Acta Scientiarum

http://periodicos.uem.br/ojs
ISSN on-line: 1807-8664
Doi: 10.4025/actascitechnol.v45i1.59806

MATHEMATICS

Some relations between the sets of f-statistically convergent difference sequences

Sarkawt Asaad[1], Rifat Çolak[2] and Yavuz Altin[2*]

[1]Graduate School of Sciences, Firat University, Elazığ, Türkiye. [2]Department of Mathematics, Firat University, Elazığ, Türkiye. *Author for correspondence. E-mail: yaltin23@yahoo.com

ABSTRACT. In this study we establish the relations between the sets of difference sequences which are statistically convergent in connection with modulus functions.

Keywords: density; modulus function; difference sequence; statistical convergence.

Received on june 24, 2021
Accepted on april 28, 2022

Introduction and preliminaries

The notion of statistical convergence reverts to the first edition of Zygmund's monograph in Zygmund (1979). The conception of statistical convergence was explicitly presented by Steinhaus (1951) and Fast (1951) and reintroduced later by Schoenberg (1959). Statistical convergence also appears as an example of density convergence introduced by Buck (1953).

To solve series summation problems, statistical convergence was studied by many researchers provided many statistical convergence results and theories in many spaces and statistical convergence has been considered in different setups, and its different speculations, expansions and variations have been concentrated by different creators up until now.

Kizmaz (1981) introduced the notion of spaces of difference sequences in, who examined the difference sequence spaces $c(\triangle)$, $c_0(\triangle)$ and $l_\infty(\triangle)$ as

$$c(\triangle) = \{(x_k) \ : \ |\triangle x_k - l| \to 0 \text{ as } k \to \infty, \text{ for some } l\}$$

$$c_0(\triangle) = \{(x_k) \ : \ |\triangle x_k| \to 0 \text{ as } k \to \infty\}$$

$$l_\infty(\triangle) = \{(x_k) \ : \ sup|\triangle x_k| < \infty\},$$

which are Banach spaces with norm

$$\|x\| = |x_1| + sup|\triangle x_k|$$

where $\triangle x = (\triangle x_k), \triangle x_k = x_k - x_{k+1}, k \in N = \{1,2,3,\dots\}$.

Et and Çolak (1995) further generalized the concept by introducing the spaces $c(\triangle^n), c_0(\triangle^n)$ and $l_\infty(\triangle^n)$.

Nakano (1953) was introduced the concept of the modulus function in 1953 and subsequently Ruckle (1973), Maddox (1987) and many researchers used a modulus function to construct some sequence spaces. Some mathematicians have studied statistical boundedness and some generalizations such as Kayan et al. (2018).

Aizpuru et al. (2014) defined a new concept of density with the help of an unbounded modulus function and as a consequence, they got a new non - matrix convergence concept, in other words, f - statistical convergence that is intermediate between statistical convergence and ordinary convergence and agrees with statistical convergence in the case identity mapping which is a modulus function.

Now, we will recall some concepts and definitions which are needful in the study.

Let N be the set of positive integers. The natural density of a set $V \subseteq N$ is defined by

$$d(V) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in V\}|$$

where $|\{k \leq n : k \in V\}|$ denotes the number of elements of V not exceeding n. Obviously, if V is finite subset of N , then $d(V) = 0$ and $d(V^c) = 1 - d(V)$. The set V is said to be statistically dense if $d(V) = 1$.

A modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that
$i)$ $f(x) = 0$ if and only if $x = 0$,
$ii)$ $f(x + y) \leq f(x) + f(y)$ for every $x, y \geq 0$,
$iii)$ f is increasing,
$iv)$ f is continuous from the right at 0 .

It is clear that any modulus f must be continuous everywhere on $[0, \infty)$. A modulus may be unbounded or bounded. For example, $f(x) = x^p$ $(0 < p \leq 1)$ is unbounded, but $f(x) = \frac{x}{x+1}$ is bounded.

Aizpuru et al. In [1] defined the f -density of a set . The $f - density$ of a set $V \subseteq N$ is defined by

$$d_f(V) = \lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : k \in V\}|),$$

whenever the limit exists, where f is an unbounded modulus function.

When $(x) = x$, the concept of f -density reduces to the natural density. It is well known that $d(V) + d(N - V) = 1$ in case of natural density. But in case of f -density, i.e., $d_f\ (V) + d_f\ (N - V) = 1$ does not hold generally, this result is no longer true. $d_f\ (V) = d_f\ (N - V) = 1$ when we take $f(x) = log(x + 1)$ and $V = \{2n : n \in N\}$. In case of f -density, we can assert that if $d_f(V) = 0$ then $d_f\ (N - V) = 1$. As in the case of natural density, finite sets also have zero f -density and so for any finite set, $d_f\ (V) + d_f\ (N - V) = 1$.

For any unbounded modulus f and $\subset N$, if $d_f\ (V) = 0$ implies that $d(V) = 0$. But converse may not be true in the sense that a set having zero natural density may have non-zero f -density with respect to some unbounded modulus f. When we take $f(x) = log(x + 1)$ and $= \{1,4,9, \dots\}$, then $d(V) = 0$ but $d_f\ (V) = \frac{1}{2}$. In case of any finite set $\subset N$, However, $d(V) = 0$ implies $d_f\ (V) = 0$ is always true, regardless of the selection of unbounded modulus f.

In this paper we discuss the relations between $S_f(\triangle)$ and $S_g(\triangle)$, $S_f(\Delta)$ and $S(\triangle)$, $BS_f(\triangle)$ and $BS_g(\triangle)$, $BS_f(\triangle)$ and $BS(\triangle)$, $S_f(\triangle)$ and $BS_g(\triangle)$ for different modulus functions f and g under certain conditions.

Main results

In this section we will give the main definitions and results of this paper.

In the sequel, we need the following fact.

Lemma 2.1 [Maddox,1987]. The limit $\lim_{t\to\infty} \frac{f(t)}{t} = \beta$ exists for any modulus .

The relation between $f -$ densities of a set of positive integers for different modulus functions is given in the following Theorem given by Çolak (2020). This helps us to establish the relations between $\triangle -$ statistically convergent and $\triangle -$ statistically bounded sequence sets defined by modulus functions.

Theorem 2.2 [Çolak ,2020]. Let f and g be two unbounded modulus functions**.** Then for a set $V \subseteq N$
(i) if

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} > 0 \tag{1}$$

then $d_g(V) = 0$ implies $d_f(V) = 0$ whenever the limit exists,
(ii) if

$$0 < \lim_{t\to\infty} \frac{f(t)}{g(t)} = \alpha < \infty \tag{2}$$

then $d_g(V) = 0 \Leftrightarrow d_f(V) = 0$ whenever the limit exists.

Corollary 2.3 [Çolak ,2020]. For any $V \subseteq N$ and any unbounded modulus f providing

$$\lim_{t\to\infty} \frac{f(t)}{t} > 0 \tag{3}$$

we have $d_f(V) = 0 \Leftrightarrow d(V) = 0$.

Now we give the following definitions by using f - density.

Definition 2.4. Let f be an unbounded modulus function. Then a sequence (x_k) is said to be
$\triangle_f$ -statistically convergent to l, if for each $\varepsilon > 0$

$d_f\ (\{k \in N\ :\ |\triangle x_k - l| \geq \varepsilon\ \}) = 0,$

i.e.,

$$\lim_{n\to\infty} \frac{1}{f(n)} f(\{k \leq n\ :\ |\triangle x_k - l| \geq \varepsilon\ \}) = 0.$$

In this case we write $S_f(\triangle) - \lim_{n\to\infty} x_k = l$ or $x_k \to l(S_f(\triangle))$.

The set of $\triangle_f-$ statistically convergent sequences will be denoted by $S_f(\triangle)$.

Definition 2.5. Let f be an unbounded modulus function. Then a sequence (x_k) is said to be $\triangle_f$ -statistically Cauchy, if there exists a number $N = N(\varepsilon)$ such that

$d_f(\{k \in N : |\triangle x_k - \triangle x_N| \geq \varepsilon\}) = 0,$

i.e.,

$$\lim_{n\to\infty} \frac{1}{f(n)} f(\{k \leq n : |\triangle x_k - \triangle x_N| \geq \varepsilon\ \}) = 0$$

for every $\varepsilon \geq 0$.

Definition 2.6. Let f be an unbounded modulus function. Then a sequence (x_k) is said to be $\triangle_f$ -statistically bounded if there exists a number $M > 0$ such that

$d_f(\{k \in N : |\triangle x_k| > M\ \}) = 0,$

i.e.,

$$\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n\ :\ |\triangle x_k| > M\}|) = 0.$$

The set of $\triangle_f-$ statistically bounded sequences will be denoted by $BS_f(\triangle)$.

Theorem 2.7. Let f and g be two unbounded modulus functions. Then

(i) if (1) holds then a sequence (x_k) is $\triangle_f-$ statistically convergent (with same limit) if it is $\triangle_g-$ statistically convergent, that is $S_g(\triangle) \subseteq S_f\ (\triangle)$.

(ii) if (2) holds then a sequence (x_k) is $\triangle_f-$ statistically convergent if and only if it is $\triangle_g-$ statistically convergent, that is $S_g(\triangle) = S_f(\triangle)$.

Proof (i) Suppose (x_k) is $\triangle_g-$ statistically convergent to l, that is $S_g(\triangle) - lim\, x_k = l$. Define $V = \{k \in N\ :\ |\triangle x_k - l| \geq \varepsilon\}$. Then

$$d_g(V) = \lim_{n\to\infty} \frac{g(|\{k \leq n : |\triangle x_k - l| \geq \varepsilon\}|)}{g(n)} = 0$$

and this implies

$$d_f(V) = \lim_{n\to\infty} \frac{f(|\{k \leq n : |\triangle x_k - l| \geq \varepsilon\}|)}{f(n)} = 0$$

if (1) holds by Theorem 2.2 (i) .

The Proof of (ii) follows from the Theorem 2.2 (ii).

Corollary 2.8. Let f be an unbounded modulus function. If (3) holds then $S_f(\triangle) = S(\triangle)$.

Proof Let the sequence (x_k) be $\triangle_f$ -statistically convergent to l. Then $d_f\ (V) = 0$ if we choose $V = \{k \in N : |\triangle x_k - l| \geq \varepsilon\ \}$. Now the proof follows from the fact for any $V \subseteq N$ and any modulus f , $d_f\ (V) = 0$ implies that $d(V) = 0$. then $S_f(\triangle) \subseteq S(\triangle)$.

To show that $S(\triangle) \subseteq S_f(\triangle)$, Assume that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and let the sequence (x_k) be $\triangle -$statistically convergent to l. Then we have

$$\lim_{n\to\infty} \frac{(|\{k \leq n\ :\ |\triangle x_k - l| \geq \varepsilon\ \}|)}{n} = 0$$

for every $\varepsilon > 0$. Now we may write

$$\frac{f(|\{k \leq n\ :\ |\triangle x_k - l| \geq \varepsilon\ \}|)}{f(n)} \leq \frac{(|\{k \leq n\ :\ |\triangle x_k - l| \geq \varepsilon\ \}|) f(1)}{n} \cdot \frac{n}{f(n)}.$$

Since $\lim_{t\to\infty}\frac{f(t)}{t}>0$, the right-hand side of above inequality tends to 0 and this implies that the left-hand side tends to 0 as $n\to\infty$. Therefore the sequence (x_k) is $\triangle_f$ -statistically convergent.

Theorem 2.9. Let f and g be two unbounded modulus functions. Then

(i) if the limit exists and (1) holds then a $\triangle_g-$ statistically Cauchy sequence is $\triangle_f-$ statistically Cauchy sequence,

(ii) if the limit exists and (2) holds then a sequence (x_k) is $\triangle_g-$ statistically Cauchy sequence if and only if it is $\triangle_f$ -statistically Cauchy sequence.

Taking $V=\{k\in N\ :\ \ |\triangle x_k-\triangle x_N|\geq\varepsilon\}$ the proof of (i) and (ii) follows from Theorem 2.2 (i) and (ii), respectively.

It is easy to show that a $\triangle_f$ - statistically convergent is $\triangle_f$ -statistically Cauchy sequence.

Theorem 2.10. Let f be an unbounded modulus function. Then a sequence of real numbers is $\triangle_f$ -statistically convergent if and only if it is a $\triangle_f$ -statistically Cauchy sequence.

Proof In order to prove that a $\triangle_f$ -statistically Cauchy sequence is $\triangle_f$ - statistically convergent we may use the technique given in the proof of Theorem 3.3 in [Aizpuru et al.(2014)].

Lemma 2.11 [1]. If $V\subset N$ is infinite, then there exists an unbounded modulus f such that $d_f(V)=1$.

We may give the following result by using Theorem 40 in [Bhardwaj et al. (2016)]].

Theorem 2.12. If for every unbounded modulus f, $(x_k)\in BS_f(\triangle)$, then $(x_k)\in\ell_\infty(\triangle)$.

Proof Let $(x_k)\in BS_f(\triangle)$. Suppose, if possible, $(x_k)\notin\ell_\infty(\triangle)$. Then for every $M>0$, we have that $V=\{k\in N\ :\ \ |\triangle x_k|>M\}$ is an infinite set and so by Lemma 2.11 there exists an unbounded modulus f such that $d_f(V)=1$ which contradicts the assumption that $(x_k)\in BS_f(\triangle)$ for every modulus f.

Theorem 2.13. Let f and g be two unbounded modulus functions. Then

(i) if (1) holds then a $\triangle_g-$ statistically bounded sequence is $\triangle_f-$ statistically bounded, that is $BS_g(\triangle)\subseteq BS_f(\triangle)$,

(ii) if (2) holds then a sequence is $\triangle_g-$ statistically bounded if and only if it is $\triangle_f-$ statistically bounded, that is $BS_g(\triangle)=BS_f(\triangle)$.

Proof Let the sequence (x_k) be $\triangle_g-$ statistically bounded. Then there exists a real number $M>0$ such that $d_g(\{k\in N:|\triangle x_k|>M\})=0$. Taking $V=\{k\in N:|\triangle x_k|>M\}$ the proof follows from Theorem 2.2 (i) and (ii).

Corollary 2.14. For any unbounded modulus f we have

(i) $BS_f(\triangle)\subseteq BS(\triangle)$,

(ii) if (3) holds $BS_f(\triangle)=BS(\triangle)$.

Proof Let the sequence (x_k) be $\triangle_f$ -statistically bounded. Then $d_f(V)=0$ if we choose $V=\{k\in N:|\triangle x_k|>M\}$ for an M large enough. Now the proof (i) follows from the fact " for any $V\subseteq N$ and any modulus f, $d_f(V)=0$ implies that $d(V)=0$ " and (ii) follows from the Corollary 2.3.

Theorem 2.15. Let f and g be two unbounded modulus functions. If (1) holds then a $\triangle_g-$ statistically convergent sequence is $\triangle_f-$ statistically bounded, that is $S_g(\triangle)\subseteq BS_f(\triangle)$.

Proof Suppose that the sequence (x_k) is $\triangle_g-$ statistically convergent to l. Let $\varepsilon>0$ be given and define $V(n)=\{k\leq n:|\triangle x_k-l|\geq\varepsilon\}$ and $Q(n)=\{k\leq n:|\triangle x_k-l|>M\}$ for a number $M>\varepsilon$ large enough. Now since clearly $|V(n)|\geq|Q(n)|$ for every $n\in N$ we have that $d_g(V)\geq d_g(Q)$ and so that $d_g(V)=0$ implies $d_g(Q)=0$. If (1) holds then $d_g(Q)=0$ implies $d_f(Q)=0$ by Theorem 2.2 (i). This means that (x_k) is $\triangle_f-$ statistically bounded.

Conclusion

In this study, considering any modulus function *f* we first defined $\triangle_f$ - statistical convergence and $\triangle_f$ - statistical boundedness of a number sequence and then defined a $\triangle_f$ - statistically Cauchy sequence. Then under some conditions on the modulus functions *f* and *g,* including the modulus *f(x)=x,* we established the relations between the sets $S_f(\triangle)$ and $S_g(\triangle)$, the relations between the sets $BS_f(\triangle)$ and $BS_g(\triangle)$ and the relations between the sets $S_f(\triangle)$ and $BS_g(\triangle)$ which are constructed via modulus functions f and g. Furthermore, under some conditions, it is also given that it is equivalent for a sequence to be $\triangle_g$ - statistically Cauchy sequence or to be $\triangle_f$ - statistically Cauchy sequence. The subject of difference sequences and their generalizations has been studied intensively since 1981. This study contributes to the subject and constitutes a resource that can provide important contributions to those who will work in this field.

There are no conflicts of interest to report. This research has not received any specific grant from funding agencies, in the commercial or non-profit sector.

References

Aizpuru A., Listán-García M. C., & Rambla-Barreno, F. (2014). Density by moduli and statistical convergence, *Questiones Mathematicae*, *37*(4), 525-530. DOI: https://doi.org/10.2989/16073606.2014.981683

Bhardwaj, V. K., Dhawan, S., & Gupta, S. (2016). Density by moduli and statistical boundedness. *Abstract and Applied Analysis, 2016*. DOI: https://doi.org/10.1155/2016/2143018

Buck, R. C. (1953). Generalized asymptotic density. *American Journal of Mathematics, 75*(2), 335-346,

Çolak, R. (2020). Some relations between the sets of f-statistically convergent sequences. In H. Dutta, Z. Hammouch, H. Bulut, & H. Baskonus, H. (Eds.) *4th International Conference on Computational Mathematics and Engineering Sciences (Advances in Intelligent Systems and Computing, Vol. 1111).* Springer, Cham. DOI: https://doi.org/10.1007/978-3-030-39112-6_25

Et, M., & Colak, R. (1995). On some generalized difference sequence spaces. *Soochow Journal of Mathematics*, *21*(4), 377-386.

Fast, H. (1951). Sur la convergence statistique. *Colloquium Mathematicum 2*, 241-244. DOI: https://doi.org/10.4064/cm-2-3-4-241-244

Kayan, E., Çolak, R., & Altin, Y. (2018). d- statistical convergence of order α and d- statistical boundedness of order α in metric spaces. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, 80*(4), 229-238

Kizmaz, H. (1981). On certain sequence spaces. *Canadian Mathematical Bulletin, 24*(2), 169-176.

Maddox, I. (1987). Inclusions between FK spaces and Kuttner's theorem. *Mathematical Proceedings of the Cambridge Philosophical Society*, *101*(3), 523-527. DOI: https://doi.org/10.1017/S0305004100066883.

Nakano, H. (1953). Concave modular. *Journal of the Mathematical Society of Japan,* (5), 29-49. DOI: https://doi.org/10.2969/jmsj/00510029

Ruckle, W. (1973). FK Spaces in Which the Sequence of Coordinate Vectors is Bounded. *Canadian Journal of Mathematics*, *25*(5), 973-978. DOI: https://doi.org/10.4153/CJM-1973-102-9

Schoenberg, I. J. (1959). The Integrability of Certain Functions and Related Summability Methods. *The American Mathematical Monthly*, *66*(5), 361–775. DOI: https://doi.org/10.1080/00029890.1959.11989303

Steinhaus, H. (1951). Sur la convergence ordinaire et la convergence asymptotique, *Colloquium Mathematicum, 2*, 73-74.

Zygmund, A. (1979). Trigonometric series. Cambridge, UK: Cambridge University Press.