Mathematical properties, application and simulation for the exponentiated generalized standardized Gumbel distribution

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ABSTRACT. In this paper, we study the two-parameter exponentiated generalized standardized Gumbel distribution, which consists of a simple generalization of the Gumbel distribution. We investigate the power of fit of the proposed distribution to real data and study via Monte Carlo simulation the behavior of the MLEs for the model parameters. We provide a comprehensive mathematical treatment and prove that the formulas related to the new model are simple and manageable.

Keywords: exponentiated generalized distribution; standardized Gumbel distribution.

Introduction

The statistical literature is replete with illustrations in which the Gumbel model is used effectively to explain real phenomena. In the area of climate modeling, for example, some applications of the Gumbel model include global warming problems, offshore modeling, and rainfall and wind speed modeling (Nadarajah, 2006). Here, it is worth mentioning some recent works that consider the Gumbel distribution in different contexts: Nadarajah (2006), Cordeiro, Ortega, and Cunha (2013) and Andrade, Rodrigues, Bourguignon, and Cordeiro (2015). The cumulative distribution function (cdf) of the Gumbel (Gu) distribution is given by Equation 1:

$$G_1(t;\mu,\sigma) = \exp\left[-\exp\left(-\frac{t-\mu}{\sigma}\right)\right], \,(1)$$

for $t \in IR$, $\mu \in IR$ and $\sigma>0$. In a recent paper, Cordeiro et al. (2013) proposed a generalization for the Gumbel model using the so-called exponentiated generalized (EG) class of distributions defined by the cdf expressed as Equation 2:

$$F(x) = \{1-[1-G(x)]^a\}^b, \,(2)$$

where $a > 0$ and $b > 0$.

The probability density function (pdf) corresponding to Equation 2 is given by Equation 3:

$$f(x) = ab[1-G(x)]^{a-1}\{1-[1-G(x)]^a\}^{b-1}g(x), \,(3)$$

where $g(x) = dG(x)/dx$ is the baseline pdf. Thus, Cordeiro et al. (2013) studied the so-called exponentiated generalized Gumbel (EGGu, for short) distribution by inserting Equation 1 into Equation 2. Later, Andrade et al. (2015) investigated in detail several mathematical properties for the EGGu model. In this paper, we will follow the methodology used by Cordeiro et al. (2013) and Andrade et al. (2015), but we will consider a standardized version of the Gumbel distribution, the so-called standardized Gumbel (SGu) model. Let T be a random variable having cdf Equation 1. The SGu distribution is defined by a linear transformation $X = (T - \mu)/\sigma$, where $\mu \in IR$ and $\sigma>0$. Without loss of generality, we can work with the SGu model, since $T = \mu + \sigma X$, where $\mu \in IR$ and $\sigma>0$. The $G(x)$ and pdf $g(x)$ of the SGu distribution are given by Equation 4 and 5:

$$G(x) = \exp[-\exp(-x)] \,(4)$$

$$g(x) = \exp[-x - \exp(-x)] \ (5)$$

respectively, for $x \in IR$. The goal is to show that our simplified version of the EGGu model has great power of fit to real data and good simulation properties, with the advantage of having only two parameters. Therefore, we define the exponentiated generalized standardized Gumbel (EGSGu) distribution by inserting Equation 4 into Equation 2. The cdf $F(x)$ and pdf $f(x)$ of the EGSGu distribution are given by Equation 6 and 7:

$$F(x) = (1 - \{1 - \exp[-\exp(-x)]\}^a)^b \ (6)$$

$$f(x) = ab \exp[-x - \exp(-x)]\left\{1 - \exp\left[-\exp(-x)\right]\right\}^{a-1}\left\{1 - \left[1 - \exp\left[-\exp(-x)\right]\right]^a\right\}^{b-1}, \ (7)$$

where $a > 0$ and $b > 0$. The two extra parameters a and b in the density Equation 7 can control both tail weights, enabling the generation of flexible distributions, with heavier or lighter tails, as appropriate. There is also an attractive physical interpretation of the model Equation 7 when a and b are positive integers: Suppose initially that a certain device is composed of b components in a parallel system. Consider also that, for each component b, there exists a series of subcomponents a independent and distributed according to the SGu model. Suppose also that each component b fails if some a subcomponent fails. Let $X_{j1}, ..., X_{ja}$ denote the lifetimes of the subcomponents within the jth component, $j = 1, ..., b$ with common cdf SGu. Let X_j denote the lifetime of the j component and let X denote the lifetime of the device. Hence:

$$P(X \le x) = P(X_1 \le x, ..., X_b \le x) = P(X_1 \le x)^b = [1 - P(X_1 > x)]^b$$
$$= [1 - P(X_{11} > x, ..., X_{1a} > x)]^b = [1 - P(X_{11} > x)^a]^b$$
$$= [1 - \{1 - P(X_{11} \le x)\}^a]^b.$$

Thus, the lifetime of the device obeys the EGSGu family of distributions. Besides this introduction, the paper is organized as follows. In the Material and Methods section, we investigated the quantile function and its applications. Next, several mathematical properties of the new model are derived and numerical studies are detailed. In the Results and Discussion section, we used a real dataset to empirically show the power of fit of the EGSGu model and presented the Monte Carlo simulation study.

Material and methods

Quantile function

As an additional characterization of X, we define in this section the quantile function (qf) of the EGSGu model. This function comes directly from the inversion of the cdf Equation 6. Thus, the qf of X is given by Equation 8:

$$Q(u) = -\log\left\{-\log\left[1 - (1 - u^{1/b})^{1/a}\right]\right\}, \ (8)$$

where $u \in (0,1)$. There are many important practical applications for Equation 8. For example, occurrences of the random variable X can be easily obtained from a uniform random variable U by $X = Q(U)$. Next, we use Equation 8 to simulate 200 EGSGu (3, 2) occurrences. Figure 1 gives the EGSGu (3, 2) pdf, histogram, exact and empirical cdfs for these simulated data.

In addition, to illustrate the practical utility of Equation 8, it should be mentioned that it can be used to determine the median of X as Med $= Q(1/2)$. Table 1 below presents a small simulation study, whose objective is to compare the empirical medians (Med) generated for different parameter values and random samples of size $n = 50, 100, 150$, with their corresponding theoretical medians (Med) obtained by $Q(1/2)$. The simulation process is performed in the software R and, to ensure the reproducibility of the experiment, we use the seed for the random number generator: set.seed (103). As expected, the difference between EMed and Med decreases when n increases.

Finally, we present a third application for qf Equation 8, which consists in obtaining in the classical measures of asymmetry and kurtosis of X the Bowley skewness (Kenney & Keeping, 1962) (B) and Moors kurtosis (Moors, 1988) (M). In Figure 2 and 3, we present 3D plots of the M and B measures for selected parameter values, respectively. These plots are obtained using the software 'Wolfram Mathematica'. Based on these plots, it is possible to conclude that changes in the additional parameters a and b have a considerable impact on the

skewness and kurtosis of the EGSGu model, thus corroborating its greater flexibility. Hence, theses plots reinforce the importance of the additional parameters.

Properties of the EGSGu distribution

In this section, we study the structural properties of the EGSGu model.

Linear representations

For an arbitrary baseline cdf $G(x)$, a random variable Y_c has the exp-G distribution with power parameter $c > 0$ say $Y_c \sim$ exp-G if its cdf and pdf are given by $H_c(x) = G(x)^c$ and $h_c(x) = cg(x)G(x)^{c-1}$ respectively. For a comprehensive discussion about the exponentiated distributions, see a recent paper by Tahir and Nadarajah (2015). Based on some results in Cordeiro and Lemonte (2014), we can express the EG cdf Equation 2 as Equation 9.

$$F(x) = \sum_{j=0}^{\infty} w_{j+1} H_{j+1}(x), \quad (9)$$

where $w_{j+1} = \sum_{m=1}^{\infty} (-1)^{j+m+1} \binom{b}{m}\binom{ma}{j+1}$ and $H_{j+1}(x) = G(x)^{j+1}$ is the exp-G cdf with power parameter $j+1$. By differentiating Equation 9, we obtain a similar linear representation for $f(x)$ as Equation 10.
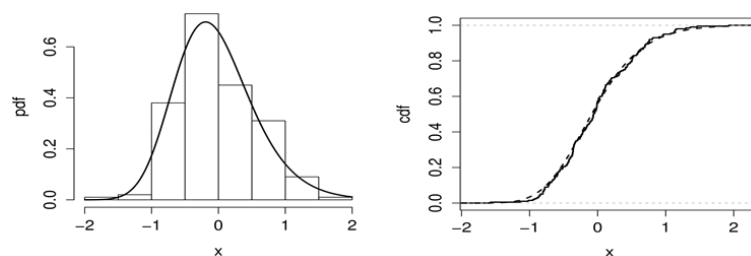

Figure 1. Plots of the EGSGu (3, 2) pdf, histogram, exact and empirical cdfs for simulated data with $n = 200$.

Table 1. Theoretical and empirical medians (for $n = 50, 100, 150$) of X for some parameter values.

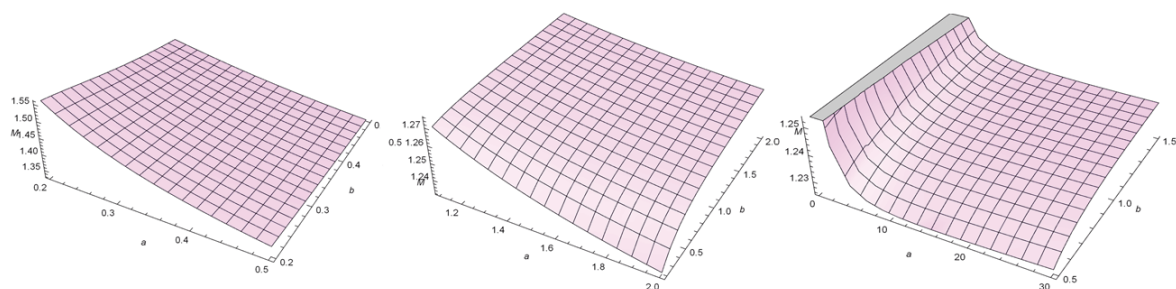| a	b	Med $\downarrow$	EMed $\rightarrow$ $n = 50$	$n = 100$	$n = 150$
1.5	3.0	0.8452	0.7252	0.8830	0.8620
3.0	3.0	0.1123	0.0342	0.1367	0.1232
3.0	1.5	-0.2355	-0.3218	-0.2087	-0.2236

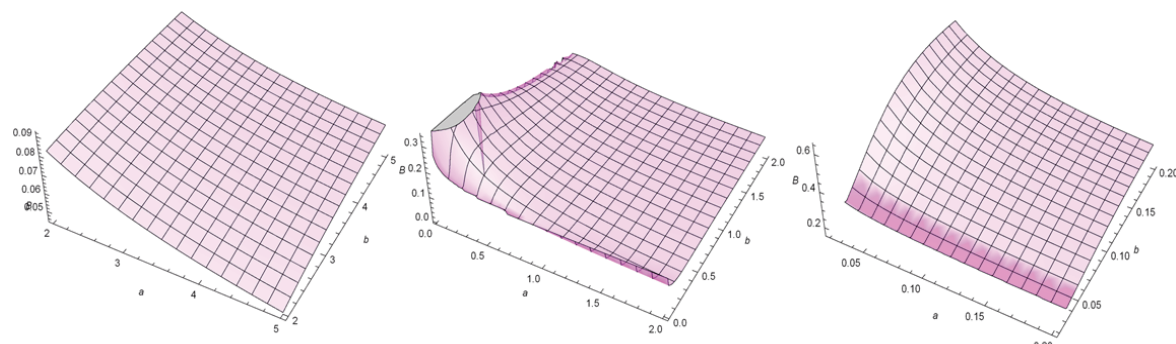**Figure 2.** Plots of the Moors kurtosis for the EGSGu distribution.

Figure 3. Plots of the Bowley skewness for the EGSGu distribution.

$$f(x) = \sum_{j=0}^{\infty} w_{j+1} h_{j+1}(x), \ (10)$$

where $h_{j+1}(x) = dH_{j+1}(x)/dx$. The expSGu pdf with power parameter $j+1$, $h_{j+1}(x)$, (for $j \geq 0$) becomes Equation 11:

$$h_{j+1}(x) = (j+1)\exp[-x - (j+1)\exp(-x)]. \ (11)$$

Combining Equation 10 and 11, we have an important result: The EGSGu density function is a linear combination of expSGu densities. This result can be used to derive some mathematical properties of X.

Moments and probability weighted moments

We provide below two ways to compute the n-th moments of X with density Equation 7. Moreover, we go beyond and also present alternative ways to calculate the probability weighted moments, say PWM and denoted by $\tau_{s,r}$, for EGSGu model. The first formula for the n-th moments of X become by using Equation 7, follow Equation 12:

$$E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx. \ (12)$$

Alternatively, combining Equation 10 and 11, we can express $E(X^n)$ in terms of expSGu moments. We write Equation 13:

$$E(X^n) = \sum_{j=0}^{\infty} w_{j+1} \int_{-\infty}^{\infty} x^n h_{j+1}(x)dx. \ (13)$$

It is very simple to calculate the n-th moment of X computationally, using the expressions Equation 12 and 13. To illustrate it, we provide next a small numerical study, comparing $E(X^n)$ from both formulas. We consider several parameter values and $n = 1, 2, 3, 4, 5, 6$. The results are shown in Table 2. This table shows that the results agree at five decimal digits of precision for both methods. All computations are obtained using the 'Wolfram Mathematica' platform. The (s, r)th PWM of X is defined by $\delta_{s,r} = E[X^s f(x)^r]$. Clearly, the ordinary moments follow as $\delta_{s,0} = E(X^s)$. Next, we derive simple expressions for the PWMs of X defined by Equation 14:

$$\delta_{s,r} = \int_{-\infty}^{\infty} x^s F(x)^r f(x)dx. \ (14)$$

Inserting Equation 6 and 7 into Equation 14, the PWMs of X can be expressed in a simple form Equation 15:

$$\delta_{s,r} = a b \int_{-\infty}^{\infty} x^s \exp[-x - \exp(-x)]\left\{1 - \exp\left[-\exp(-x)\right]\right\}^{a-1}$$
$$\times \left\{1 - \left[1 - \exp\left[-\exp(-x)\right]\right]^a\right\}^{b(r+1)-1} dx. \quad (15)$$

Under simple algebraic manipulation, we can write $\delta_{s,r}$ as Equation 16:

$$\delta_{s,r} = a b \sum_{k,\ell=0}^{\infty} \frac{(-1)^{k+\ell}}{\ell+1} \binom{b(r+1)-1}{k} \binom{a(k+1)-1}{\ell} \int_{-\infty}^{\infty} x^s h_{\ell+1}(x)dx, \ (16)$$

where $h_{l+1}(x)$ is the expSGu density with parameter $(l+1)$. Equation 16 reveals that the PWMs of X can be expressed in terms of the ordinary moments of $X \sim$ expSGu($l+1$).

Table 3 gives from Equation 15 the values of $\delta_{s,r}$ for $X \sim$ EGSGu(a, b) and some values of s and r. All computations are performed using the 'Wolfram Mathematica Software'. Based on the values in Table 3, we conclude that, for fixed r, the PWMs increase when s increases. The opposite happens when we fix parameter s and r increases.

Stochastic comparisons of sample maximum

In this section, we compare the maximum of two independent and heterogeneous samples each following EG class with the same baseline distributions. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the order statistics to the random variables

$X_1, X_2, ..., X_n$ with $X_{1:n}$ and $X_{n:n}$ the sample minimum and sample maximum, respectively. The study of Parallel systems plays a prominent role in reliability theory and is equivalent to the study of the largest order statistics. The following two definitions and notation will be needed to prove our main result.

Definition 1: (Shaked & Shanthikumar, 2007): Let Y and Z be two random variables with, respectively, absolutely continuous distribution functions $M(\cdot)$ and $H(\cdot)$ densities $m(\cdot)$ and $h(\cdot)$, and reversed hazard rate functions $\tilde{r}(\cdot)$ and $\tilde{s}(\cdot)$. Hence, Y is said to be smaller than Z in (i) 'usual stochastic order', denoted by $Y \leq_{st} Z$, if $M(t) \geq H(t)$, $\forall t$; (ii) 'reversed hazard rate order', denoted by $Y \leq_{rhr} Z$, if $\tilde{r}(t) \leq \tilde{s}(t)$, $\forall t$.

We consider in our main result stochastic comparison in terms of the reversed hazard rate order. Note that reversed hazard rate order implies usual stochastic order. Let I^n be an n-dimensional Euclidean space with $I \subseteq IR$.

Definition 2: (Marshall, Olkin, & Arnold, 2011): Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ be two vectors in I^n. The vector $\mathbf{x}$ is said:

(i)   to majorize the vector $\mathbf{y}$ (say, $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{i:n-1} \leq \sum_{i=1}^{j} y_{i:n-1}$, $j = 1,2,...,n-1$, and $\sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}$;

(ii) to weakly supermajorize the vector $\mathbf{y}$ (say, $\mathbf{x} \overset{w}{\pm} \mathbf{y}$) if $\sum_{i=1}^{j} x_{i:n} \leq \sum_{i=1}^{j} y_{i:n}$, for $j = 1, 2, ..., n$;

(iii) to weakly submajorize the vector $\mathbf{y}$ (say, $\mathbf{x} \pm_w \mathbf{y}$) if $\sum_{i=j}^{n} x_{i:n} \geq \sum_{i=j}^{n} y_{i:n}$, for $j = 1, 2, ..., n$.

Notation: Let us include the following notations.
(i) $D = \{(x_1, x_2, ..., x_n) \in IR^n : x_1 \geq x_2 \geq \cdots \geq x_n\}$;
(ii) $E = \{(x_1, x_2, ..., x_n) \in IR^n : x_1 \leq x_2 \leq \cdots \leq x_n\}$;
(iii) $D_+ = \{(x_1, ..., x_n) \in IR^n : x_1 \geq \cdots \geq x_n > 0\}$;
(iv) $E_+ = \{(x_1, x_2, ..., x_n) \in IR^n : 0 < x_1 \leq \cdots \leq x_n\}$.

Table 2. First sixth moments of X for several a and b values.

a	b	E(X)	E(X²)	E(X³)	E(X⁴)	E(X⁵)	E(X⁶)
1	1	0.57722	1.97811	5.44487	23.5615	117.839	715.067
	2	1.27036	3.25876	10.7232	45.3600	233.071	1418.73
	3	1.67583	4.45333	15.3804	66.3318	345.015	2113.31
	4	1.96351	5.50031	19.6637	86.4083	454.276	2798.98
2	1	-0.11593	0.69747	0.16651	1.76299	2.60769	11.4045
	2	0.34165	0.72200	1.03513	2.52093	6.50153	20.6720
	3	0.59545	0.92435	1.59047	3.63454	9.74663	30.7523
	4	0.77030	1.14191	2.07825	4.75215	12.8901	40.7867
3	1	-0.40361	0.61140	-0.45472	0.93635	-0.68052	2.31902
	2	0.02993	0.37204	0.10596	0.51195	0.57386	1.68848
	3	0.17170	0.36710	0.31147	0.60373	1.02392	2.32992
	4	0.30846	0.41307	0.45178	0.75980	1.38992	3.06458
4	1	-0.57351	0.67294	-0.70212	1.00506	-1.28616	2.13690
	2	-0.24490	0.33249	-0.14312	0.29771	-0.09027	0.46581
	3	-0.07069	0.24739	0.01646	0.20673	0.12820	0.38704
	4	0.04630	0.22709	0.09714	0.20835	0.23144	0.45403

Table 3. The PWM of X for fixed $r = 1$ and several values of a, b and s.

a	b	$\delta_{1,1}$	$\delta_{2,1}$	$\delta_{3,1}$	$\delta_{4,1}$	$\delta_{5,1}$	$\delta_{6,1}$
1	1	0.63518	1.62938	5.36162	22.6800	116.536	709.365
	2	0.98176	2.75015	9.83185	43.2041	227.138	1399.49
	3	1.18449	3.62849	13.6947	62.1381	332.862	2072.90
	4	1.32833	4.35138	17.1322	79.8149	434.481	2731.50
2	1	0.17082	0.36100	0.51756	1.26046	3.25077	10.3360
	2	0.38515	0.57095	1.03912	2.37608	6.44506	20.3933
	3	0.50519	0.77208	1.48086	3.45325	9.50317	30.2954
	4	0.58839	0.94653	1.88113	4.47512	12.4662	40.0248
3	1	-0.01497	0.18602	0.05298	0.25598	0.28693	0.84429
	2	0.15423	0.20654	0.22589	0.37990	0.69496	1.53229
	3	0.24681	0.26923	0.33986	0.54915	1.03731	2.27785
	4	0.31014	0.33295	0.44066	0.71657	1.36863	3.01745
4	1	-0.12245	0.16625	-0.07156	0.14885	-0.04514	0.23290
	2	0.02315	0.11354	0.04857	0.10418	0.11572	0.22702
	3	0.10163	0.12300	0.09790	0.13451	0.18804	0.32561
	4	0.15486	0.14437	0.13469	0.17284	0.25175	0.43164

Consider X a random variable with continuous distribution function $G(x)$ and pdf $g(x)$. For $I = 1, 2, ..., n$, consider also $X_i \sim EG(a_i, b_i, G)$ and $Y_i \sim EG(c_i, d_i, G)$ two sets of n independent random variables where the baseline $G(x)$ is homogenous and common to both sets of random variables. Suppose that $F_{n:n}(\cdot)$ and $H_{n:n}(\cdot)$ are the cdfs of $X_{n:n}$ and $Y_{n:n}$, respectively. Hence,

$$F_{n:n}(x) = \prod_{i=1}^{n} \left\{ 1 - [1 - G(x)]^{a_i} \right\}^{b_i} \quad \text{and} \quad H_{n:n}(x) = \prod_{i=1}^{n} \left\{ 1 - [1 - G(x)]^{c_i} \right\}^{b_i}.$$

We can also notice that if $\tilde{r}_{n:n}(\cdot)$ and $\tilde{s}_{n:n}(\cdot)$ are, respectively, reversed hazard rate functions of $X_{n:n}$ and $Y_{n:n}$, then $\tilde{r}_{n:n}(x) = \sum_{i=1}^{n} \dfrac{a_i b_i g(x)[1 - G(x)]^{a_i - 1}}{1 - [1 - G(x)]^{a_i}}$ and $\tilde{s}_{n:n}(x) = \sum_{i=1}^{n} \dfrac{c_i d_i g(x)[1 - G(x)]^{c_i - 1}}{1 - [1 - G(x)]^{c_i}}$.

The following lemmas will be needed to prove our main result.

Lemma 1 (Marshall et al., 2011, p. 86): Let $\varphi: I \to IR$. Then, $(a_1, a_2, ..., a_n) \pm_w (b_1, b_2, ..., b_n)$ implies $\varphi(a_1, a_2, ..., a_n) \geq \varphi(b_1, b_2, ..., b_n)$ if, and only if, φ is decreasing and Schur-convex on I^n.

Lemma 2 (Kundu, Chowdhury, Nanda, & Hazra, 2016): Let $\varphi(\mathbf{x}) = \sum_{i=1}^{n} u_i g(x_i)$ with $\mathbf{x} \in D$ and let $I \subseteq IR$ be an interval. Consider a function $g: I \to IR$. If $\mathbf{u} = (u_1, u_2, ..., u_n) \in E_+$ and $g(\cdot)$ is decreasing and convex then $\varphi(\mathbf{x})$ is Schur-convex on $\mathbf{D}$.

Consider the following vectors belonging to I^n: $\mathbf{a} = (a_1, a_2, ..., a_n)$, $\mathbf{b} = (b_1, b_2, ..., b_n)$, and $\mathbf{c} = (c_1, c_2, ..., c_n)$.

Theorem (Main result of the section): Suppose $X_i \sim EG(a_i, b_i, G)$ and $Y_i \sim EG(c_i, d_i, G)$ with X_i and Y_i two sets of mutually independent random variables and $I = 1, 2, ..., n$. Also suppose that $\mathbf{a}, \mathbf{c} \in \mathbf{D}_+$ and $\mathbf{b} \in \mathbf{E}_+$ Then, $\mathbf{a} \overset{w}{\pm} \mathbf{c}$ implies $X_{n:n} \geq_{rhr} Y_{n:n}$.

Proof. Let $\bar{G}(x) = 1 - G(x)$. Differentiating $v(a_i) = \dfrac{a_i g(x) \bar{G}^{a_i - 1}(x)}{1 - \bar{G}^{a_i}(x)}$ partially with respect to a_i, we have

$$v'(a_i) = \frac{g(x) \bar{G}^{a_i - 1}(x)\left[1 - \bar{G}^{a_i}(x) + \log \bar{G}^{a_i}(x) \right]}{\left(1 - \bar{G}^{a_i}(x) \right)^2}$$

$$= \frac{g(x) \bar{G}^{a_i - 1}(x)(-1)\left\{ \left[\bar{G}^{a_i}(x) - 1 \right] - \log \bar{G}^{a_i}(x) \right\}}{\left(1 - \bar{G}^{a_i}(x) \right)^2}.$$

We note that $\log \bar{G}^{a_i}(x) \leq \bar{G}^{a_i}(x) - 1$ since $\log x \leq x - 1$, for all $x > 0$. Hence, $v'(a_i) \leq 0$ and $v(a_i)$ is decreasing in a_i. By differentiating again $v'(a_i)$ with respect to a_i, we obtain $v''(a_i) = \dfrac{g(x) \bar{G}^{a_i - 1}(x) \log \bar{G}^{a_i}(x)}{\left(1 - \bar{G}^{a_i}(x) \right)^3} v_1(a_i)$, with

$v_1(a_i) = 2 - 2\bar{G}^{a_i}(x) + \log \bar{G}^{a_i}(x) + a_i \bar{G}^{a_i}(x) \log \bar{G}(x)$. Taking the derivative of $v_1(a_i)$ with respect to a_i, we obtain $v_1'(a_i) = v_2(a_i) \log \bar{G}(x)$, where $v_2(a_i) = 1 - \bar{G}^{a_i}(x) + a_i \bar{G}^{a_i}(x) \log \bar{G}(x)$. Finally, by differentiating $v_2(a_i)$, also with respect to a_i, we have $v_2'(a_i) = a_i \bar{G}^{a_i}(x)[\log \bar{G}(x)]^2 > 0$.

It is easy to see, for $a_i > 0$, the following chain of implications, $v_2'(a_i) > 0 \Rightarrow v_2(a_i) > v_2(0) = 0 \Rightarrow v_1'(a_i) < 0 \Rightarrow v_1(a_i)$ *decreasing in* a_i $\Rightarrow v_1(a_i) < v_1(0) = 0 \Rightarrow v''(a_i) > 0 \Rightarrow v(a_i)$ *convex in* a_i.

Thus, by Lemma 1 and Lemma 2, the proof is obvious.

Considering that the theorem holds for any continuous baseline G we naturally have the following corollary.

Corollary (Result applied to EGSGu distribution): Suppose $X_i \sim EGSGu(a_i, b_i)$ and $Y_i \sim EGSGu(c_i, b_i)$ with X_i and Y_i two sets of mutually independent random variables and $i = 1, 2, ..., n$. Also suppose that $\mathbf{a}, \mathbf{c} \in \mathbf{D}_+$ and $\mathbf{b} \in \mathbf{E}_+$ and Then, $\mathbf{a} \overset{w}{\pm} \mathbf{c}$ implies $X_{n:n} \geq_{rhr} Y_{n:n}$.

Dual generalized order statistics

The dual generalized order statistics (dgos) were introduced in Burkschat, Cramer, and Kamps (2003) as a model for descending ordered random variables and admits as special cases reversed ordered order statistics, lower k-records and lower Pfeifer records (Arnold, Balakrishnan, & Nagaraja, 1998). In this section, we present general expressions for dgos from the EG class. Next, we present results for the EGSGu distribution. We derive an explicit expression for the density of the mth $(1 \leq m \leq n)$ dual generalized order

statistic $X^*(m, n, v, k)$, say $f_{x^*(m,n,v,k)}(x)$, in a random sample of size n from the EG class. By definition we have Equation 17:

$$f_{X^*(m,n,v,k)}(x) = \frac{C_{m-1}}{(m-1)!} F^{\gamma_m-1}(x) g_v^{m-1}(F(x)) f(x) \quad (17)$$

where

$$C_{m-1} = \prod_{i=1}^{m-1} \gamma_i, \ \gamma_m = k + (n-m)(v+1) \geq 1, \ g_v(u) = t_v(u) - t_v(1), \ u \in [0,1) \quad \text{and} \quad t_v(u) = \begin{cases} -\dfrac{1}{v+1} u^{v+1}, & \text{if } v \neq -1 \\ -\log u, & \text{if } v = -1 \end{cases} \quad \text{with} \quad k \geq 1,$$

$v \in IR$. According to Equation 17, we can rewrite it in two cases, follow Equation 18:

$$f_{X^*(m,n,v,k)}(x) = \begin{cases} \dfrac{C_{m-1}}{(m-1)!} F^{\gamma_m-1}(x) \left\{ \dfrac{1}{v+1}[1 - F^{v+1}(x)] \right\}^{m-1} f(x), & \text{if } v \neq -1 \\ \dfrac{C_{m-1}}{(m-1)!} F^{\gamma_m-1}(x) \{ -\log[F(x)] \}^{m-1} f(x), & \text{if } v = -1. \end{cases} \quad (18)$$

Case I: For $v \neq -1$.

Using the binomial expansion in the first sentence of Equation 18 and inserting cdf Equation 2 and pdf Equation 3, we readily obtain Equation 19:

$$f_{X^*(m,n,v,k)}(x) = \frac{C_{m-1}(v+1)^{1-m}}{(m-1)!} \sum_{p=0}^{m-1} (-1)^p \, a \, b \binom{m-1}{p} [1 - G(x)]^{a-1} \quad (19)$$
$$\times \left\{ 1 - [1 - G(x)]^a \right\}^{b[\gamma_m + p(v+1)]-1} g(x).$$

For any real non-integer β, $(|z| < 1)$, Equation 20:

$$(1-z)^{\beta-1} = \sum_{q=0}^{\infty} \frac{(-1)^q \Gamma(\beta)}{\Gamma(\beta-q)q!} z^q. \quad (20)$$

By applying the last equation twice in Equation 19 and after simple algebraic manipulation, we write $f_{x^*(m,n,v,k)}(x)$ as Equation 21:

$$f_{X^*(m,n,v,k)}(x) = \sum_{s=0}^{\infty} \xi(v) h_{s+1}(x), \quad (21)$$

where

$$\xi(v) = \sum_{p=0}^{m-1} \sum_{q=0}^{\infty} \frac{(-1)^{p+q+s} abC_{m-1}(v+1)^{1-m} \Gamma(a[1+q]) \Gamma(b[\gamma_m + p(v+1)])}{(s+1)q!s!(m-1)! \Gamma(a[1+q]-s) \Gamma(b[\gamma_m + p(v+1)]-q)} \binom{m-1}{p}.$$

Case II: For $v = -1$.

By expanding the logarithm function in power series and then using an equation for a power series raised to a positive integer given in Gradshteyn and Ryzhik (2007) (Section 0.314), we have Equation 22,

$$\{ -\log[-\log[F(x)]] \}^{m-1} = \{ -\log[1 - \bar{F}(x)] \}^{m-1} = \sum_{p=0}^{\infty} \sum_{q=0}^{p+m-1} (-1)^q c_{m-1,p} \binom{p+m-1}{q} F^q(x), \quad (22)$$

where $\bar{F}(x) = 1 - F(x)$ and the coefficients $c_{m-1,p}$ (for $p = 1, 2, ...$) are determined from the recurrence equation $c_{m-1,p} = (pa_0)^{-1} \sum_{v=1}^{p} [vm - p] a_v c_{m-1,p-v}$, with $c_{m-1,0} = a_0^{m-1}$. From Equation 22, the second sentence of Equation 18 reduces to $f_{X^*(m,n,v,k)}(x) = \dfrac{C_{m-1}}{(m-1)!} \sum_{p=0}^{\infty} \sum_{q=0}^{p+m-1} (-1)^q c_{m-1,p} \binom{p+m-1}{q} F^{\gamma_m+q-1}(x) f(x)$.

Analogously, inserting Equation 2 and 3 in the previous equation and applying Equation 20 twice, the pdf $f_{X^*(m,n,v,k)}(x)$ can be expressed as Equation 23,

$$f_{X^*(m,n,v,k)}(x) = \sum_{j=0}^{\infty} \xi'(v) h_{j+1}(x), \quad (23)$$

where

$$\xi'(v) = \sum_{s,p=0}^{\infty} \sum_{q=0}^{p+m-1} \frac{(-1)^{q+s+j} \, ab \, C_{m-1} c_{m-1,p} \Gamma(a[s+1]) \Gamma(b[\gamma_m + q])}{(j+1) \, j! \, s! \, (m-1)! \Gamma(a[s+1]-j)\Gamma(b[\gamma_m+q]-s)} \binom{p+m-1}{q}.$$

Hence, cases I (Equation 21) and II (Equation 23) can be summarized as (for baseline *G* distribution) Equation 24:

$$f_{X^*(m,n,v,k)}(x) = \begin{cases} \sum_{s=0}^{\infty} \xi'(v) h_{s+1}(x), & \text{if } v \neq -1 \\ \sum_{j=0}^{\infty} \xi'(v) h_{j+1}(x), & \text{if } v = -1. \end{cases} \quad (24)$$

Based on the previous equation, we can easily obtain the dgos of EGSGu distribution since it depends on exp-G densities. Furthermore, several mathematical quantities of the EG dgos can be obtained from those quantities of exp-G distributions. We provide an example for the dgos moments from EGSGu distribution. Before that, an additional comment on the moments of the expSGu is in order. Inspired by Andrade et al. (2015), we can find a formula for the *t*th moment of $Z_c \sim$ expSGu with power parameter $c > 0$. By setting $w = \exp\{-z\}$, and using Andrade et al. (2015)'s Equation 14, the *t*th moment of $Z_c \sim$ expSGu can be written as Equation 25:

$$E[Z_c^t] = (-1)^t c \left\{ \left(\frac{\partial}{\partial \mu}\right)^t \left[c^{-\mu} \Gamma(\mu)\right]\Big|_{\mu=1} \right\}. \quad (25)$$

Thus, the *t*th moment dgos of the EGSGu distribution can be expressed from Equation 24 and 25 as:

$$E\left[X^{*t}(m,n,v,k)\right] = \begin{cases} \sum_{s=0}^{\infty} (s+1)(-1)^t \xi(v) \left\{ \left(\frac{\partial}{\partial \mu}\right)^t \left[(s+1)^{-\mu} \Gamma(\mu)\right]\Big|_{\mu=1} \right\}, & \text{if } v \neq -1; \\ \sum_{s=0}^{\infty} (j+1)(-1)^t \xi'(v) \left\{ \left(\frac{\partial}{\partial \mu}\right)^t \left[(j+1)^{-\mu} \Gamma(\mu)\right]\Big|_{\mu=1} \right\} & \text{if } v = -1. \end{cases}$$

It is well known that if $v = 0$, $k = 1$ then $X^*(m, n, v, k)$ reduces to the $(n-m+1)$th order statistics $X_{n-m+1:n}$ from the sample $X_1, ..., X_n$, and when $v = -1$, then $X^*(m, n, v, k)$ reduces to the *m*th lower *k*-record value. The main result of this section is given by Equation 24.

Results and discussion

Real data illustration

We adjusted the EGSGu model given in Equation 7, which contains just two parameters, and compared the results with other important models in the literature. We considered the EGSGu distribution and two Gumbel Lehmann's alternatives sub-models, denoted by EISGu and EIISGu, respectively. In addition, we have adjusted the models proposed by Mahmoudi (2011) (BGP distribution), Nadarajah and Eljabri (2013) (KwGP distribution), and Silva, Ortega, and Cordeiro (2010) (BMW distribution). Their respective densities are given by

$$f_{BGP}(x) = \frac{\sigma^{-1}}{B(a,b)} u^{b+\xi} (1-u)^{a-1}, \quad f_{KwGP}(x) = ab\sigma^{-1} u^{1+\xi}(1-u)^{a-1}[1-(1-u)^a]^{b-1}$$

and $f_{BMW}(x) = \frac{\alpha x^{\gamma-1}(\gamma+\lambda x)\exp(\lambda x)}{B(a,b)}[1-\exp\{-\alpha x^\gamma \exp(\lambda x)\}]^{a-1} \exp\{-b\alpha x^\gamma \exp(\lambda x)\}$,

where $u = [1+\xi(x-t)/\sigma]^{\frac{-1}{\xi}}$ and $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}$ is the beta function.

The data set was obtained from Murthy, Xie, and Jiang (2004), and consists of the times between failures for repairable items. In Table 4 we provide the MLEs (and their standard errors in parentheses) for all fitted models.

Table 4 also lists the values of the Akaike information criterion (AIC), Bayesian information criterion (BIC) and corrected Akaike information criterion (Caic) statistics. In general, it is considered that lower

values of these criteria indicate better fit to the data. With the exception of the Caic of the EISGu model, the figures in Table 4 reveal that the EGSGu model has the lowest AIC, BIC and Caic values among all fitted models. Thus, the proposed EGSGu distribution is the best model to explain these data. Plots of the estimated pdf and cdf of the EGSGu distribution and the histogram of the data are displayed in Figure 4. These plots clearly reveal that the EGSGu model fits the data adequately and then it can be chosen for modeling these data.

Simulated data illustration

We investigated, by means of a simulation study, the behavior of the MLEs for the parameters of the EGSGu model by generating from qf Equation 8 with samples sizes n = 50, 100, 150, 200 and selected values for a and b. The simulation process is based on 10,000 Monte Carlo replications, performed in the software R using the simulated-annealing (SANN) maximization method in the *maxLik* script. The results of these new simulations are presented in Table 5 and 6, which contain the estimates and their estimated asymptotic variances in parentheses. These results reveal that, for all estimates, in general, the biases and variances decrease as the sample size increases.

Table 4. MLEs (and the corresponding standard errors in parentheses), AIC, BIC and CAIC statistics for number of successive failures for the air conditioning system.

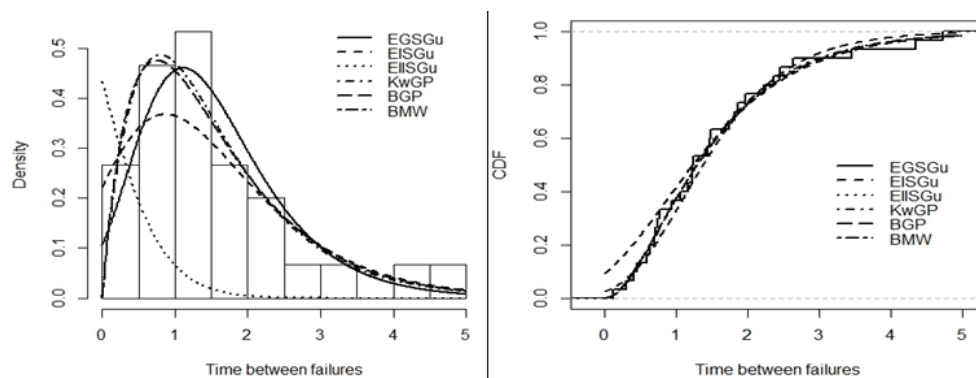| Distributions	$\hat{a}$	$\hat{b}$				AIC	BIC	CAIC
EGSGu	1.340/ (0.247)	4.800/ (1.670)				86.181	88.983	86.826
EISGu	1/ (----)	3.118/ (0.569)				86.334	87.735	86.477
EIISGu	0.589/ (0.108)	1/ (----)				103.73	105.13	103.87
	$\hat{a}$	$\hat{b}$	$\hat{\xi}$	$\hat{\sigma}$				
KwGP	1.970/ (0.837)	14.432/ (12.687)	1.119/ (2.371)	4.972/ (4.870)		87.283	92.888	88.883
BGP	1.979/ (0.460)	13.850/ (1.983)	0.156/ (1.960)	11.061/ (3.852)		87.257	92.862	88.857
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{\gamma}$	$\hat{\lambda}$			
BMW	2.901/ (1.903)	0.344/ (0.168)	2.374/ (0.164)	0.854/ (0.210)	0.093/ (0.109)	89.239	96.245	91.739

Figure 4. Estimated pdf and cdf of the EGSGu model for the times between failures for repairable items.

Table 5. MLEs for several a and b parameter values (variances in parentheses).

a b	n = 50		n = 100		n = 150		n = 200	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
1 1	1.0521	1.0601	1.0262	1.0287	1.0170	1.0192	1.0131	1.0145
	(0.040)	(0.0447)	(0.0185)	(0.0189)	(0.0121)	(0.0122)	(0.0089)	(0.0091)
1 2	1.0416	2.1546	1.0210	2.0733	1.0135	2.0487	1.0105	2.0368
	(0.0280)	(0.2526)	(0.0131)	(0.1029)	(0.0087)	(0.0661)	(0.0064)	(0.0484)
1 3	1.0380	3.2710	1.0191	3.1277	1.0123	3.0845	1.0096	3.0640
	(0.0240)	(0.7163)	(0.0113)	(0.2837)	(0.0075)	(0.1811)	(0.0055)	(0.1314)
1 4	1.0361	4.4045	1.0182	4.1898	1.0117	4.1254	1.0091	4.0947
	(0.0219)	(1.5130)	(0.0103)	(0.5873)	(0.0068)	(0.3728)	(0.0050)	(0.2687)
1 5	1.0350	5.5516	1.0176	5.2585	1.0114	5.1705	1.0088	5.1284
	(0.0206)	(2.6931)	(0.0097)	(1.0349)	(0.0064)	(0.6539)	(0.0047)	(0.4695)
2 1	2.1041	1.0601	2.0524	1.0287	2.0340	1.0193	2.0262	1.0146
	(0.1599)	(0.0446)	(0.0739)	(0.0189)	(0.0485)	(0.0122)	(0.0355)	(0.0091)
2 2	2.0832	2.1545	2.0419	2.0732	2.0271	2.0488	2.0210	2.0368
	(0.1122)	(0.2526)	(0.0526)	(0.1028)	(0.0348)	(0.0661)	(0.0255)	(0.0484)
2 3	2.0760	3.2711	2.0382	3.1277	2.0247	3.0847	2.0192	3.0640
	(0.0960)	(0.7158)	(0.0451)	(0.2837)	(0.0299)	(0.1812)	(0.0219)	(0.1314)

a	b								
2	4	2.0723 (0.0875)	4.4044 (1.5131)	2.0363 (0.0412)	4.1900 (0.5872)	2.0234 (0.0273)	4.1254 (0.3726)	2.0182 (0.0200)	4.0947 (0.2685)
2	5	2.0699 (0.0821)	5.5510 (2.6869)	2.0352 (0.0387)	5.2581 (1.0351)	2.0227 (0.0257)	5.1702 (0.6539)	2.0176 (0.0189)	5.1282 (0.4691)
3	1	3.1562 (0.3598)	1.0601 (0.0447)	3.0786 (0.1663)	1.0287 (0.0189)	3.0509 (0.1093)	1.0192 (0.0122)	3.0392 (0.0799)	1.0145 (0.0091)
3	2	3.1248 (0.2525)	2.1546 (0.2527)	3.0628 (0.1182)	2.0732 (0.1028)	3.0405 (0.0782)	2.0487 (0.0661)	3.0315 (0.0573)	2.0369 (0.0484)
3	3	3.1141 (0.2159)	3.2710 (0.7162)	3.0573 (0.1014)	3.1277 (0.2837)	3.0571 (0.0673)	3.0847 (0.1811)	3.0287 (0.0493)	3.0639 (0.1314)
3	4	3.1085 (0.1968)	4.4044 (1.5079)	3.0545 (0.0927)	4.1898 (0.5872)	3.0352 (0.0615)	4.1255 (0.3726)	3.0273 (0.0450)	4.0947 (0.2686)
3	5	3.1047 (0.1846)	5.5492 (2.6673)	3.0527 (0.0871)	5.2580 (1.0340)	3.0341 (0.0579)	5.1702 (0.6537)	3.0265 (0.0424)	5.1285 (0.4689)
4	1	4.2082 (0.6396)	1.0601 (0.0446)	4.1048 (0.2956)	1.0287 (0.0189)	4.0679 (0.1945)	1.0192 (0.0122)	4.0523 (0.1421)	1.0146 (0.0091)
4	2	4.1663 (0.0487)	2.1544 (0.2527)	4.0838 (0.2101)	2.0732 (0.1028)	4.0541 (0.1390)	2.0487 (0.0661)	4.0419 (0.1019)	2.0368 (0.0484)
4	3	4.1521 (0.3839)	3.2711 (0.7160)	4.0765 (0.1804)	3.1277 (0.2838)	4.0495 (0.1195)	3.0848 (0.1811)	4.0383 (0.0876)	3.0640 (0.1314)
4	4	4.1447 (0.3502)	4.4046 (1.5122)	4.0727 (0.1647)	4.1899 (0.5872)	4.0470 (0.1092)	4.1255 (0.3724)	4.0365 (0.0801)	4.0946 (0.2686)
4	5	4.1398 (0.3283)	5.5499 (2.6737)	4.0703 (0.1549)	5.2581 (1.0347)	4.0455 (0.1028)	5.1704 (0.6539)	4.0353 (0.0754)	5.1284 (0.4691)

Table 6. MLEs for several *a* and *b* parameter values (variances in parentheses).

a	b	n = 10		n = 20		n = 30		n = 40	
		$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
1	1	1.2922 (0.3835)	1.4124 (0.8753)	1.1364 (0.1291)	1.1665 (0.1816)	1.0893 (0.0763)	1.1032 (0.0876)	1.0679 (0.0533)	1.0780 (0.0608)
1	2	1.2207 (0.2182)	3.0650 (4.5985)	1.1079 (0.0861)	2.4438 (1.1979)	1.0709 (0.0521)	2.2699 (0.5299)	1.0542 (0.0369)	2.2022 (0.3524)
1	3	1.1844 (0.1543)	4.6307 (8.5029)	1.0975 (0.0714)	3.7774 (3.2781)	1.0646 (0.0443)	3.4785 (1.5574)	1.0495 (0.0315)	3.3564 (1.0175)
1	4	1.1540 (0.1154)	6.0016 (11.2756)	1.0901 (0.0617)	5.1139 (6.0177)	1.0611 (0.0399)	4.7132 (3.2699)	1.0471 (0.0286)	4.5328 (2.1581)
1	5	1.1261 (0.0895)	7.1680 (12.9804)	1.0825 (0.0535)	6.3945 (8.6183)	1.0581 (0.0365)	5.9510 (5.4658)	1.0453 (0.0267)	5.7225 (3.7875)
2	1	2.5842 (1.5318)	1.4121 (0.8739)	2.2728 (0.5161)	1.1665 (0.1815)	2.1786 (0.3052)	1.1032 (0.0876)	2.1358 (0.2130)	1.0780 (0.0608)
2	2	2.4408 (0.8700)	3.0571 (4.4186)	2.2158 (0.3442)	2.4432 (1.1851)	2.1418 (0.2087)	2.2701 (0.5337)	2.1083 (0.1476)	2.2021 (0.3523)
2	3	2.3679 (0.6192)	4.6131 (8.1938)	2.1946 (0.2844)	3.7732 (3.2076)	2.1290 (0.1768)	3.4772 (1.5407)	2.0989 (0.1259)	3.3562 (1.0163)
2	4	2.3042 (0.4616)	5.9499 (10.6918)	2.1784 (0.2433)	5.0913 (5.6459)	2.1220 (0.1592)	4.7113 (3.2351)	2.0940 (0.1143)	4.5316 (2.1459)
2	5	2.2427 (0.3550)	7.0540 (11.9683)	2.1610 (0.2085)	6.3515 (8.1286)	2.1161 (0.1459)	5.9501 (5.5010)	2.0905 (0.1066)	5.7206 (3.7591)
3	1	2.1358 (0.2130)	1.0780 (0.0608)	3.4091 (1.1614)	1.1665 (0.1816)	3.2680 (0.6863)	1.1032 (0.0875)	3.2037 (0.4793)	1.0780 (0.0608)
3	2	3.6590 (1.9302)	3.0530 (4.3600)	3.3236 (0.7739)	2.4429 (1.1783)	3.2126 (0.4694)	2.2701 (0.5335)	3.1626 (0.3322)	2.2023 (0.3523)
3	3	3.5521 (1.3818)	4.6151 (8.1101)	3.2920 (0.6405)	3.7749 (3.2347)	3.1937 (0.3982)	3.4778 (1.5472)	3.1484 (0.2832)	3.3562 (1.0168)
3	4	3.4579 (1.0376)	5.9566 (10.5722)	3.2681 (0.5486)	5.0951 (5.6934)	3.1830 (0.3587)	4.7108 (3.2458)	3.1411 (0.2575)	4.5327 (2.1610)
3	5	3.3682 (0.7999)	7.0695 (11.7716)	3.2416 (0.4708)	6.3449 (7.9364)	4.2310 (0.5797)	5.9381 (5.2623)	3.1355 (0.2393)	5.7184 (3.7228)
4	1	5.1642 (6.0063)	1.4101 (0.8514)	4.5455 (2.0640)	1.1665 (0.1815)	4.3571 (1.2202)	1.1032 (0.0876)	4.2715 (0.8520)	1.0780 (0.0607)
4	2	4.8702 (3.3608)	3.0296 (4.0290)	4.4313 (1.3750)	2.4427 (1.1782)	4.2836 (0.8342)	2.2699 (0.5277)	4.2168 (0.5907)	2.2024 (0.3527)
4	3	4.7246 (2.3826)	4.5771 (7.7049)	4.3891 (1.1352)	3.7720 (3.1640)	4.2584 (0.7086)	3.4784 (1.5609)	4.1979 (0.5032)	3.3560 (1.0154)
4	4	4.6035 (1.7977)	5.9348 (10.3640)	4.3588 (0.9799)	5.1031 (5.7925)	4.2443 (0.6392)	4.7129 (3.2742)	4.1880 (0.4573)	4.5314 (2.1404)
4	5	4.4874 (1.3902)	7.0556 (11.5550)	4.3254 (0.8448)	6.3667 (8.2116)	4.3254 (0.8448)	6.3667 (8.2116)	4.1805 (0.4245)	5.7169 (3.7004)

Conclusion

In this article, we propose and study a new two-parameter distribution with real support called the exponentiated generalized standardized Gumbel distribution (EGSGu). Our proposal includes both Lehmann's type I and II transformations as special cases. We study some mathematical properties of the

new model. The model's parameters are estimated by the maximum likelihood method. A simulation study reveals that the estimators have desirable properties. We empirically prove that the new distribution provides a better fit to a real dataset than other competitive models.

Acknowledgements

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