Acta Scientiarum
http://periodicos.uem.br/ojs/acta
ISSN on-line: 1807-8672
Doi: 10.4025/actascitechnol.v47i1.70124

PHYSICS

Understanding of quasilinear hyperbolic systems through the investigation of their asymptotic solutions

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ABSTRACT. The main aim of the study is to deepen understanding of quasilinear hyperbolic systems through the investigation of their asymptotic solutions, with specific objectives related to theoretical analysis, wave dynamics characterization, and practical applications in physics and engineering. The methods of mathematical analysis employed include asymptotic analysis, Taylor series expansions, and the formulation of transfer equations. The paper considers systems of quasilinear hyperbolic equations in partial derivatives of the first order with two independent variables. The main results of the paper are: 1) high-frequency asymptotic solutions of small amplitude for quasilinear hyperbolic systems of the first order were obtained. For fixed values of t and x_k, values of the modulus $|\sigma_0(t) - \sigma_0(t^0)|, |\sigma_0^v(t) - \sigma_0^v(t^0)|$ are limited by p→∞, because the transfer equations depend on p. Thus, the moduli of the decomposition coefficients are bounded at p→∞ and at fixed u and x_k; 2) It has been established that for u_i^0= const, A_{ik}^K, B_i, independent of t, x_k the solution of the equation is greatly simplified because the coefficient a0 is constant. For the linear function $\phi(t, x)$, $b_{00'}$ is also constant. Practical applications of the results lie in fields such as fluid dynamics, wave propagation, and materials science, where understanding dispersion phenomena is crucial.

Keywords: asymptotics; differential equation; boundary conditions; quasilinear hyperbolic equations; Taylor series.

Received on October **26**, 2023.
Accepted on April 11, 2024.

Introduction

The relevance of applying systems of differential equations in various fields of science, technology, and natural and social sciences is beyond doubt. Hyperbolic partial differential equations describe wave phenomena in physical models in which information is transmitted with finite velocities (Abdel-Rehim, 2021). Equilibrium and diffusion phenomena, as well as the conservation of mass, momentum, and energy, are represented by two additional broad categories of partial derivative equations: elliptic equations and parabolic equations. These equations capture various physical phenomena such as the motion of continuous media, disintegration of discontinuities, explosion, and high-speed fluid flow.

Literature review

These issues have been considered by many authors (Gugat & Herty, 2020). Despite advances in the physical and mathematical sciences, at the moment there is still no general theory providing the existence, uniqueness and qualitative behaviour of solutions (Ebert & Reissig, 2018). In process engineering and hydraulics, the one-dimensional Euler system is successfully used, which allows for simulating flow regimes under non-stationary conditions with high accuracy. However, in the literature, there are also data on the use of this system for modelling nonlinear processes (Sahoo, Sekhar, & Sekhar, 2019). The application of the finite difference method to the approximate solution of hyperbolic partial differential equations will be considered in the example of a simple wave equation, which is a mathematical model of the problem of vibrations of an elastic string with fixed edges (Beketaeva, Naimanova, Shakhan, & Zadauly, 2023; Macías-Díaz, 2021). It is known that the most important property that clear differential schemes should have is stability, that is, the ability of the scheme not to accumulate computational noise (Sharma & Venkatraman, 2012). In the case of a hyperbolic equation, stability analysis is usually performed concerning the output data based on the spectrum of eigenvalues of the transition operator to a new time layer, based on which the adopted difference schemes are selected (Sharma & Venkatraman, 2012). The Janenko method (Sahoo et al., 2019), is used to find the exact solution of the gas dynamics equations. The general form of the asymptotic expansion in the vicinity of individual characteristics was set with the accuracy up to the second-order terms on the parameter determining the distance from the characteristics (Sahoo et al., 2019). The authors of (Moussati & Dalle, 2006) used the systematic reduction approach to obtain separate classes of solutions for this system. It is also known about obtaining invariant-like solutions to this problem for a class of fast-type materials and second-kind fluids (Wang & Wang, 2020a,2020b) and solving the fast diffusion equation, relaxation processes, nonlinear diffusion-convention equations, and the problem for transport flow. In Suleimanov and Shavlukov (2021) the authors explain the evolutionary behaviour of the shock wave for flat and nonplanar space. In the paper (Abdel-Rehim, 2021) the authors present a new classical proof of the results (Gugat & Herty, 2020). Much attention is paid to the study of nonlinear evolution equations: Klein-Gordon, Schrödinger, etc., which are special cases of classical linear equations. In Abdel-Rehim (2021), Gugat and Herty (2020) the method of obtaining asymptotic solutions for the quadratic quasilinear wave equation is presented. The authors Moussati and Dalle (2006) used the method of approximate Lie symmetries to solve one-dimensional quasilinear hyperbolic systems; Wang and Wang (2020a, 2020b) also used high-frequency asymptotic solutions for similar problems. It is known that integrals in most cases are almost not calculated explicitly. Therefore, such problems require a lot of effort and powerful computing resources, especially when the integrand contains a large parameter (Moussati & Dalle, 2006). It should be noted that differential equations with a small parameter describing the above-mentioned phenomena and processes are usually nonlinear or quasilinear. To solve such problems, the main method is asymptotic analysis, which allows one to construct approximate (asymptotic) solutions of the equations. Asymptotic solutions allow approximating the exact solutions of quasilinear hyperbolic systems (Wang & Wang, 2020a, 2020b).

Sharma (2010) found asymptotic solutions for singularly perturbed hyperbolic systems of first-order equations in which a small parameter is contained only in some derivatives. In Symak, Sabadash, Gumnitsky, and Hnativ (2021) the singularly perturbed hyperbolic systems of linear and quasilinear systems are considered, for which asymptotic solutions of the linear hyperbolic system of first-order equations for the case when a small parameter is contained in some derivatives are constructed by the boundary function method. Systems of equations for functions of both regular and singular parts of the asymptotics have been studied and it is shown that the terms of the asymptotic singular part have the properties of the boundary layer functions (Sharma, 2010).

Research gap

Even though quasilinear hyperbolic equations describe many processes, at the moment it is important to obtain solutions to this problem asymptotically (Kintonova, Sabitov, Povkhan, Khaimulina, & Gabdreshov, 2023; Beketaeva, Bruel, & Naimanova, 2019). The asymptotic behavior of solutions to these problems can provide valuable insights into the long-term behavior of physical systems described by such equations. Here are a few reasons why studying asymptotic solutions of quasilinear hyperbolic problems is important:

1. Asymptotic solutions describe the behavior of solutions as time or space tends to infinity or to some other critical point. Understanding this behavior is crucial for predicting the long-term evolution of physical systems described by hyperbolic equations.
2. In many practical applications, it may not be feasible to solve hyperbolic equations exactly. Asymptotic analysis provides a framework for deriving approximate solutions that are valid under certain limiting conditions. Understanding the accuracy and limitations of these approximations is essential for practical engineering and scientific applications.
3. Hyperbolic systems often exhibit discontinuous solutions known as shock waves. Understanding the asymptotic behavior of these shock waves is essential for developing accurate shock-capturing numerical schemes and for predicting the behavior of systems in extreme conditions.
4. Asymptotic analysis can provide insights into the stability of solutions to hyperbolic equations. Stability analysis is crucial for determining whether small perturbations to a system's initial conditions lead to significant changes in the solution over time.
5. Asymptotic matching techniques allow for the construction of composite solutions that capture the behavior of a solution in different regions of space or time. These techniques are particularly useful for problems with multiple scales or discontinuities.

Given the importance of understanding the long-term behavior and stability of solutions to hyperbolic equations, further research into the asymptotic solutions of quasilinear hyperbolic problems is warranted.

This research can lead to improved predictive models, more accurate numerical methods, and a deeper understanding of the underlying physical phenomena.

The main aim of the study is to deepen understanding of quasilinear hyperbolic systems through the investigation of their asymptotic solutions, with specific objectives related to theoretical analysis, wave dynamics characterization, and practical applications in physics and engineering. This investigation encompasses several specific objectives: to derive asymptotic solutions for the given quasilinear hyperbolic equations; to characterize the dynamics of wave propagation described by the hyperbolic equations; to analyze weak discontinuities in the solutions of the equations; to demonstrate the practical relevance of the asymptotic solutions in physics, particularly in the study of shock waves in gas and particulate media; to provide a theoretical justification for the constructed asymptotic solutions; to compare its findings with existing results in the literature, highlighting similarities, differences, and areas of agreement or disagreement. Lastly, the study may aim to discuss the methodological approaches used in deriving the asymptotic solutions, including any assumptions made, techniques applied, and challenges encountered.

Material and methods

System of equations and solution development

Let us consider a quasilinear system of first-order differential equations in general form (Sharma, 2010):

$$A_{ik}^{K}(u,x)\frac{\partial u_k}{\partial x_k}+pB_i(u,x)=0 \qquad (1)$$

Equation (1) for K=0,1.....n and i, k=1, ... n by definition is a symmetric hyperbolic equation. Moreover, $A_{ik}^{0}=I_{ik}, x_k=t$. The equation coefficients are regular functions of the variables u_k, as well as the variables x_k around the U and D. The area of existence of a sufficiently regular solution $u_{ik}^{K}\in U$ of the equation (1). If $u_{ik}^{K}=const$ and $B_i(u^0)=0$, then u_{ik}^{0} is a solution to the system (1).

The symmetric hyperbolicity condition is essential for transitioning from a traveling wave equation to a well-defined transport equation. This condition ensures the stability and well-posedness of the system. The asymptotic behavior of the solution tends towards hyperbolic equations, but not necessarily symmetric ones. This indicates a distinction between the behavior of solutions in the long-term limit and the symmetry of the equations.

The asymptotic solution of (1) will be presented in the following form:

$$u_i(x)=u_i^0(x)+\sum_{v=1}^{N}g_i^{\gamma}(x)S_v(\phi)+R_i(x) \qquad (2)$$

Moreover: $S_v(\phi)=(ip)^{-v}e^{ip\phi}$,

where $\phi(x)$ – phase function.

The authors assume that the coefficients of the series $g_i^{\gamma}(x)$ and the rest of the series are sufficiently regular in the range D.

Equation coefficients (1) $A_{ik}^{K}(u,x)$ and $B_i(u,x)$ are represented in the form of a Taylor series in the vicinity of u_{ik}^{0} at the steady-state value x_k. All equations containing the remainder of the series $R_i(x)$ are grouped in expressions $\tilde{R}_{ik}^{K}(x)$ and $\tilde{R}_i(x)$.

$$S_v'=S_{v-1}, pS_{v_{+M}}=-iS_{v-1},$$
$$S_vS_M=e^{i\phi p}S_{v_{+M}} \qquad (3)$$

$$A^K(u)=A^K(u^0)+\sum_{n=1}^{N}\frac{1}{n!}A^{K(n)}\sum^n \quad +\tilde{R}^K \qquad (4)$$

$$B(u)=B(u^0)+\sum_{n=1}^{N}\frac{1}{n!}B^{(n)}\sum^n \quad +\tilde{R} \qquad (5)$$

$\tilde{R}^K$ and $\tilde{R}$ are the remainder of $\tilde{R}_{ik}^{K}(x)$. $A^{K(n)}$ and $B^{(n)}$ are the nth-order derivatives of the functions $A_{ik}^{K}(u,x)$ and $B_i(u,x)$. $\sum^n$ is a quantitative expression of $\sum i$. $\sum i=\Delta u_i-R_i$.

After substituting (2), (4) and (5) into (1), one can obtain:

$$L_i(u)=\frac{\partial}{\partial t}(\Sigma_i \quad +R_i)+A_{ik}{}^{K}(u^0)\frac{\partial}{\partial x_k}(\Sigma_i \quad +R_i)+\left(\sum-i^k+\tilde{R}_i^k\right)\frac{\partial u_k}{\partial x_k}+p\left(\sum-i+\tilde{R}_i\right) \qquad (6)$$

After establishing the solution (2) and finding the coefficients of equation (1), one can obtain:

$$\sum_{v=1}^{M}E_i^{v}S_v+N_i[R]=0 \qquad (7)$$

To fulfill the following equation it is sufficient to establish that E_i^v was 0 for y=0,1,....,N-1 and $E_i^v S_\nu + N_i[R] = 0$, where one gets a series of recurrent equations:

$$A_{ik} {g_k}^1 = 0 \quad (8)$$

$$A_{ik} {g_k}^{v+1} + D_{ik} g_k^v + H_{ik}^{(v)} g_k^v + G_i^{(v)} = 0 \quad (9)$$

$$E_i^v S_\nu + N_i[R] = \quad (10)$$

$D_{ik} = A_{ik}(u^0, x)\frac{\partial}{\partial x_k}$ is a differential operator; $A_{ik} = {A_{ik}}^k + iB_i^k$ - dispersion matrix; $A_{ik} = D_{ik}\phi$ denotes the matrix of characteristic equations (1); B_i^k denotes the derivative $\partial B_i/\partial u_k$for $u_k = u_k^0$;

Functions $H_{ik}^{(v)}, G_i^{(v)}$ depend also on the multipliers $g_k^v, \ldots., {g_k}^{v+1}$. $H_{ik}^{(v)}$ depends linearly on $e^{ip\phi}$.

Let us apply the definition $\partial\phi/\partial x_k = k_k$, herewith $k_k = -\omega$; $A_{ik} = k_k {A_{ik}}^k$ and $A_{ik} = k_k {A_{ik}}^k - iB_i^k$ or:

$$A_{ik} = -\omega I_{ik} + \overline{A}_{ik} \quad (11)$$

$$\overline{A}_{ik} = k_k {A_{ik}}^k - iB_i^k, k = 1, \ldots\ldots, m$$

The final condition for the existence of a non-zero value of the solution ${g_k}^1$ of equation (8) is the disappearance of the determinant $Q = det(A_{ik})$. Thus, one can obtain the following dispersion equation:

$$Q(-\omega, k) = 0 \quad (12)$$

k – variable of k_k.

For valid values of k_k, matrix $\overline{A}_{ik}$ is hermitian under the assumption that the matrix B_i^k is antisymmetrical. Then equation (12) has n real roots for each real number k_k. If the matrices ${A_{ik}}^k$ are asymmetric and the matrix B_i^k is not antisymmetric, then it is sufficient to assume that for every real number k_k there is n real roots k_k. Fulfillment of this condition ensures the correctness of the following procedure.

Let us write the roots of equation (10) as:

$$\omega^{(i)} = H^{(i)}(t, x, k) \quad (13)$$

for i=1,...,n; x denotes x_k.

It is assumed that $H^{(i)}$ are sufficiently regular functions. Equation (12) is a first-order differential equation defining the phase function $\phi(t, x)$. The branches of this function are defined by equation (13).

The characteristic bands of equation (13) satisfy the system of canonical Hamilton equations.

$$\dot{x}_k = \frac{\partial H}{\partial k_k} \quad (14)$$

$$\dot{k}_k = -\frac{\partial H}{\partial k_k} \quad (15)$$

where the dot indicates the usual derivatives on t. Curves $x_k = x_k(t)$ are the rays of the asymptotic solution of the system, and $\gamma_k = \dot{x}_k$ is the radial velocity. The phase function can be described by the equation:

$$\dot{\phi} = -\omega + k_k\gamma_k = L \quad (16)$$

The Lagrangian and the Hamiltonian are related by the following relationship, which arises from (16):

$$L = -H + k_k \frac{\partial H}{\partial k_k} \quad (17)$$

We assume that ω is a q multiple of the dispersion equation. Dispersion matrix $A_{ik}(t, x, \omega, k)$ includes q zero right and left vectors l_i^0 r_i^o.

From equations (9) and (10) it follows that: $g_k^1 = \sigma_0 r_k^0$ $g_k^v = \sigma_0^v r_k^0 + h_k^v, v = 2, \ldots, M$ (18)

The coefficient σ_0 is determined on the rays at the asymptotic solution of the transport equation.

$$\dot{\sigma}_0 + a_{0'} e^{ip\phi}\sigma_{0'}\sigma_0 + b_{00'}\sigma_{00'}\sigma_{0'} = 0 \quad (19)$$

$$a_0 = r_k^0 \frac{\partial H}{\partial u_k}, u_k = u_k^0 \quad (20)$$

$$b_{00^{\bullet}} = m_{00^{\bullet}}^{-1}\left(l_i^{0'} D_{ik} r_k^0 + l_i^{0'} A_{i1}^{kv} r_k^0 \frac{\partial u_1^0}{\partial x_n}\right) \tag{21}$$

$m_{00^{\bullet}}^{-1} = l_i^0 r_i^{0'}$ inverse matrix.

$$A_{il}^{kk} = \frac{\partial A_{il}^k}{\partial u_k}, u_k = u_k^0. \tag{22}$$

Function h_k^v (16) is the root of the equation:

$$-A_{ik} h_k^v = D_{ik} g_k^{v-1} + H_{ik}^{(v-1)} g_k^{(h-1)} + G_i^{(v-1)} \tag{23}$$

Coefficient σ_0^v is defined in the rays through the equation:

$$\dot{\sigma}_0^v + b_{00^{\bullet}}^{(v)} \sigma_{0^{\bullet}}^v + c_0^{(v)} = 0 \tag{24}$$

where

$$b_{00^{\bullet}}^{(v)} = m_{00^{\bullet}}^{-1}\left(l_i^{0''} D_{ik} r_k^{0'} + l_i^{0''} H_{ik}^{(v)} r_k^{0'}\right); \tag{25}$$

$$c_0^{(v)} = m_{00^{\bullet}}^{-1}\left(l_i^{0''} D_{ik} h_k^v + l_i^{0'} H_{ik}^{(v)} h_k^v + l_i^{0''} G_i^{(v)}\right), \tag{26}$$

for O, O', O''= 1, 2, …., q.

Results

This section includes 3 necessary subsections (Figure 1).

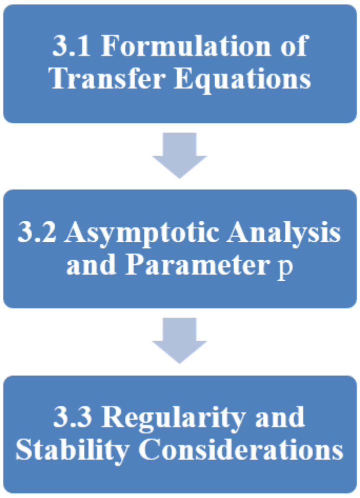

Figure 1. Flow-chart of section results.

Formulation of transfer equations

Formal application of the travelling wave expansion to the system of quasilinear dispersion equations in first-order partial derivatives leads to the transfer equations determining the expansion coefficients. Both the travelling wave expansion and the system of equations contain a large parameter p. The study will show that the travelling wave expansion is asymptotic under the condition p →∞. The general solution of the first transfer equation can be found under simplifying assumptions, a few examples illustrate the difficulties of applying the theory to the well-known equations of mathematical physics.

Functions $x_k(t)$ and $k_k(t)$, which are solutions of the canonical Hamilton equations (14) and (15) are regular in the vicinity $t=t^0$ due to the regularity of the function $H(t, x, k)$. The same goes for the phase function defined on the asymptote (ray) by the integral:

$$\phi(t, x) = \int_{t^0}^{t} L\,(s, x, k) ds + \phi(t^0, x^0) \tag{27}$$

it is regular in the vicinity of $t=t^0$ for $t>t^0$.

From the above equation $x_k = x_k(t)$, $k_k = k_k(t)$, $x_k' = x_k(t')$. The solution $\phi(t, x)$ of the dispersion equation (13) with the initial value $\phi(t^0, x) = \phi^0(x)$ is obtained by the equation in the initial value of *x0 k* for the function $x_k(t)$. Finally, it is argued that the function is regular in some vicinity of the initial surface $t = t^0$at $t > t0$, if only the initial function $\phi^0$is regular.

Asymptotic analysis and parameter p

The coefficients of transfer equations (19) and (24) are regular functions of the variable t. It follows that there exist one-valued solutions to these equations that are regular in some vicinity $t = t^0$. Functions, σ_0, σ_0^v depend on the parameter p, since the coefficients of the transfer equations depend /linearly/ on the function $e^{ip\phi}$. However, it can be shown that the modules $|\sigma_0(t) - \sigma_0(t^0)|, |\sigma_0^v(t) - \sigma_0^v(t^0)|$ limited to p→∞. $\sigma_0(t), \sigma_0^v(t)$ should be understood as functions $\sigma_0(t,x), \sigma_0^v(t,x)$, where $x_k = x_k(t)$ is a solution to the canonical equation. Let us write equation (19) as:

$$\dot{\sigma}_0 = f(t, \sigma_1, \dots, \sigma_q) \quad (28)$$

Let the area of determination of the function f_0 be defined by the inequalities $|t - t^0| \leq T$ and $|\sigma_0 - \sigma_0{}^0| \leq S$, where T and S — are positive constants, the functions of f are continuous. Moreover, they satisfy the Lipschitz condition by the dependences $\sigma_0, |f_0| \leq M, M > 0$, and the Lipschitz constant K, and the constant M do not depend on the parameter p, hence the conclusion about the presence of an environment for the point t^0, defined by the inequality:

$$|t - t^0| \leq S = min\left(\frac{1}{K}, \frac{S}{M}, T\right) \quad (29)$$

in which the system of equations (28) has a single-valued solution with an initial value in $\sigma_0(t^0) = \sigma_0^0$ and $|\sigma_0(t) - \sigma_0^0| \leq S$ for each p. The same conclusion is true for the system of transfer equations (24).

Varying the initial value $x^0{}_k$, we get the regular functions $\sigma_0(t,x), \sigma_0^v(t,x)$ in some vicinity of the initial surface $t = t^0$at $t \rangle t^0$, if only the initial data are regular. After taking into account equation (23) and the final formulation, these expansion coefficients $g_i^0(t,x), g_i^v(t,x)$ of the form (18) are regular in some vicinity of the initial surface with regular initial data.

Regularity and stability considerations

As follows from the theory of partial differential equations (Chaturvedi, Gupta, & Singh, 2019; Bachir, Giacomoni, & Warnault, 2021), there exists a single-valued solution of the initial problem for the system of equations (1) regular in some vicinity of the initial surface. Let us assume that the initial data have an asymptotic expansion of the form (2), i.e.:

$$u_i(t^0,x) = u_i^0(t^0,x) + \sum_{v=1}^{N} g_i^v(t^0,x) S_v(\phi^0) + R_i(t^0,x) \quad (30)$$

$g_i^v(t^0,x)$ are fairly regular and $R_i(t^0,x) = o(S_N)$ at p→∞, originating in the region of D_0, primary area. D_0 denotes the intersection of the area D of the initial surface. Then the residue of the expansion (2) is regular in some vicinity of the initial surface at t > t^0, because the coefficients of the decomposition of the function u_i^0 and functions u are regular, and $R_i(t^0,x) = o(S_N)$ at p→∞ as well since $u_i \to u_i^0$. The above considerations show that the following is true:

In some vicinity of the initial surface, there is an asymptotic expansion of the form (2) at p→∞, provided that the initial function has an asymptotic expansion of this form. In the vicinity of the initial surface, the expansion of the quasilinear hyperbolic equation has the property that successive terms of the expansion have increasing order, and the residual of the expansion has the highest order with respect to the degrees of 1/p at p →∞. In the case of quasilinear hyperbolic equations, the expansion coefficients of the equation depend on the parameter p included in the functions. However, it should be emphasized that this dependence has the form of a function.

In summary, the Results section delves into applying traveling wave expansion to quasilinear dispersion equations, focusing on their behavior as the parameter p tends to infinity. It underscores the regularity of solutions and coefficients near initial surfaces, crucial for stability and well-posedness. By formalizing transfer equations, it elucidates the asymptotic nature of the expansion and the existence of expansions contingent on initial conditions. Dependencies on p are noted, particularly in coefficient functions, with limits as p→∞ revealing specific patterns. The discussion extends to the properties of expansion terms, where successive terms exhibit increasing orders, and the residual term dominates at large p. This analysis elucidates the intricacies of quasilinear systems, bridging theoretical formulations with their asymptotic behaviors, offering insights into how solutions evolve and stabilize under certain conditions, enriching our understanding of dispersion phenomena in mathematical physics.

Discussion

The application of the travelling wave type expansion to the dispersive quasilinear equations, as has been done in this paper, causes additional difficulties. Namely, the transfer equations explicitly contain the parameter p, which is included in the Sy functions and in the system of partial derivative dispersion equations under consideration.

In Sahoo et al. (2019), a wave-type expansion was applied to the quasilinear equations. In this case, the Sy^M functions are fixed since the transfer equations depend on the choice of the variable {Sy}. Applying this expansion to linear dispersion equations (Macías-Díaz, 2021) shows that quasilinear hyperbolic equations depend on the sequence {Sy}. The properties of quasilinear equations due to the presence of discontinuities in the solutions of derivatives are similar to those of linear systems: weak discontinuities arise only on characteristic surfaces and correspond to ordinary differential equations (transfer equations). However, there is a fundamental difference. Now the characteristic surfaces, radii, and transfer equation coefficients depend on the solution. Thus, values for weak discontinuities can be found along the ray of the asymptotic solution if the solution near this ray is known. Knowing the initial solution of the system of equations is also necessary when studying the weak gap decay problem (Yadav, Singh, & Arora, 2021). This means that the initial problem should be solved with initial conditions taking into account the weak discontinuities on and near the surface under study (Gabdreshov, Magzymov, & Yensebayev, 2023). The weak discontinuity damping method, which should be emphasized, remains the same as for the linear equations, except that to determine it one must additionally know the initial data on the surface D. While the results on weak discontinuities in the solutions of linear hyperbolic equations can be considered as classical, for example, in Abdel-Rehim (2021), the properties of weak discontinuities in the case of quasilinear equations are the subject of recent studies. First of all, let us mention Gupta and Singh (2022) in which the weak discontinuities for the system of two quasilinear equations of the first order with dissociating variables were investigated. By introducing new dependent variables and keeping the system of equations in normal form, he obtained the transfer equation for the chosen characteristic. It turned out that the transfer equation for discontinuities of first-order derivatives is a nonlinear ordinary differential equation, while the transfer equations for discontinuities of higher derivatives are linear differential equations. From the form of the transfer equation of the first derivatives discontinuities, one could conclude that these discontinuities cannot arise if they are not given by initial conditions (Abdel-Rehim, 2021), and become unbounded after a finite time. Transfer equations describing the discontinuities of the first derivatives through the properties of a hyperbolic system of differential equations with two independent variables were developed by the authors (Ebert & Reissig, 2018). In deriving these equations, they used the term extended system, whose solutions are the functions contained in the original system of equations and their first derivatives. They also investigated discontinuities of first derivatives for the case of a system of quasilinear equations with two independent variables (Baikov, Gazizov, & Ibragimov, 1988). This paper contains considerations about the critical time after which the discontinuity of the solution occurs. It is shown that for an exceptional characteristic, the critical time is infinite, i.e., the discontinuities of the first derivatives for such a characteristic are bounded. The case of a system of quasilinear equations with m independent variables was analyzed in Yadav et al. (2021). Here new variables are introduced near the chosen characteristic surface so that one of them is the distance from the surface (let us call it the nominal variable). Then the coefficients of the system of equations were decomposed near the unperturbed state. After substituting the two sweeps into the original system of equations, the transfer equations for the discontinuities of the first and second derivatives on the rays corresponding to the chosen characteristic surface were obtained. A mathematical description of shock waves in an ideal gas consisting of solid particles is used to describe space phenomena, pneumatic transport, supersonic aircraft motion in dust storms, gas flows during volcanic eruptions, nuclear reactions, etc. The present study used the method of multiple time scales to obtain a high-frequency asymptotic solution of small amplitude. The obtained solutions may be relevant for solving the transport equations for shock wave propagation and other hyperbolic systems. In a mixture of gas and particulate matter, the study of the shock wave is of greater importance because of its wide application in several fields, such as supersonic vehicles in sandstorm conditions, supersonic flights in polluted air, nuclear reactions, aerospace engineering, etc. In the works by

Abdel-Rehim (2021) and Sharma (2010) the propagation of a shock wave in a dusty gas medium with different densities were discussed, in Wang and Wang (2020a, 2020b), Symak et al. (2021) - the results of studies of a weak shock wave in a supersonic flow of dusty gas. Dusty gas consists of finely dispersed particles of the solid gas phase. The concentration of particles is no more than 5% vol. In Sharma and Venkatraman (2012), the multiple timescale scheme by Wang and Wang (2020a, 2020b) was used to study the interaction of waves in a nonequilibrium gas flow. In Symak et al. (2021) and Yadav et al. (2021), to analyze the evolution of weak shock waves in non-ideal magnetogasdynamics and non-ideal radiative gas flow, the asymptotic method was used. In Sharma (2010), a theoretical substantiation of shock wave propagation in terms of radiation magnetogasdynamics is given. Shock wave propagation in a mixture of gas and dust particles has also been widely studied by several authors (Chaturvedi et al., 2019; Bachir et al., 2021; Li & Rao, 2019). In Sharma and Venkatraman (2012), the authors applied methods of relatively undistorted waves and weakly nonlinear geometric optics to study the fluxes of a nonideal relaxing gas at unboundedly small disturbance amplitudes.

The findings we have acquired may be juxtaposed with the outcomes of the study by Angeles (2023). The link between the equations of motion of an inviscid compressible fluid in space and an objective Cattaneo-type extension for the heat flow was examined by the authors. The equations were expressed in quasilinear form, and we ascertain which of the provided formulations for the heat flow permits the hyperbolicity of the system. A physically acceptable concept of well-posedness for the Cauchy problem of a system of equations necessitates the inclusion of this property.

The solvability of the Ionkin problem for differential equations with a single space variable was investigated by Kozhanov (2024). The equations include a variety of classifications, such as parabolic and quasiparabolic, hyperbolic and quasihyperbolic, pseudoparabolic and pseudohyperbolic, elliptic and quasielliptic equations, as well as several additional varieties. The splitting technique is used to show the following theorems for the given equations: the presence of regular solutions, which are solutions that have weak derivatives according to S. L. Sobolev and take place in the relevant equation.

This consolidated Table 1 provides a comprehensive comparison between the current study and recent research, covering various aspects such as methodology, parameter analysis, regularity considerations, and discussion points.

Table 1. Comparison of results with recent research.

Aspect	Current study	Recent Research
Formulation of Transfer Equations	Traveling wave expansion	Wave-type expansion
	Explicit inclusion of parameter p	Variable dependence
	Asymptotic under p→∞	Applied to linear dispersion equations
Asymptotic analysis of parameter p	Linear dependence on p	Variable dependence
	Regular functions of t	Regularity not explicitly discussed
	Stability addressed with p→∞	Stability not explicitly discussed
Regularity and Stability Considerations	Stability near initial surfaces	Stability not explicitly discussed
	Emphasized for well-posedness	Regularity implications not discussed
	Successive terms with increasing order	Not discussed in context of expansions

Conclusion

The present study obtained asymptotic solutions of a quasilinear hyperbolic system of first-order equations. The main results and the significance of the paper are:

– The study obtained high-frequency asymptotic solutions of small amplitude for quasilinear hyperbolic systems of the first order.

– The justification of the constructed asymptotics is given.

– The connection between the solution of the hyperbolic system of quasilinear equations of the first order with one spatial variable and the solution of the corresponding problem for the system with no time derivatives is proved.

– The theorem on the asymptoticity of the solution of the nonlinear boundary value problem for the hyperbolic system of the first order for p→∞ is proved, provided that the initial function has an asymptotic development of this form.

Moreover, it is proved that the quasilinearity of the system of equations (1), appearing already in the first term of the asymptotic expansion (2), at p→∞ in the first approximation should be omitted. At least that is what happens when u_i^0= const. Indeed, expression P containing multiplier A is of order 1/p at p→∞. On the other hand, the function B depends only on the variance matrix $A_{ik}(u^0)$, which is the zero vector, i.e., from the value of the matrix $B_i^k(u^0)$. However, this relationship of function b with the nonlinearity of the non-differential term in equation (1) is of little importance. This is because (1) can be replaced by a system of equations with an undifferentiated linear term of the form $B_{ik}u_k$, where $B_{ik}u_k = B_i^k(u^0)$.

Acknowledgmentes

Dayong Nie was supported by the "Innovative Entrepreneurship Teaching Reform Research Project of YRCTI" [Grant Number: 2023CXCYJG004].

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