{2}\mathbf{T}+k_1'\mathbf{E}_1+k_2'\mathbf{E}_2+k_3'\mathbf{E}_3, \tag{63}$$

$$0=k_1''-\frac{k_1}{2}\left(k_1^2-k_2^2+k_3^2+\Omega\right), \tag{64}$$

$$0=k_2''-\frac{k_2}{2}\left(k_1^2-k_2^2+k_3^2+\Omega\right), \tag{65}$$

$$0=k_3''-\frac{k_3}{2}\left(k_1^2-k_2^2+k_3^2+\Omega\right). \tag{66}$$

If we assume that we have Equation 67:

$$k_1^2-k_2^2+k_3^2+\Omega=\log s, \tag{67}$$

then we can solve the differential equation system and get the following plot for the sample solution family (Figure 3).

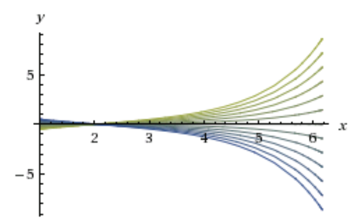

Figure 3. Sample solution family.

Theorem 3.8: Constant vector field's energy by using Sasaki metric is stated by Equation 68:

$$\varepsilon nergy_3(J)=\frac{s}{2}. \tag{68}$$

Corollary 3.9: For a unit speed spacelike curve with the given Bishop characters we have the Equation 69:

$$\varepsilon nergy_3(J)+G=\varepsilon nergy_3(\mathbf{T}). \tag{69}$$

Case 4: Let α be a unit speed vector with the Bishop characterization of spacelike vectors T, E$_1$, E$_3$ and timelike vector E$_3$. For a vector field V which is an infinitesimal variation of the curve α, we have constant vector field J and some restrictions as the following Equation 70, 71, 72 and 73:

$$J=\frac{-\Omega-\kappa^2}{2}\mathbf{T}+k_1'\mathbf{E}_1+k_2'\mathbf{E}_2+k_3'\mathbf{E}_3, \tag{70}$$

$$0=k_1''-\frac{k_1}{2}\left(k_1^2+k_2^2-k_3^2+\Omega\right), \tag{71}$$

$$0=k_2''-\frac{k_2}{2}\left(k_1^2+k_2^2-k_3^2+\Omega\right), \tag{72}$$

$$0=k_3''-\frac{k_3}{2}\left(k_1^2+k_2^2-k_3^2+\Omega\right). \tag{73}$$

If we assume that we have Equation 74:

$$k_1^2+k_2^2-k_3^2+\Omega=\arcsin s, \tag{74}$$

then we can solve the differential equation system and get the following plot for the sample solution family (Figure 4).

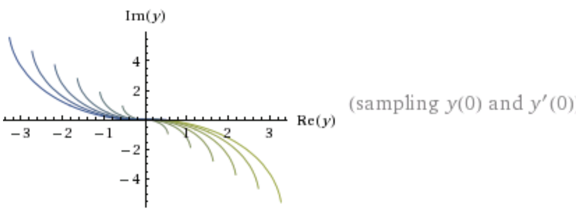

Figure 4. Sample solution family.

Theorem 3.10: Constant vector field's energy by using Sasaki metric is stated by Equation 75:

$$\varepsilon nergy_4(J)=\frac{s}{2}. \tag{75}$$

Corollary 3.11: For a unit speed spacelike curve with the given Bishop characters we have the following Equation 76:

$$\varepsilon nergy_4(J)+G=\varepsilon nergy(\mathbf{T}). \tag{76}$$

Conclusion

In this section, we deal with the concept of non-elastic curve and their energy for different type of curves in E_1^4.

Case 1: Let $\boldsymbol{\alpha}\in E_1^4$ be a unit speed timelike curve defined on any fixed interval $[y_1, y_2]$ so that it has the Bishop characterization same as in Equation 2. For a vector field V, which is an infinitesimal variation of the curve $\boldsymbol{\alpha}$, by using Equation 13 and 14 we get Equation 77:

$$0=\left\langle \boldsymbol{\alpha}'',V'\right\rangle-\left\langle V,\Gamma\boldsymbol{\alpha}'+\boldsymbol{\alpha}'''\right\rangle+\int_{y_1}^{y_2}\left\langle V,E(\boldsymbol{\alpha})\right\rangle dt, \tag{77}$$

where $E(\boldsymbol{\alpha})=\boldsymbol{\alpha}''''+\left(\Gamma\boldsymbol{\alpha}'\right)'$, for some function $\Gamma(t)$. As opposed to Equation 20, if we assume that the curve is not elastica then for $\Gamma(s)\neq\frac{-3}{2}\kappa^2+\frac{\Omega}{2}$, for some constant Ω, we will have Equation 78 and 79:

$$E(\boldsymbol{\alpha})=\left(k_1''+k_1\left(k_1^2+k_2^2+k_3^2+\Gamma\right)\right)\mathbf{E}_1+\left(k_2''+k_2\left(k_1^2+k_2^2+k_3^2+\Gamma\right)\right)\mathbf{E}_2 \tag{78}$$

$$+\left(k_3''+k_3\left(k_1^2+k_2^2+k_3^2+\Gamma\right)\right)\mathbf{E}_3+3\left(k_1k_1'+k_2k_2'+k_3k_3'+\frac{\Gamma'}{3}\right)\mathbf{T} \tag{79}$$

for non-elastic curve $\boldsymbol{\alpha}$, which is parametrized by the arc-length s.

Theorem 4.1:Energy of non-elastic curve by using Sasaki metric is stated by Equation 80, 81 and 82:

$$\varepsilon nergy_1\left(E(\boldsymbol{\alpha})\right)=\frac{-s}{2}+\frac{1}{2}\int_0^s\left(\left(k_1''+k_1\left(k_1^2+k_2^2+k_3^2+\Gamma\right)\right)^2\right. \tag{80}$$

$$+\left(k_2''+k_2\left(k_1^2+k_2^2+k_3^2+\Gamma\right)\right)^2 \tag{81}$$

$$\left.+\left(k_3''+k_3\left(k_1^2+k_2^2+k_3^2+\Gamma\right)\right)^2-9\left(k_1k_1'+k_2k_2'+k_3k_3'+\frac{\Gamma'}{3}\right)^2\right)ds, \tag{82}$$

Example 1:If we takethe values given in the Equation 83:

$$k_1=s^2,\ k_2=s^3,\ k_3=0,\ \Gamma=1, \tag{83}$$

then we have a following graph for the energy of non-elastic timelike particle (Figure 5).

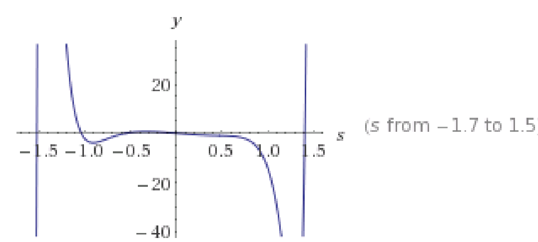

Figure 5. Energy of non-elastic timelike particle.

Corollary 4.2 For a unit speed timelike curve with the given Bishop character we have the followingrelations given by Equation 84, 85 and 86:

$$\varepsilon nergy_1\left(E(\boldsymbol{\alpha})\right)-\varepsilon nergy_1(J)=\frac{1}{2}\int_0^s\left(\left(k_1''+k_1\left(k_1^2+k_2^2+k_3^2+\Gamma\right)\right)^2\right. \tag{84}$$

$$+\left(k_2''+k_2\left(k_1^2+k_2^2+k_3^2+\Gamma\right)\right)^2 \tag{85}$$

$$\left.+\left(k_3''+k_3\left(k_1^2+k_2^2+k_3^2+\Gamma\right)\right)^2-9\left(k_1k_1'+k_2k_2'+k_3k_3'+\frac{\Gamma'}{3}\right)^2\right)ds. \tag{86}$$

Case 2: Let $\boldsymbol{\alpha}\in E_1^4$ be a unit speed spacelike curve defined on any fixed interval $[y_1, y_2]$ so that it has the Bishop characterization same as in Equation 3, 4 and 5; respectively. For a vector field V, which is an infinitesimal variation of the curve $\boldsymbol{\alpha}$, by using Equation 13 and 14 we get Equation 87:

$$0=\left\langle \boldsymbol{\alpha}'',V'\right\rangle+\left\langle V,\Gamma\boldsymbol{\alpha}'-\boldsymbol{\alpha}'''\right\rangle+\int_{y_1}^{y_2}\left\langle V,E(\boldsymbol{\alpha})\right\rangle dt, \tag{87}$$

where $E(\boldsymbol{\alpha})=\boldsymbol{\alpha}''''-\left(\Gamma\boldsymbol{\alpha}'\right)'$, for some function $\Gamma(t)$. As opposed to Equation 47, if we assume that the curve is not elastica then for $\Gamma(s)=\frac{3}{2}\kappa^2+\frac{\Omega}{2}$, for some constant Ω, we will have Equation 88 and 89:

$$E(\boldsymbol{\alpha})=\left(k_1''+k_1\left(\kappa^2-\Gamma\right)\right)\mathbf{E}_1+\left(k_2''+k_2\left(\kappa^2-\Gamma\right)\right)\mathbf{E}_2 \tag{88}$$

$$+\left(k_3'' + k_3\left(\kappa^2 - \Gamma\right)\right)\mathbf{E}_3 + 3\left(k_1 k_1' + k_2 k_2' + k_3 k_3' - \frac{\Gamma'}{3}\right)\mathbf{T} \quad (89)$$

for non-elastic curve $\boldsymbol{\alpha}$, which is parametrized by the arc-length s.

Theorem 4.3: Energy of non-elastic curve that has the Bishop characterization as in Equation 3, 4 and 5 can be given respectively by using Sasaki metric as the following way by Equation 90, 91 and 92:

$$\varepsilon nergy_2\left(E\left(\boldsymbol{\alpha}\right)\right) = \frac{1}{2}s + \frac{1}{2}\int_0^s \Big(-\left(k_1'' + k_1\left(-k_1^2 + k_2^2 + k_3^2 - \Gamma\right)\right)^2 + \left(k_2'' + k_2\left(-k_1^2 + k_2^2 + k_3^2 - \Gamma\right)\right)^2 + \left(k_3'' + k_3\left(-k_1^2 + k_2^2 + k_3^2 - \Gamma\right)\right)^2 + 9\left(k_1 k_1' + k_2 k_2' + k_3 k_3' - \frac{\Gamma'}{3}\right)^2\Big)ds, \quad (90)$$

$$\varepsilon nergy_3\left(E\left(\boldsymbol{\alpha}\right)\right) = \frac{1}{2}s + \frac{1}{2}\int_0^s \Big(\left(k_1'' + k_1\left(k_1^2 - k_2^2 + k_3^2 - \Gamma\right)\right)^2 - \left(k_2'' + k_2\left(k_1^2 - k_2^2 + k_3^2 - \Gamma\right)\right)^2 + \left(k_3'' + k_3\left(k_1^2 - k_2^2 + k_3^2 - \Gamma\right)\right)^2 + 9\left(k_1 k_1' + k_2 k_2' + k_3 k_3' - \frac{\Gamma'}{3}\right)^2\Big)ds, \quad (91)$$

$$\varepsilon nergy_4\left(E\left(\boldsymbol{\alpha}\right)\right) = \frac{1}{2}s + \frac{1}{2}\int_0^s \Big(\left(k_1'' + k_1\left(k_1^2 + k_2^2 - k_3^2 - \Gamma\right)\right)^2 + \left(k_2'' + k_2\left(k_1^2 + k_2^2 - k_3^2 - \Gamma\right)\right)^2 - \left(k_3'' + k_3\left(k_1^2 + k_2^2 - k_3^2 - \Gamma\right)\right)^2 + 9\left(k_1 k_1' + k_2 k_2' + k_3 k_3' - \frac{\Gamma'}{3}\right)^2\Big)ds. \quad (92)$$

Example 2: If we take the values given in the Equation 93:

$$k_1 = s^2,\ k_2 = s^3,\ k_3 = 0,\ \Gamma = 1, \quad (93)$$

then we have a following graph respectively for the energy of non-elastic spacelike particle with the Bishop characterization Equation 3, 4 and 5 (Figure 6).

Corollary 4.4: For a unit speed spacelike curve with the given Bishop characters as in Equation 3, 4 and 5 we have the following Equation 94, 95 and 96, respectively:

$$\varepsilon nergy_2\left(E\left(\boldsymbol{\alpha}\right)\right) - \varepsilon nergy_2\left(J\right) = \frac{1}{2}\int_0^s \Big(-\left(k_1'' + k_1\left(-k_1^2 + k_2^2 + k_3^2 - \right.\right.$$
$$+\left(k_2'' + k_2\left(-k_1^2 + k_2^2 + k_3^2 - \Gamma\right)\right)^2 + \left(k_3'' + k_3\left(-k_1^2 + k_2^2 + k_3^2 - \Gamma\right)\right)^2 + 9\left(k_1 k_1' + k_2 k_2' + k_3 k_3' - \frac{\Gamma'}{3}\right)^2\Big)ds, \quad (94)$$

$$\varepsilon nergy_3\left(E\left(\boldsymbol{\alpha}\right)\right) - \varepsilon nergy_3\left(J\right) = \frac{1}{2}\int_0^s \Big(\left(k_1'' + k_1\left(k_1^2 - k_2^2 + k_3^2 - \right.\right.$$
$$-\left(k_2'' + k_2\left(k_1^2 - k_2^2 + k_3^2 - \Gamma\right)\right)^2 + \left(k_3'' + k_3\left(k_1^2 - k_2^2 + k_3^2 - \Gamma\right)\right)^2 + 9\left(k_1 k_1' + k_2 k_2' + k_3 k_3' - \frac{\Gamma'}{3}\right)^2\Big)ds, \quad (95)$$

$$\varepsilon nergy_4\left(E\left(\boldsymbol{\alpha}\right)\right) - \varepsilon nergy_4\left(J\right) = \frac{1}{2}\int_0^s \Big(\left(k_1'' + k_1\left(k_1^2 + k_2^2 - k_3^2 - \right.\right.$$
$$+\left(k_2'' + k_2\left(k_1^2 + k_2^2 - k_3^2 - \Gamma\right)\right)^2 - \left(k_3'' + k_3\left(k_1^2 + k_2^2 - k_3^2 - \Gamma\right)\right)^2 + 9\left(k_1 k_1' + k_2 k_2' + k_3 k_3' - \frac{\Gamma'}{3}\right)^2\Big)ds. \quad (96)$$

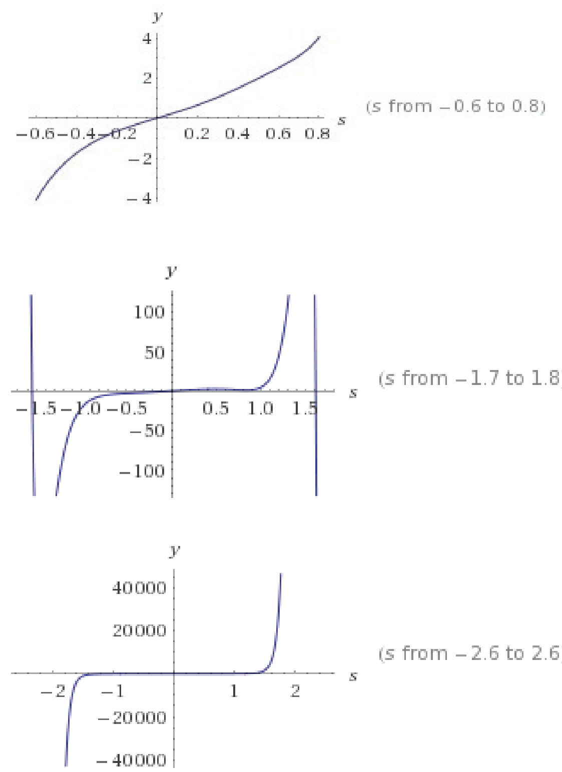

Figure 6. Energy of non-elastic spacelike particle.

Corollary 4.5: If the energy of non-elastic curve is constant for each $\varepsilon nergy_i\left(E\left(\boldsymbol{\alpha}\right)\right)$, where i = 1, 2, 3, 4 then we have following statements given by Equation 97, 98, 99 and 100:

If $\frac{d}{ds}\left(\varepsilon nergy_1\left(E\left(\boldsymbol{\alpha}\right)\right)\right) = 0,$ then

$$\left(k_1'' + k_1\left(k_1^2 + k_2^2 + k_3^2 + \Gamma\right)\right)^2 + \left(k_2'' + k_2\left(k_1^2 + k_2^2 + k_3^2 + \Gamma\right)\right)^2 + \left(k_3'' + k_3\left(k_1^2 + k_2^2 + k_3^2 + \Gamma\right)\right)^2 - 9\left(k_1 k_1' + k_2 k_2' + k_3 k_3' + \frac{\Gamma'}{3}\right)^2 = 1. \quad (97)$$

If $\frac{d}{ds}\left(\varepsilon nergy_2\left(E\left(\boldsymbol{\alpha}\right)\right)\right) = 0,$ then

$$-\left(k_1'' + k_1\left(-k_1^2 + k_2^2 + k_3^2 - \Gamma\right)\right)^2 + \left(k_2'' + k_2\left(-k_1^2 + k_2^2 + k_3^2 - \Gamma\right)\right)^2 + \left(k_3'' + k_3\left(-k_1^2 + k_2^2 + k_3^2 - \Gamma\right)\right)^2 + 9\left(k_1 k_1' + k_2 k_2' + k_3 k_3' - \frac{\Gamma'}{3}\right)^2 = -1 \quad (98)$$

If $\frac{d}{ds}\left(\varepsilon nergy_3\left(E\left(\boldsymbol{\alpha}\right)\right)\right) = 0,$ then

$$\left(k_1'' + k_1\left(k_1^2 - k_2^2 + k_3^2 - \Gamma\right)\right)^2 - \left(k_2'' + k_2\left(k_1^2 - k_2^2 + k_3^2 - \Gamma\right)\right)^2 + \left(k_3'' + k_3\left(k_1^2 - k_2^2 + k_3^2 - \Gamma\right)\right)^2 + 9\left(k_1 k_1' + k_2 k_2' + k_3 k_3' - \frac{\Gamma'}{3}\right)^2 = -1 \quad (99)$$

If $\frac{d}{ds}\left(\varepsilon nergy_4\left(E(\boldsymbol{\alpha})\right)\right)=0,$ then

$$\left(k_1^{''}+k_1\left(k_1^2+k_2^2-k_3^2-\Gamma\right)\right)^2+\left(k_2^{''}+k_2\left(k_1^2+k_2^2-k_3^2-\Gamma\right)\right)^2 \quad (100)$$
$$-\left(k_3^{''}+k_3\left(k_1^2+k_2^2-k_3^2-\Gamma\right)\right)^2+9(k_1k_1^{'}+k_2k_2^{'}+k_3k_3^{'}-\frac{\Gamma^{'}}{3})^2=-1$$

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