Acta Scientiarum

http://periodicos.uem.br/ojs
ISSN on-line: 1807-8664
Doi: 10.4025/actascitechnol.v45i1.61519

Normal-tangent-logarithm-(G_1, G_2): a class of probabilistic distributions depending on two baselines

Natália Moraes Cordeiro[1], Frank Gomes-Silva[1*], Cícero Carlos Ramos de Brito[2], Jader da Silva Jale[1] and Josimar Mendes de Vasconcelos[1]

[1]Departamento de Estatística e Informática, Universidade Federal Rural de Pernambuco, Rua Dom Manuel de Medeiros, s/n, 52171-900, Dois Irmãos, Recife, Pernambuco, Brazil. [2]Instituto Federal de Educação, Ciência e Tecnologia de Pernambuco, Recife, Pernambuco, Brazil. *Author for correspondence. E-mail: franksinatrags@gmail.com

ABSTRACT. Based on the normal distribution, a new generator of continuous distributions is presented using the monotonic functions $\tan((\pi/2)G_1)$ and $\log(1 - G_2)$, such that G_1 and G_2 are the baselines. A study of identifiability of the proposed class is exhibited as well as the series expansions for its cumulative distribution function and probability density function. Additionally, some mathematical properties of the class are discussed, namely, the raw moments, the central moments, the moment generating function, the characteristic function, the derivatives of the log-likelihood function, and a study of the support. A numerical analysis comprising a simulation study and an application to real data is presented. Comparisons between the proposed model and other well-known models evince its potentialities and modeling benefits.

Keywords: normal distribution; goodness-of-fit; identifiability; maximum likelihood; Monte Carlo simulation.

Received on November 11, 2021.
Accepted on August 22, 2022.

Introduction

Probability distributions play statistics a fundamental role. Probability models are important tools to deal with real problems since they can provide powerful models to describe natural and social phenomena. In recent times, several methods to generate new distributions have been presented in order to create distributions with higher flexibility than the classical ones.

A review of some relevant methods is presented in Lee, Famoye, and Alzaatreh (2013) and a detailed list of generalized classes of continuous distributions widely found in the statistical literature is cited in Tahir and Nadarajah (2015). Some notable examples are the $T - X$ family of continuous distributions (Alzaatreh, Lee, & Famoye, 2013), the beta-G (Eugene, Lee, & Famoye, 2002), the McDonald-G (Alexander, Cordeiro, Ortega, & Sarabia, 2012), the exponencialized exponential-Poisson (Ristić & Nadarajah, 2014), the logistic-G (Tahir, Cordeiro, Alzaatreh, Mansoor, & Zubair, 2016a), the new Weibull-G (Tahir et al., 2016b), the new gamma-G (de Brito, Rêgo, de Oliveira, & Gomes-Silva, 2017) and the normal-G (Silveira et al., 2019).

Probability mixture models are often used in data modeling with more than one mode. Bimodal distributions are quite useful tools because they model relevant variables in nature. For instance, the sizes of the weaver ant workers are bimodally distributed (Nichols & Padgett, 2006) as well as the number of cases per year of Hodgkin's lymphoma (Mauch, Armitage, Diehl, Hoppe, & Weiss, 1999). In inferential terms, this modeling can be difficult since mixture models can admit a reasonable amount of parameters. It is well-known that probability mixture models can lead to identifiability problems making parametric inferences becomes a hard task. The two-component normal mixture model is a classic example of this problem that can be found (Teicher, 1961).

In this work we employ a method to generate classes of probability distributions (de Brito, Rêgo, de Oliveira, & Gomes-Silva, 2019) to build a class whose cumulative distribution function (cdf) is written as a composition of the standard normal cdf and two baselines, namely, G_1 and G_2. One of the new features of this method is working with multiple baselines. Since this method considers the Lebesgue integral instead of the Riemann integral, one can choose either continuous or discrete distributions to be baselines. The proposed class is called Normal-tangent-logarithm-(G_1, G_2), NTL-(G_1, G_2) for short, and besides furnish a more parsimonious model, it holds interesting properties, like the parametric identifiability (under certain conditions) and the bimodality of some special cases. The name of the class alludes to the monotonic functions that are used in its definition.

The method to generate distributions and classes of probabilistic distributions presented by ~~de~~ Brito et al. (2019) is given by:

$$F_{G_1,\ldots,G_m}(x) = \int_{l_1(\cdot)(x)}^{\mu_1(\cdot)(x)} dF(t)$$

where

$\mu_1(\cdot)(x) = \mu_1(G_1, G_2, \ldots, G_m)(x)$ and $l_1(\cdot)(x) = l_1(G_1, G_2, \ldots, G_m)(x)$ are compositions known probability models.

This method extends the process of building probability distributions, allowing the classes to be designed from classic distributions and predefined univariate monotonic functions. Thus, considering the monotonic functions $\mu_1(G_1, G_2, \ldots, G_m)(x) = \tan\left(\frac{\pi}{2} G_1(x)\right)$, $l_1(G_1, G_2, \ldots, G_m)(x) = \log\left(1 - G_2(x)\right)$ and the standard normal probability density function (pdf), the cdf of the new class is given by:

$$H_{G_1,\,G_2}(x) = \int_{\log(1-G_2(x))}^{\tan\left(\frac{\pi}{2}G_1(x)\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \Phi\left(\tan\left(\frac{\pi}{2} G_1(x)\right)\right) - \Phi\left(\log\left(1 - G_2(x)\right)\right), \quad (1)$$

where G_1 and G_2 are the baselines. Given that $g_1(x)$ and $g_2(x)$ are the pdfs associated with $G_1(x)$ and $G_2(x)$ respectively, the pdf associated with $H_{G_1,\,G_2}$ is given by:

$$h_{G_1,\,G_2}(x) = H'_{G_1,G_2}(x) = \Phi'\left[\tan\left(\frac{\pi}{2} G_1(x)\right)\right] - \Phi'\left[\log\left(1 - G_2(x)\right)\right]$$

$$= \frac{\pi}{2} g_1(x) \sec^2\left(\frac{\pi}{2} G_1(x)\right) \phi\left[\tan\left(\frac{\pi}{2} G_1(x)\right)\right] + \frac{g_2(x)}{1-G_2(x)} \phi\left[\log\left(1 - G_2(x)\right)\right]. \quad (2)$$

Equation (1) can be reduced to new unibaseline ("classic generators") class by setting $G_1 = G_2$. It is worth noting that for each pair (G_1, G_2) of baselines we have a completely new class and that $H_{G_1,\,G_2} \neq H_{G_2,\,G_1}, \forall\, G_1 \neq G_2$. Based on Equation (2), the proposed model is quite competitive with the two-component probability mixture models.

The remainder of the paper is organized as follows. Identifiability, support, series representation, row and central moments, moment generating function (mgf) and characteristic function (cf), and estimation and inference are derived in Section 2. Applications to both simulated and real data sets are addressed in Section 3. Finally, we offer a conclusion in Section 4.

Mathematical properties

Identifiability

Discussing the identifiability of a probability distribution is considerably important, because the parametric estimation of unidentifiable distributions may not be trustworthy. Under specific conditions, the (sole) theorem presented in Silveira et al. (2021) assures that the submodels of the normal-(G_1, G_2) class of probability distributions enjoy the property of identifiability. Since the NTL-(G_1, G_2) clearly resembles the normal-(G_1, G_2) class, we bring an adapted version of the aforementioned theorem for the new class in the following lines.

Theorem 1. Let $G_1(x|\theta_1)$ and $G_2(x|\theta_2)$ be the baselines of the NTL-(G_1, G_2) class, whose cdf is defined by the expression.

$$H_{G_1,\,G_2}(x|\theta) = \Phi\left(\tan\left(\frac{\pi}{2} G_1(x)\right)\right) - \Phi\left(\log\left(1 - G_2(x)\right)\right).$$

Additionally,

$$\theta_1 = (\theta_{11}, \ldots, \theta_{1n}) \in \Theta_1,\ \theta_2 = (\theta_{21}, \ldots, \theta_{2m}) \in \Theta_2$$

and

$$\theta = (\theta_{11}, \ldots, \theta_{1n}, \theta_{21}, \ldots, \theta_{2m}) \in \Theta$$

where Θ_1, Θ_2 and Θ are the parametric spaces associated with G_1, G_2 and $H_{G_1,\,G_2}$ respectively. If G_1 and G_2 are identifiable, then $H_{G_1,\,G_2}$ is identifiable.

Proof. Assume that

$$\Phi\left(\tan\left(\frac{\pi}{2} G_1(x|\theta_1)\right)\right) = \Phi\left(\tan\left(\frac{\pi}{2} G_1(x|\theta_1^*)\right)\right),$$

where

$\{\theta_1, \theta_1^*\} \subset \Theta_1$ and $\theta_1 \neq \theta_1^*$.

Since Φ is injective, $G_1(x|\theta_1) = G_1(x|\theta_1^*)$; this is a contradiction, because it denies the identifiability of G_1. Thus if $\theta_1 \neq \theta_1^*$, then

$$\Phi\left(\tan\left(\frac{\pi}{2}G_1(x|\theta_1)\right)\right) \neq \Phi\left(\tan\left(\frac{\pi}{2}G_1(x|\theta_1^*)\right)\right).$$

Analogously, it is also true for $\theta_2, \theta_2^* \subset \Theta_2$. If $\theta_2 \neq \theta_2^*$ then

$$\Phi(\log[1 - G_2(x|\theta_2)]) \neq \Phi(\log[1 - G_2(x|\theta_2^*)]).$$

Now consider $\{\theta, \theta^*\} \subset \Theta$ and assume that

$H_{G_1, G_2}(x|\theta) = H_{G_1, G_2}(x|\theta^*)$. If $\theta_1 = \theta_1^*$ and $\theta_2 \neq \theta_2^*$,

we can infer from the equation

$$H_{G_1, G_2}(x) = \Phi\left(\tan\left(\frac{\pi}{2}G_1(x)\right)\right) - \Phi\big(\log\big(1 - G_2(x)\big)\big)$$

that $G_2(x|\theta_2) = G_2(x|\theta_2^*)$, namely, an absurd. Likewise, if $\theta_1 \neq \theta_1^*$ and $\theta_2 = \theta_2^*$, we get to a similar contradiction, where $G_1(x|\theta_2) = G_1(x|\theta_2^*)$. If $\theta_1 \neq \theta_1^*$ and $\theta_2 \neq \theta_2^*$ then the assumption fails since $H_{G_1, G_2}(x|\theta) \neq H_{G_1, G_2}(x|\theta^*)$ for almost all values of x within the support. Thus, H_{G_1, G_2} is identifiable.

Support of the NTL-(G_1, G_2) class

Almost all probability distributions that appear in the statistical literature depend on one single baseline. In such case, the distribution emerged from the class and the associated baseline usually share the same support. Defining the support of the NTL-(G_1, G_2) though demands further attention.

As mentioned in the first paragraph of Section (1), we have that

$$\mu_1[G_1(x), G_2(x)] = \tan\left(\frac{\pi}{2}G_1(x)\right)$$

and

$$l_1[G_1(x), G_2(x)] = \log[1 - G_2(x)].$$

It can be observed that

1. $S_\Phi = (-\infty, +\infty)$, the support of the normal distribution, is a convex set;

2. (a) $\mu_1[G_1(+\infty), G_2(+\infty)] = \mu_1(1,1) = \tan\left(\frac{\pi}{2}G_1(+\infty)\right) = +\infty = \sup[x \in \mathbb{R} : \Phi(x) < 1]$;

(b) $l_1[G_1(+\infty), G_2(+\infty)] = l_1(1,1) = \log[1 - G_2(+\infty)] = -\infty = \inf[x \in \mathbb{R} : \Phi(x) > 0]$;

(c) $\mu_1[G_1(x), G_2(x)]$ and $l_1[G_1(x), G_2(x)] = \log[1 - G_2(x)]$ are monotonic functions.

These statements satisfy the hypotheses established by Theorem (T4) in de Brito et al. (2019). Therefore, the support of $H_{G_1, G_2}(\cdot)$ is the union of the supports of both baselines, that is, $S_{H_{G_1, G_2}} = S_{G_1} \cup S_{G_2}$.

Figures 1 and 2 evince the flexibility of the proposed class considering different baselines and it also illustrates the important result of the support shown in this subsection.

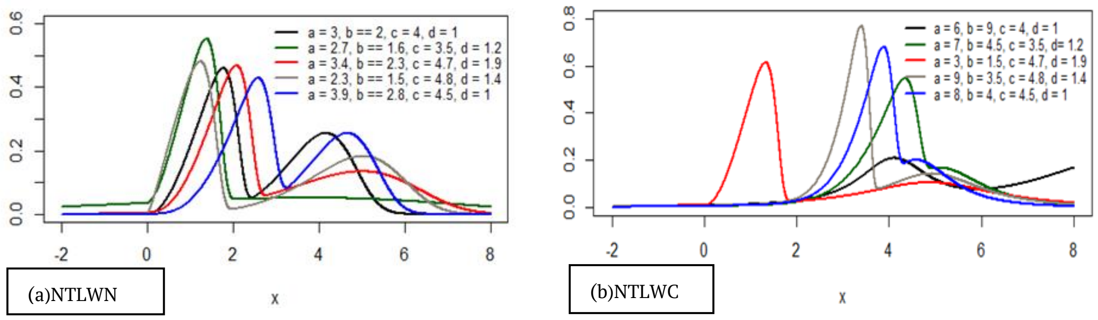

Figure 1. Support behavior for some submodels of the proposed class.

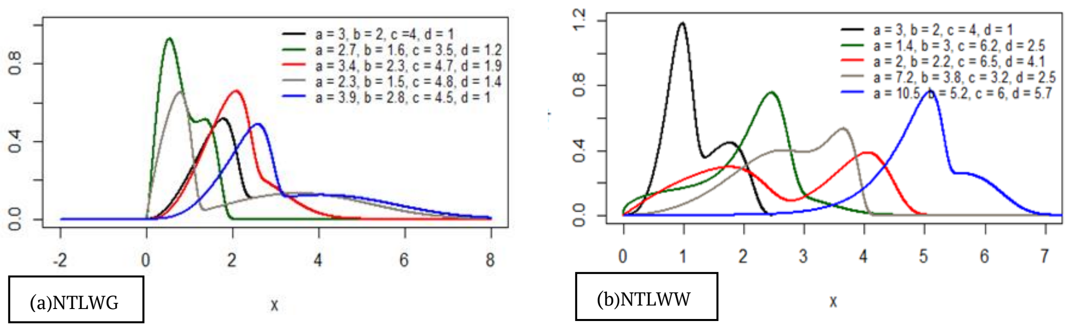

Figure 2. Support behavior for some submodels of the proposed class.

The chosen baselines are Weibull (W), normal (N), Cauchy (C) and gamma (Ga). In some cases (Figure 1a and b), we have $S_{G_1} \neq S_{G_2}$. Furthermore, the pdfs can have bimodal shape.

Series representation

This section plays a fundamental role in terms of the development of subsequent mathematical properties. The normal cdf can be written in terms of the error function as follows:

$$\Phi(z) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)\right], \tag{3}$$

such that $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\frac{s^2}{2}}\,ds$. Since $\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)$ can be linearly represented by:

$$\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{z}{\sqrt{2}}\right)^{2n+1}}{n!\,(2n+1)}$$

$$= \sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\frac{z^{2n+1}}{n!(2n+1)}. \tag{4}$$

Replacing (4) in (3), we get:

$$\Phi(z) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\frac{z^{2n+1}}{n!\,(2n+1)}.$$

Thus, we have that:

$$\Phi\left[\tan\left(\frac{\pi}{2}G_1(x)\right)\right] = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\frac{\left[\tan\left(\frac{\pi}{2}G_1(x)\right)\right]^{2n+1}}{n!(2n+1)} \tag{5}$$

Since,

$$\tan(y) = \sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k+2}(2^{2k+2}-1)B_{2k+2}}{(2k+2)!}y^{2k+1} \tag{6}$$

for $|y| < \frac{\pi}{2}$ and $B_p = \sum_{l=0}^{p-1}\binom{p}{l}\frac{B_l}{k-l+1}$
(Bernoulli numbers). Thus, rewriting (5) from Eq. (6) we have

$$\Phi\left[\tan\left(\frac{\pi}{2}G_1(x)\right)\right] = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n$$

$$\times\frac{\left[\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k+2}(2^{2k+2}-1)B_{2k+2}}{(2k+2)!}\left[\frac{\pi}{2}G_1(x)\right]^{2k+1}\right]^{2n+1}}{n!\,(2n+1)}$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\left(\frac{\pi}{2}\right)^{4kn+2k+2n+1}$$

$$\times\frac{[G_1(x)]^{2n+1}\left[\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k+2}(2^{2k+2}-1)B_{2k+2}\left[\frac{\pi}{2}G_1(x)\right]^{2k}}{(2k+2)!}\right]^{2n+1}}{n!(2n+1)}. \tag{7}$$

According to Gradshteyn, Ryzhik, Jeffrey, and Zwillinger (2007), a power series raised to a positive integer *N* can be expanded as:

$$\left[\sum_{j=0}^{\infty} a_j y^j\right]^N = \sum_{k=0}^{\infty} c_j y^j,$$

where $c_0 = a_0^N$ and $c_j = \frac{1}{ja_0}\sum_{s=1}^{j}(sN - j + s)a_s c_{j-s}$, for $j \geq 1$ and $N \in \mathbb{N}$. In this way, (7) can be rewritten as:

$$\Phi\left[\tan\left(\frac{\pi}{2}G_1(x)\right)\right] = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{4kn+2k+2n+1}}{2^{4kn+2k+3n+1}}\frac{[G_1(x)]^{2n+1}\sum_{k=0}^{\infty} c_k\{[G_1(x)]^2\}^k}{n!\,(2n+1)}$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^n\pi^{4kn+2k+2n+1}c_k}{n!\,(2n+1)\,2^{4kn+2k+3n+1}}[G_1(x)]^{2n+2k+1},$$

where

$$c_0 = a_0^{2n+1},\ c_k = \frac{1}{ka_0}\sum_{s=1}^{k}(s(2n+1) - k + s)a_s c_{k-s},\ a_0 = 1$$

and

$$a_s = \frac{(-1)^k 2^{2k+2}(2^{2k+2}-1)B_{2k+2}}{(2k+2)!},$$

for

$k \geq 1$ and $n \in \mathbb{N}$.

Analogously, we have that:

$$\Phi[\log(1 - G_2(x))] = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\frac{\log(1-G_2(x))^{2n+1}}{n!(2n+1)}. \quad (8)$$

Furthermore,

$$\log(1 - y) = -\sum_{j=0}^{\infty}\frac{y^{j+1}}{j+1},\ \text{for } |y| < 1.$$

Therefore, (8) can be rewritten as:

$$\Phi[\log(1 - G_2(x))] = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\frac{\left[-\sum_{j=0}^{\infty}\frac{[G_2(x)]^{j+1}}{j+1}\right]^{2n+1}}{n!\,(2n+1)}$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^{3n+1}}{n!(2n+1)2^n}\left[\sum_{j=0}^{\infty}\frac{[G_2(x)]^{j+1}}{j+1}\right]^{2n+1}. \quad (9)$$

Equation (9) can be rewritten similarly as we did with (7), that is, expanding a power series raised to a positive integer. Thus:

$$\Phi[\log(1 - G_2(x))] = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^{3n+1}}{n!\,(2n+1)2^n}[G_2(x)]^{2n+1}\left[\sum_{j=0}^{\infty}\frac{[G_2(x)]^j}{j+1}\right]$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{3n+1}}{n!(2n+1)2^n}d_j\,[G_2(x)]^{2n+j+1}, \quad (10)$$

where

$$d_0 = 1,\ d_j = \frac{1}{j}\sum_{s=1}^{j}(2s(n+1) - j)a_s d_{j-s}$$

and

$$a_s = \frac{1}{2s+1}$$

for ≥ 1 and $N \in \mathbb{N}$.

Using the expressions obtained in (7) and (10), we can rewrite (2) as follows:

$$H_{G_1,G_2}(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^n \pi^{4kn+2k+2n+1} c_k}{n!\,(2n+1)\,2^{4kn+2k+3n+1}} [G_1(x)]^{2n+2k+1}$$

$$- \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{3n+1} d_j}{n!(2n+1)2^n} [G_2(x)]^{2n+j+1}. \quad (11)$$

Defining

$$\beta_{1,n,k} = \frac{(-1)^n \pi^{4kn+2k+2n+1} c_k}{n!(2n+1)\,2^{4kn+2k+3n+1}}$$

and

$$\beta_{2,n,j} = \frac{(-1)^{3n+1} d_j}{n!(2n+1)2^n},$$

Equation (11) can be written as follows:

$$H_{G_1,G_2}(x) = \frac{1}{\sqrt{2\pi}} \left\{ \sum_{n,k=0}^{\infty} \beta_{1,n,k} [G_1(x)]^{2n+2k+1} - \sum_{n,j=0}^{\infty} \beta_{2,n,j} [G_2(x)]^{2n+j+1} \right\}, \quad (12)$$

namely, the cdf of the new class expressed as a linear combination of exponentiated baselines. In case of continuous G_1 and G_2 the pdf associated to (12), namely, $h_{G_1,G_2}(x) = H'_{G_1,G_2}(x)$, can be expressed as follows:

$$h_{G_1,G_2}(x) = \frac{1}{\sqrt{2\pi}} \left\{ \sum_{n,k=0}^{\infty} \beta_{1,n,k}\, g_{1,2n+2k+1}(x) - \sum_{n,j=0}^{\infty} \beta_{2,n,j}\, g_{2,2n+j+1}(x) \right\},$$

where

$$g_{1,2n+2k+1}(x) = [2n+2k+1] g_1(x) [G_1(x)]^{2n+2k}$$

and

$$g_{2,2n+j+1}(x) = [2n+j+1] g_2(x) [G_2(x)]^{2n+j}.$$

Raw moments

The (i, j, k)th Probability Weighted Moments (PWM) introduced by Greenwood, Landwehr, Matalas, and Wallis (1979), are a generalization of the usual moments for probability models. PWM are an alternative for parametric estimating when it is not possible through the methods of moments and maximum likelihood. The PWM is given by:

$$\tau_{i,j,k} = \mathbb{E}\{X^i [F(X)]^j [1-F(X)]^k\} = \int_0^1 \mathbb{Q}[F(x)]^i F(x)^j [1-F(x)]^K dF(x)$$

where $i, j, k \in \mathbb{R}$ and $\mathbb{Q}(\cdot)$ denotes the quantile function of $F(\cdot)$ cdf. The advantage of computing moments in terms of PWM is that in most distributions, or at least the main ones, these quantities are defined in the literature (Cordeiro & Nadarajah, 2011). The expressions seen in the next sections are written in terms of PWMs.

Moments are very important in Statistics, since they characterize the probability distributions and determine measures of central tendency, dispersion, skewness, and kurtosis. The expressions for the raw moments of the NTL-(G_1, G_2) class are presented in the following lines.

It is known that

$$\mu_m = \mathbb{E}(X^m) = \int_{-\infty}^{+\infty} x^m dH_{G_1,G_2}(x). \quad (13)$$

Thus, inserting (12) in (13) we get to:

$$\mu_m = \int_{-\infty}^{+\infty} x^m \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \beta_{1,n,k} (2n+2k+1) g_1(x) [G_1(x)]^{2n+2k}\, dx$$

$$- \int_{-\infty}^{+\infty} x^m \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \beta_{2,n,j} (2n+j+1) g_2(x) [G_2(x)]^{2n+j}\, dx. \quad (14)$$

Making

$$\int_{-\infty}^{+\infty} x^m g_1(x) [G_1(x)]^{2n+2k} dx = \tau_{m,2n+2k,0}$$

and

$$\int_{-\infty}^{+\infty} x^m g_2(x)[G_2(x)]^{2n+j} dx = \eta_{m,2n+j,0},$$

Equation (14) can be expressed by:

$$\mu_m = \frac{1}{\sqrt{2\pi}} \sum_{n,k=0}^{\infty} (2n+2k+1)\beta_{1,n,k}\, \tau_{m,2n+2k,0} - \frac{1}{\sqrt{2\pi}} \sum_{n,j=0}^{\infty} (2n+j+1)\beta_{2,n,j}\, \eta_{m,2n+j,0}. \quad (15)$$

Equation (15) reveals that the moments of any NTL-(G_1, G_2) distribution is an infinite weighted sum of PWMs of the baselines G_1 and G_2.

The mean μ for the NTL-(G_1, G_2) class can be obtained making $m = 1$ in (15). Thus:

$$\mu_1 = \mu = \frac{1}{\sqrt{2\pi}} \sum_{n,k=0}^{\infty} (2n+2k+1)\beta_{1,n,k}\, \tau_{1,2n+2k,0} - \frac{1}{\sqrt{2\pi}} \sum_{n,j=0}^{\infty} (2n+j+1)\beta_{2,n,j}\, \eta_{1,2n+j,0}.$$

Central moments

The m-th central moment is denoted by:

$$\mu'_m = E[(X-\mu)^m] = \int_{-\infty}^{+\infty} (x-\mu)^m dH(x),$$

and

$$\mu'_m = \sum_{r=0}^{m} \binom{m}{r} (-1)^r \mu^r \mu_{m-r}. \quad (16)$$

However,

$$\mu_{m-r} = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (2n+2k+1)\beta_{1,n,k}\tau_{m-r,2n+2k,0}$$

$$-\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (2n+j+1)\beta_{2,n,j}\eta_{m-r,2n+j,0} \quad (17)$$

where

$$\int_{-\infty}^{+\infty} x^{m-r} g_1(x)[G_1(x)]^{2n+2k} dx = \tau_{m-r,2n+2k,0}$$

and

$$\int_{-\infty}^{+\infty} x^{m-r} g_2(x)[G_2(x)]^{2n+j} dx = \eta_{m-r,2n+j,0}.$$

Thus, inserting (17) in (16), we get to:

$$\mu'_m = \sum_{r=0}^{m} \binom{m}{r} (-1)^r \mu^r \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (2n+2k+1)\beta_{1,n,k}\tau_{m-r,2n+2k,0}$$

$$-\sum_{r=0}^{m} \binom{m}{r} (-1)^r \mu^r \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (2n+j+1)\beta_{2,n,j}\eta_{m-r,2n+j,0}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{m} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{m}{r} (-1)^r \mu^r (2n+2k+1)\beta_{1,n,k}\tau_{m-r,2n+2k,0}$$

$$-\frac{1}{\sqrt{2\pi}} \sum_{r=0}^{m} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \binom{m}{r} (-1)^r \mu^r (2n+j+1)\beta_{2,n,j}\eta_{m-r,2n+j,0}. \quad (18)$$

Moreover, considering that

$$\delta_{1,m,r,n,k} = \binom{m}{r} (-1)^r \mu^r (2n+2k+1)\beta_{1,n,k}$$

and

$$\delta_{2,m,r,n,j} = \binom{m}{r} (-1)^r \mu^r (2n+j+1)\beta_{2,n,j}, \quad (19)$$

Equation (18) can be written as:

$$\mu'_m = \frac{1}{\sqrt{2\pi}}\sum_{r=0}^{m}\sum_{n,k=0}^{\infty}\delta_{1,m,r,n,k}\tau_{m-r,2n+2k,0} - \frac{1}{\sqrt{2\pi}}\sum_{r=0}^{m}\sum_{n,j=0}^{\infty}\delta_{2,m,r,n,j}\eta_{m-r,2n+j,0}.$$

For $m = 2$, Equation (19) represents the expansion of the variance for the NTL-(G_1, G_2) class, which is given by:

$$\sigma^2 = \mu'_2 = \frac{1}{\sqrt{2\pi}}\sum_{r=0}^{2}\sum_{n,k=0}^{\infty}\delta_{1,2,r,n,k}\tau_{2-r,2n+2k,0} - \frac{1}{\sqrt{2\pi}}\sum_{r=0}^{2}\sum_{n,j=0}^{\infty}\delta_{2,2,r,n,j}\eta_{2-r,2n+j,0}.$$

Moment generating function and Characteristic function

The mgf is considerably useful, but there are cases in which it does not exist. In such cases, one can use the cf, that always exists. We present in this section the expansions for both functions.

The mgf is defined by:

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} dH_{G_1,G_2}(x). \quad (20)$$

Inserting (12) in (20), we have:

$$M_X(t) = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(2n+2k+1)\beta_{1,n,k}\int_{-\infty}^{+\infty} e^{tx} g_1(x)[G_1(x)]^{2n+2k}dx$$

$$-\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(2n+j+1)\beta_{2,n,j}\int_{-\infty}^{+\infty} e^{tx} g_2(x)[G_2(x)]^{2n+j}dx. \quad (21)$$

Given that

$$e^{tx} = \sum_{m=0}^{\infty}\frac{t^m x^m}{m!}, \quad (22)$$

we can insert (22) in (21) to obtain:

$$M_X(t) = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(2n+2k+1)\beta_{1,n,k}t^m}{m!}\int_{-\infty}^{+\infty} x^m g_1(x)[G_1(x)]^{2n+2k}dx$$

$$-\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(2n+j+1)\beta_{2,n,j}t^m}{m!}\int_{-\infty}^{+\infty} x^m g_2(x)[G_2(x)]^{2n+j}dx. \quad (23)$$

Making

$$\int_{-\infty}^{+\infty} x^m g_1(x)[G_1(x)]^{2n+2k}dx = \tau_{m,2n+2k,0}$$

and

$$\int_{-\infty}^{+\infty} x^m g_2(x)[G_2(x)]^{2n+j}dx = \eta_{m,2n+j,0},$$

the Equation (23) may be written as:

$$M_X(t) = \frac{1}{\sqrt{2\pi}}\left\{\sum_{k,m,n=0}^{\infty}\frac{(2n+2k+1)\beta_{1,n,k}t^m}{m!}\tau_{m,2n+2k,0} - \sum_{j,m,n=0}^{\infty}\frac{(2n+j+1)\beta_{2,n,j}t^m}{m!}\eta_{m,2n+j,0}\right\},$$

namely, the linear representation of the mgf.

The expression for the linear representation of the cf is derived similarly and it is given by:

$$\varphi X(t) = \frac{1}{\sqrt{2\pi}}\left\{\sum_{k,m,n=0}^{\infty}\frac{(2n+2k+1)\beta_{1,n,k}i^m t^m}{m!}\tau_{m,2n+2k,0} - \sum_{j,m,n=0}^{\infty}\frac{(2n+j+1)\beta_{2,n,j}i^m t^m}{m!}\eta_{m,2n+j,0}\right\}.$$

Estimation and Inference

The maximum likelihood method is a mathematical procedure to find estimators for the parameters of a statistical model. The method consists in finding the points that maximize the likelihood function. Such points also maximize the log-likelihood function, which is generally more tractable than the likelihood function itself. In this section, we bring expressions for the log-likelihood function and the score vector considering $\mathbf{x} = (x_1, \dots, x_n)$ an observed sample of size n from a random variable following a distribution belonging to the NTL-(G_1, G_2) class.

Let $\boldsymbol{\theta_1} = (\theta_{11}, \dots, \theta_{1s})$ be the parametric vector of $G_1(x) = G_1(x|\theta_1)$, $\theta_2 = (\theta_{21}, \dots, \theta_{2m})$ the parametric vector of $G_2(x) = G_2(x|\theta_2)$ and $f_{G_1,G_2}(x) = f_{G_1,G_2}(x|\theta)$, where $\theta = (\theta_{11}, \dots, \theta_{1s}, \theta_{21}, \dots, \theta_{2m})$. The corresponding log-likelihood function can be written as:

$$\ell(\boldsymbol{\theta}; X) = \sum_{i=1}^{n} \left\{ \log\left\{\frac{\pi}{2} g_1(x_i; \boldsymbol{\theta_1}) \sec^2\left[\frac{\pi}{2} G_1(x_i; \boldsymbol{\theta_1})\right] \phi\left[\frac{\pi}{2} G_1(x_i; \boldsymbol{\theta_1})\right]\right\} + \frac{g_2(x_i; \boldsymbol{\theta_2})}{1 - G_2(x_i; \boldsymbol{\theta_2})} \phi\{\log[1 - G_2(x_i; \boldsymbol{\theta_2})]\} \right\}.$$

The maximum likelihood estimates for θ can also be obtained solving the system $U(\theta; X) = o_{s+m}$, where o_{s+m} is the null vector of size $(s + m) \times 1$ and $U(\theta; X) = \triangle_\theta\, \ell(\theta; X) = (u_j)_{1 \le j \le s+m}$ is the score vector, whose elements are given by:

$$u_j = \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial \theta_{2j}} \log\left\{\frac{\pi}{2} g_1(x_i; \boldsymbol{\theta_1}) \sec^2\left[\frac{\pi}{2} G_1(x_i; \boldsymbol{\theta_1})\right] \phi\left[\frac{\pi}{2} G_1(x_i; \boldsymbol{\theta_1})\right]\right\} + \frac{g_2(x_i; \boldsymbol{\theta_2})}{1 - G_2(x_i; \boldsymbol{\theta_2})} \phi\{\log[1 - G_2(x_i; \boldsymbol{\theta_2})]\} \right\}$$

for $1 \le j \le s$, and

$$u_j = \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial \theta_{2k}} \log\left\{\frac{\pi}{2} g_1(x_i; \boldsymbol{\theta_1}) \sec^2\left[\frac{\pi}{2} G_1(x_i; \boldsymbol{\theta_1})\right] \phi\left[\frac{\pi}{2} G_1(x_i; \boldsymbol{\theta_1})\right]\right\} + \frac{g_2(x_i; \boldsymbol{\theta_2})}{1 - G_2(x_i; \boldsymbol{\theta_2})} \phi\{\log[1 - G_2(x_i; \boldsymbol{\theta_2})]\} \right\}$$

for $1 \le j \le m$.

Applications to the NTL-(G_1, G_2) class

We bring in this section some applications to the proposed class.

Theoretical application

Let NTL-Gompertz-IGamma (NTLGoIGa) denote the distribution generated by the NTL-(G_1, G_2) class when G_1, g_1, G_2 and g_2 in (1) are replaced by the cdf and the pdf of the Gompertz (Go) distribution, the cdf and the pdf of the Inverse Gamma (IGa) distribution respectively. Thus, the NTLGoIGa cdf is given by:

$$H_{Go,IGa}(x) = \int_{\log\left(1 - \frac{\Gamma\left(\alpha, \frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)}^{\tan\left[\frac{\pi}{2}\left(1 - e^{-\frac{\theta}{\lambda}\left(e^{\lambda x - 1}\right)}\right)\right]} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

Similarly, the NTLGoIGa pdf is obtained by (2). It is given by:

$$h_{Go,GaI}(x) = \frac{\pi}{2} \theta e^{\lambda x} e^{-\frac{\theta}{\lambda}\left(e^{\lambda x - 1}\right)} \sec^2\left(\frac{\pi}{2}\left(1 - e^{-\frac{\theta}{\lambda}\left(e^{\lambda x - 1}\right)}\right)\right) \phi\left[\tan\left(\frac{\pi}{2}\left(1 - e^{-\frac{\theta}{\lambda}\left(e^{\lambda x - 1}\right)}\right)\right)\right] + \frac{\frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{x}\right)^{\alpha+1} e^{\frac{-\beta}{x}}}{1 - \frac{\Gamma\left(\alpha, \frac{\beta}{x}\right)}{\Gamma(\alpha)}} \phi\left[\log\left(1 - \frac{\Gamma\left(\alpha, \frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)\right],$$

which is a theoretical application of the class.

Graphs of the pdf and the failure rate considering different values of the parameters θ, λ, α, and β are presented in Figure 3.

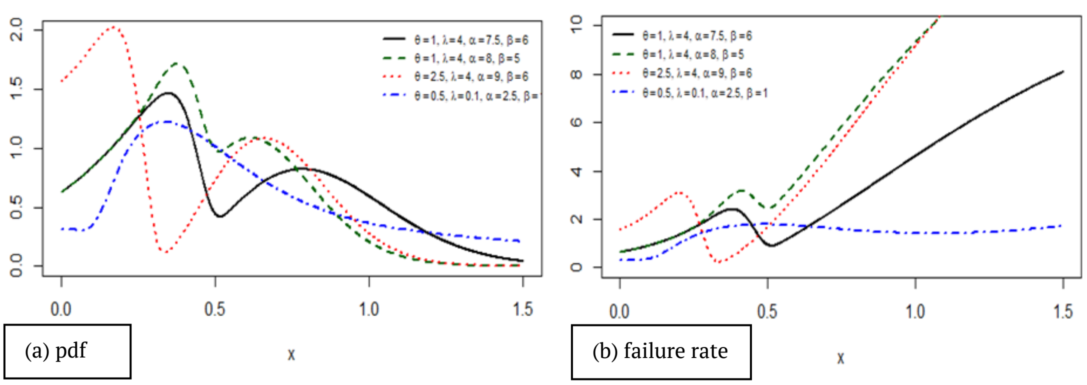

Figure 3. pdf and failure rate for NTLGolDa.

According to Figure 3, the shape of the pdf generated by the class can be unimodal or bimodal. It confirms the flexibility of the class. Additionally, Figure 3b displays that the failure rate presents a non-monotonic behavior.

NTLGoIGa distribution applied to simulated data

In order to assess the performance of the maximum likelihood estimates for the NTLGoIGa distribution, we present a Monte Carlo simulation study. It was implemented using the software for statistical computing R version 3.4.4 (R Language, 2018) and it was considered the L-BFGS-B method of optimization. We generated pseudo-random samples of size $n = 50, 100, 200, 500$ using the method of acceptance-rejection of von Neumann (1951). We calculated the bias and the mean squared error (MSE) for different values of the parameters, as shown in Table 1. The bias and MSE are obtained as follows:

$$\mathrm{Bias}_i = \frac{1}{10000}\sum_{j=1}^{10000}\left(\hat{k}_{ij} - k_i\right)$$

and

$$\mathrm{MSE}_i = \frac{1}{10000}\sum_{j=1}^{10000}\left(\hat{k}_{ij} - k_i\right)^2,$$

where k_i is the i-th element of the parametric vector $k = (k_1, \ldots k_r)$ and $\hat{k}_{ij}$ is the estimative for at the j-th replication.

The values presented in Table 1 indicate that the bigger the sample size, the smaller the MSE, as expected.

Table 1. Bias and MSE for NTLGoIGa model estimates.

	Real Value				Bias				MSE			
n	θ	λ	α	β	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
50	1.3	0.02	1.5	3.2	0.425	0.136	1.136	0.176	0.310	0.093	1.916	1.605
	0.5	0.1	1.7	4	0.555	0.235	1.022	0.556	0.436	0.214	1.830	3.783
	0.5	0.04	2.8	6.2	0.321	0.109	0.418	1.752	0.387	0.130	0.734	1.673
	0.7	0.2	1.4	3	0.406	0.136	1.366	0.649	0.421	0.041	2.702	2.544
	0.8	0.02	1.2	1.7	0.180	0.037	0.965	1.560	0.099	0.011	1.237	2.258
100	1.3	0.02	1.5	3.2	0.287	0.064	1.109	0.111	0.307	0.018	1.778	1.507
	0.5	0.1	1.7	4	0.438	0.119	0.556	0.296	0.276	0.096	1.783	2.265
	0.45	0.04	2.8	6.2	0.115	0.026	0.256	1.172	0.094	0.008	0.456	1.512
	0.7	0.2	1.4	3	0.292	0.126	1.234	0.634	0.247	0.023	2.201	2.416
	0.8	0.02	1.2	1.7	0.132	0.024	0.890	1.510	0.070	0.002	0.883	2.170
200	1.3	0.02	1.5	3.2	0.191	0.049	1.058	0.099	0.226	0.006	1.598	1.461
	0.5	0.1	1.7	4	0.283	0.043	0.330	0.197	0.124	0.018	0.358	1.884
	0.45	0.04	2.8	6.2	0.047	0.002	0.178	1.047	0.042	0.004	-0.306	1.135
	0.7	0.2	1.4	3	0.184	0.118	1.176	0.565	0.152	0.021	1.909	1.301
	0.8	0.02	1.2	1.7	0.072	0.020	0.864	1.478	0.065	0.001	0.879	2.088
500	1.3	0.02	1.5	3.2	0.071	0.046	0.981	0.042	0.218	0.006	1.287	1.307
	0.5	0.1	1.7	4	0.167	0.018	0.003	0.135	0.044	0.0005	0.050	1.278
	0.45	0.04	2.8	6.2	0.037	0.001	0.107	0.804	0.028	0.0008	0.303	1.037
	0.7	0.2	1.4	3	0.164	0.086	1.032	0.235	0.134	0.021	1.358	1.214
	0.8	0.02	1.2	1.7	0.001	0.017	0.024	1.373	0.064	0.0009	0.002	1.942

NTL-(G_1, G_2) applied to breaking stress of carbon fibers data (in Gba)

The data presented in Table 2 are defined in Nichols and Padgett (2006), corresponding to 100 observations of breaking stress of carbon fibers in Gba.

We made comparisons between the fits of the NTLGoIGa, NTL-Normal-IGamma (NTLNIGa), NTL-IGamma-IWeibull (NTLIGaIW), NTL-IGamma-Cauchy (NTLIGaC) distributions and their analogous mixtures Gompertz-IGamma (GoIGa), Normal-IGamma (NIGa), IGamma-IWeibull (IGaIW) and IGamma-Cauchy (IGaC), where IW (IGa) stands for inverse Weibull (inverse Gamma). Furthermore, we also made comparisons considering the Go, IGa, W and N distributions. Two-component normal and weibull mixture models (denoted by NN and WW, respectively) are considered in this application.

Table 2. Data defined by Nichols and Padgett (2006) consisting of 100 break observations of carbon fibers by stress (in Gba).

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11	4.42	2.41	3.19	3.22
1.69	3.28	3.09	1.87	3.15	4.90	3.75	2.43	2.95	2.97	3.39	2.96	2.53	2.67
2.93	3.22	3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85	2.56	3.56	3.15	2.35
2.55	2.59	2.38	2.81	2.77	2.17	2.83	1.92	1.41	3.68	2.97	1.36	0.98	2.76
4.91	3.68	1.84	1.59	3.19	1.57	0.81	5.56	1.73	1.59	2.00	1.22	1.12	1.71
2.17	1.17	5.08	2.48	1.18	3.51	2.17	1.69	1.25	4.38	1.84	0.39	3.68	2.48
0.85	1.61	2.79	4.70	2.03	1.80	1.57	1.08	2.03	1.61	2.12	1.89	2.88	2.82
2.05	3.65												

We determined the Maximum Likelihood estimator (MLE), their standard errors (SE), the Akaike information criterion (AIC), the Consistent Akaike information criterion (CAIC), the Bayesian information criterion (BIC), the Hannan-Quinn information criterion (HQIC), and the modified statistics of Anderson-Darling (A*) and Cramér-Von Mises (W*) (Chen & Balakrishnan, 1995) with the software R version 3.4.4 (R Language, 2018).

Some descriptive statistics of the cited data are summarized in Table 3. They present positive skewness and the distribution is leptokurtic.

Table 3. Descriptive Statistics.

Mean	Median	Mode	Variance	Asymmetry	Kurtosis	Minimum	Maximum
2.62	2.7	2.17	1.02	0.37	0.17	0.39	5.56

Table 4 brings the MLEs of the parameters and the corresponding SEs.

Table 4. Estimates and respective standard errors (in parentheses).

Distributions	Estimates				
NTLGoIGa	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	
	1.142	0.022	3.108	6.726	
	(0.201)	(0.013)	(0.647)	(1.837)	
GoIGa	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{w}$
	1.784	0.007	4.616	9.313	0.490
	(0.259)	(0.005)	(1.021)	(2.314)	(0.061)
NTLNIGa	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\beta}$	
	3.047	1.062	3.062	6.670	
	(0.121)	(0.144)	(0.692)	(2.017)	
NIGa	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{w}$
	99.999	99.999	5.802	13.005	0.010
	(0.121)	(0.144)	(0.691)	(0.045)	(0.007)
NTLIGaIW	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	
	4.075	1.617	4.3423	0.328	
	(0.612)	(1.232)	(0.684)	(0.019)	
GaIW	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{w}$
	9.994	25.585	1.780	0.944	0.827
	(1.283)	(0.010)	(0.879)	(1.476)	(0.097)
NTLIGaC	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\varepsilon}$	$\hat{\kappa}$	
	1.879	4.591	2.895	0.489	
	(0.370)	(1.184)	(0.120)	(0.118)	
IGaC	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\varepsilon}$	$\hat{\kappa}$	$\hat{w}$
	3.355	7.104	-3.503	0.003	0.866
	(1.912)	(5.421)	(0.080)	(0.087)	(0.049)
WW	$\hat{a}_1$	$\hat{b}_1$	$\hat{a}_1$	$\hat{b}_2$	$\hat{w}$
	2.454	2.854	8.541	3.134	0.783
	(0.200)	(0.109)	(2.380)	(0.126)	(0.093)
Go	$\hat{\theta}$	$\hat{\lambda}$			
	0.791	0.076			
	(0.077)	(0.017)			
GaI	$\hat{\alpha}$	$\hat{\beta}$			
	4.448	9.519			
	(0.606)	(1.375)			

W	$\hat{a}$	$\hat{b}$
	2.792	2.943
	(0.214)	(0.111)
N	$\hat{\sigma}$	$\hat{\mu}$
	2.621	1.008
	(0.100)	(0.071)

The small values suggest that the estimates are fairly precise. For all models, except for the WW, the maximization was performed via the L-BFGS-B algorithm, since it presented the best results and convergence. On the other hand, the maximization for the mixtures WW and NN was realized via EM algorithm, although without convergence for NN model.

Table 5 presents AIC, CAIC, BIC, HQIC, and the modified versions of the Anderson-Darling and Cramér-von Mises statistics (A* and W* respectively) (Chen & Balakrishnan, 1995).

Table 5. Information Criteria and Test Statistics.

Distributions	AIC	AICc	BIC	HQCI	A*	W*
NTLGoIGa	286.398	300.819	296.819	290.615	0.197	0.027
GoIGa	292.055	310.081	305.081	297.327	0.324	0.048
NTLNIGa	287.435	301.856	297.856	291.652	0.280	0.046
NIGa	323.049	341.076	336.076	328.321	2.070	0.337
NTLIGaIW	292.803	307.224	303.224	297.020	0.379	0.044
IGaIW	302.633	320.659	315.659	307.904	0.803	0.138
NIGaC	292.428	306.849	302.849	296.646	0.400	0.053
IGaC	298.114	316.14	311.14	303.385	0.658	0.130
WW	289.327	307.354	302.354	294.599	0.272	0.034
Go	302.25	309.46	307.46	304.359	1.767	0.229
IGa	321.474	328.684	326.684	323.583	2.871	0.522
W	287.059	294.269	292.269	289.167	0.420	0.063
N	289.541	296.751	294.751	291.649	0.471	0.058
C	324.431	331.641	329.641	326.539	1.676	0.173

The information criteria can be used as relative goodness-of-fit measures, such that the smallest values characterize the best fitted models. In spite of not being the best indication to make comparisons between non-nested models, the information criteria of the NTLGoIGa model figured among the most competitives compared to the remaining models. On the other hand, A* and W* are commonly used to investigate the goodness-of-fit of probabilistic models, either nested or not. Similarly, to the information criteria, the smaller the values of A* and W*, the better the fit.

According to the cited statistics, the NTLGoIGa distribution outperforms most of the fitted models, including the mixture WW, which is a very competitive model for bimodal data. It is worth pointing that the NTLNIGa, NTLIGaIW and NTLIGaC models beat their corresponding mixture versions, namely, NIGa, IGaIW and IGaC considering all statistics.

Figure 4 displays the histogram of the data overlapped by the fitted densities of the NTLGoIGa and the best three models according to A* and W*.

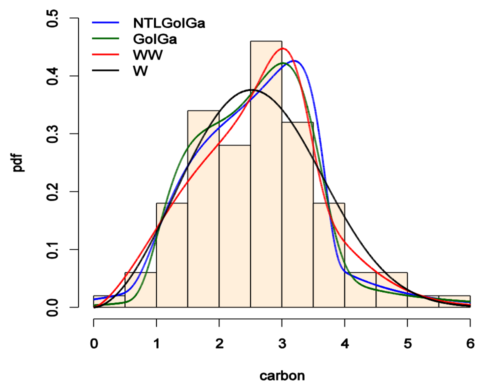

Figure 4. Histogram.

The density function of the NTLGoIGa and the GoIGa mixture model density have very close shapes. We emphasize that numerical analysis, rather than graphical analysis, should be used to choose the best model. In this way, the statistics suggest that the proposed model outperforms the mixture.

Conclusion

We built a class of probability distributions whose cumulative can be written as a composite function of two baselines. The new class was designed using the method for generating classes of probability distributions by Brito et al. (2019). The class is called Normal-tangent-logarithm-(G_1, G_2) and, under certain conditions, it produces identifiable distributions. Some structural properties were presented, that is, the linear expansion of cumulative and density functions, raw and central moments, moment generating function, characteristic function and general coefficient. We also presented the derivatives of the log-likelihood function of the class and the study of the support. Moreover, we studied the NTL-Gompertz-IGamma (NTLGoIGa) distribution, which was generated by the class. In such study, applications to simulated and real data sets were done. The fitted model was compared to other competitive distributions, including the (convex) mixture of two Weibulls. The information criteria AIC, BIC, CAIC and HQIC and the modified statistics of Anderson-Darling and Cramér-von Mises were considered in the comparisons. The results suggest that the NTLGoIGa distribution defeats the remainder ones considered in the numerical analysis.

References

Alexander, C., Cordeiro, G. M., Ortega, E. M., & Sarabia, J. M. (2012). Generalized beta-generated distributions. *Computational Statistics & Data Analysis*, *56*(6), 1880-1897. DOI: https://doi.org/10.1016/j.csda.2011.11.015

Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, *71*(1), 63-79. DOI: https://doi.org/10.1007/s40300-013-0007-y

Brito, C. R., Rêgo, L. C., Oliveira, W. R., & Gomes-Silva, F. (2019). Method for generating distributions and classes of probability distributions: The univariate case. *Hacettepe Journal of Mathematics and Statistics*, *48*(3), 897-930. Retrieved on Oct. 10, 2021 from https://dergipark.org.tr/en/pub/hujms/issue/45735/577353

Brito, C. C. R., Rêgo, L. C., de Oliveira, W. R., & Gomes-Silva, F. (2017). A new class of gamma distribution. *Acta Scientiarum. Technology, 39*(1), 79-89. DOI: https://doi.org/10.4025/actascitechnol.v39i1.29890

Cordeiro, G. M., & Nadarajah, S. (2011). Closed-form expressions for moments of a class of beta generalized distributions. *Brazilian Journal of Probability and Statistics*, *25*(1), 14-33. DOI: https://doi.org/10.1214/09-BJPS109

Chen, G., & Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, *27*(2), 154-161. DOI: https://doi.org/10.1080/00224065.1995.11979578

Eugene, N., Lee, C., & Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods*, *31*(4), 497-512. DOI: https://doi.org/10.1081/STA-120003130

Gradshteyn, I. S., Ryzhik, I. M., Jeffrey, A., & Zwillinger, D. (2007). *Table of integrals, series and products* (7th ed.). New York. NY: Academic Press.

Greenwood, J. A., Landwehr, J. M., Matalas, N. C., & Wallis, J. R. (1979). Probability weighted moments: definition and relation to parameters of several distributions expressable in inverse form. *Water Resources Research*, *15*(5), 1049-1054. DOI: https://doi.org/10.1029/WR015i005p01049

Lee, C., Famoye, F., & Alzaatreh, A. Y. (2013). Methods for generating families of univariate continuous distributions in the recent decades. *Wiley Interdisciplinary Reviews: Computational Statistics*, *5*(3), 219-238. DOI: https://doi.org/10.1002/wics.1255

Mauch, P., Armitage, J., Diehl, V., Hoppe, R., & Weiss, L. (1999). *Hodgkin's Disease*. Boston, MA: Lippincott Williams & Wilkins.

Nichols, M. D., & Padgett, W. J. (2006). A bootstrap control chart for Weibull percentiles. *Quality and Reliability Engineering International*, *22*(2), 141-151. DOI: https://doi.org/10.1002/qre.691

R Core Team (2018). *R: A language and environment for statistical computing*. Vienna: AT: R Foundation for Statistical Computing.

Ristić, M. M., & Nadarajah, S. (2014). A new lifetime distribution. *Journal of Statistical Computation and Simulation*, *84*(1), 135-150. DOI: https://doi.org/10.1080/00949655.2012.697163

Silveira, F. V., Gomes-Silva, F., Brito, C. C., Cunha-Filho, M., Gusmão, F. R., & Xavier-Júnior, S. F. (2019). Normal-G class of probability distributions: Properties and applications. *Symmetry*, *11*(11), 1-17. DOI: https://doi.org/10.3390/sym11111407

Silveira, F. V., Gomes-Silva, F., Brito, C. C., Jale, J. S., Gusmão, F. R., Xavier-Júnior, S. F., & Rocha, J. S. (2021). Modelling wind speed with a univariate probability distribution depending on two baseline functions. In *arXiv preprint arXiv:2101.03622* (p. 1-22). DOI: https://doi.org/10.48550/arXiv.2101.03622

Tahir, M. H., & Nadarajah, S. (2015). Parameter induction in continuous univariate distributions: Well-established G families. *Anais da Academia Brasileira de Ciências*, *87*(2), 539-568. DOI: https://doi.org/10.1590/0001-3765201520140299

Tahir, M. H., Cordeiro, G. M., Alzaatreh, A., Mansoor, M., & Zubair, M. (2016a). The logistic-X family of distributions and its applications. *Communications in Statistics-Theory and Methods*, *45*(24), 7326-7349. DOI: https://doi.org/10.1080/03610926.2014.980516

Tahir, M. H., Zubair, M., Mansoor, M., Cordeiro, G. M., Alizadehz, M., & Hamedani, G. G. (2016b). A new Weibull-G family of distributions. *Hacettepe Journal of Mathematics and Statistics*, *45*(2), 629-647.

Teicher, H. (1961). Identifiability of mixtures. *The Annals of Mathematical Statistics*, *32*(1), 244-248. DOI: https://doi.org/10.1214/aoms/1177705155

Von Neumann, J. (1951). 13. various techniques used in connection with random digits. *Applied Math Series*, *12*, 36-38.