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MATHEMATICS

f-lacunary statistical convergence in topological groups

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ABSTRACT. In this study, we will define the concept of f-lacunary statistical convergence in topological groups with the help of an unbounded modulus function. In addition, we will examine some inclusion theorems for different unbounded modulus functions in topological groups.

Keywords: Topological groups; statistical convergence; lacunary statistical convergence.

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Introduction and preliminaries

The idea of statistical convergence goes back to the first edition of monograph of Zygmund (1977). This notion has firstly been defined for real and complex sequences by Fast (1951). Schoenberg (1959) proved that the statistical limit is a linear functional on some sequence spaces. Šalăt (1980) showed that the set of bounded statistically convergent sequences is a closed subspace of the space of bounded sequences. Statistical convergence is closely related to both probability and measure theory. Subsequently, this idea was linked to summability, by various authors (Maddox, 1978; Kolk, 1991). Let us briefly explain the concept.

Let $\mathbb{N}$ be the set of all natural numbers and let $A \subseteq \mathbb{N}$. Suppose that χ_A is the characteristic function on A defined as

$$\chi_A(x) = \begin{cases} 1, & if\ x \in A \\ 0, & if\ x \notin A \end{cases}$$

The density of A is defined, whenever the following limit exists, as

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_A(j)$$

A sequence $z = (z(k))$ is said to be statistically convergent to number l if, for every positive number ε, $\delta(|k \in \mathbb{N}: |z(k) - l| \geq \varepsilon|)$ has natural density zero. The number l is called the statistical limit of $(z(k))$ and written as $S - \lim z(k) = l$. We denote the space of all statistically convergent sequences by S, see Fridy (1985).

By a lacunary sequence we mean increasing integer sequence $\theta = (k_r)$ such that $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r (see Freedman et al. 1978). Later, using the concept of statistical convergence, the concept of lacunary statistical convergence was defined by Fridy and Orhan (1993 a). The relationships between lacunary statistical convergence and summability were also examined by Fridy and Orhan (1993 b).

The concept of summability in topological groups was studied for the first time by Prullage (1967,1968,1969a,1969b). Later, this concept was studied by Çakalli and Thorpe (1990,1996). Then, the concept of statistical convergence in topological groups was given by Çakalli (1996). In addition, the concept of lacunary statistical convergence in topological groups was given by Çakalli (1995). Çakalli also examined the relationship between statistical convergence and lacunary statistical convergence in topological groups. Later, (Bulut, 2000; Bulut & Çakalli, 2003) introduced the concepts of both statistical and lacunary statistical h-regularity of series methods in topological groups.

We first quote the definition of a modulus function.

The idea of a modulus function was structured by Nakano (1953). Later, Ruckle (1973), Maddox (1986) and Mölder (2004) have used this concept to construct some sequence spaces.

Let f be a real-valued function defined on $[0,\infty)$ which satisfies:

$i)$ $f(x) = 0$ if and only if $x = 0$,

$ii)$ $f(x+y) \leq f(x) + f(y)$ for every $x, y \in \mathbb{R}^+$,

$iii)$ f is increasing,

$iv)$ f is continuous from the right at 0.

It is clear that every such function is continuous. A modulus may be unbounded or bounded. Some examples of modulus functions are $f(x) = \frac{x}{x+1}$, $f(x) = \log(x+1)$ and $f(x) = x^p$ with $0 < p \leq 1$.

After that Connor (1988) explained the relationship between statistically convergent and strong summable with the help of modulus function.

Aizpuru (2014) defined and studied the concepts of f-density and f-statistical convergence. f-statistical convergence for lacunary sequences was studied by Bhardwaj and Dhawan (2016), Şengül and Et (2018). Moreover, statistical convergence theorems for two different unbounded modulus functions were given by Çolak (2020). Garcia-Pacheco (2020) studied some properties of the concept of topological modules. Later, García-Pacheco and Kama (2022) continued this work and defined f-statistical convergence on topological modules. According to f-modulus for lacunary convergence, some inclusion theorems were also studied by Romero de la Rosa (2023).

Recently, the relationship between statistical convergence and topology has been studied by mathematicians (Caserta et al., 2011; Di Maio & Kocinac, 2008; Savaş, 2015; Uluçay & Ünver, 2019).

Main results

In this section, we will define the concept of f-lacunary statistical convergence in topological groups using an unbounded modulus function. We also examined the relationships between f-statistical convergence and f-lacunary statistical convergence in topological groups. In addition, we will examine some inclusion theorems for different unbounded modulus functions.

We will use Z in this study to denote an abelian topological Hausdorff group written additively. It satisfies the first axiom of countability.

García-Pacheco and Kama (2022) defined the notion f-statistical convergence in uniform spaces using any modulus function. We gave the definition of the concept of f-statistical convergence in topological groups by taking an unbounded modulus function in (Sarikaya & Altin, 2024). Now let's give this definition.

Definition 1. [Sarikaya & Altin, 2024] A sequence $z = (z(k))$ in Z is said to be f-statitically convergent or $S_f(Z)$-convergent to l of Z if for each neighbourhood U of 0

$$\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : z(k) - l \notin U\}|) = 0,$$

where f is unbounded modulus function. In this case we write $stat(Z,f) - \lim_{k\to\infty} z(k) = l$. The set of f-statistically convergent in Z will be denoted by $S_f(Z)$. In case of $l = 0$, we shall write $S_{f,0}(Z)$.

Definition 2. Let $z = (z(k))$ be a sequence in Z and $\theta = (k_r)$ be a lacunary sequence. A sequence $z = (z(k))$ is said to be f-lacunary statistically convergent or $S_f^\theta(Z)$-convergent to l of Z if for each neighbourhood U of 0

$$\lim_{r\to\infty} \frac{1}{f(h_r)} f(|\{k \in I_r : z(k) - l \notin U\}|) = 0,$$

where f is unbounded modulus function. In this case we write $stat_\theta(Z,f) - \lim_{k\to\infty} z(k) = l$. The set of f-lacunary statistically convergent in Z will be denoted by $S_f^\theta(Z)$. In case of $l = 0$, we shall write $S_{f,0}^\theta(Z)$.

Theorem 1. Let $\theta = (k_r)$ be a lacunary sequence and f be an unbounded modulus function. If a sequence $z = (z(k))$ is f-lacunary statistically convergent in Z, then $stat_\theta(Z,f) - \lim_{k\to\infty} z(k)$ is unique.

Proof. Let $z = (z(k))$ be f-lacunary statistically convergent in Z. Suppose that $(z(k))$ has two different f-lacunary statistical limits, l_1, l_2 say. Since Z is Hausdorff space there exists a neighbourhood U of 0 such that $l_1 - l_2 \notin$ U. Then we may choose a neighbourhood V of 0 such that $V + V \subset U$, (Arnautov et al., 1996). Write

$y(k) = l_1 - l_2$ for all $k \in \mathbb{N}$. Therefore for all $r \in \mathbb{N}$,

$\{k \in I_r: y(k) \notin U\} \subset \{k \in I_r: l_1 - z(k) \notin V\} \cup \{k \in I_r: z(k) - l_2 \notin V\}$.

Now it follows from this inclusion that, for all $r \in \mathbb{N}$ and since f is increasing,

$$\frac{1}{f(h_r)} f(|\{k \in I_r: y(k) \notin U\}|) \leq \frac{1}{f(h_r)} f(|\{k \in I_r: l_1 - z(k) \notin V\}|)$$
$$+\frac{1}{f(h_r)} f(|\{k \in I_r: z(k) - l_2 \notin V\}|).$$

Since $stat_\theta(Z, f) - \lim_{k\to\infty} z(k) = l_1$ and $stat_\theta(Z, f) - \lim_{k\to\infty} z(k) = l_2$ we get

$$\lim_{r\to\infty} \frac{1}{f(h_r)} f(|\{k \in I_r: y(k) \notin U\}|) \leq \lim_{r\to\infty} \frac{1}{f(h_r)} f(|\{k \in I_r: l_1 - z(k) \notin V\}|)$$
$$+\lim_{r\to\infty} \frac{1}{f(h_r)} f(|\{k \in I_r: z(k) - l_2 \notin V\}|).$$

Hence $1 \leq 0 + 0 = 0$. This contradiction shows that $l_1 = l_2$.

Theorem 2. Let $\theta = (k_r)$ be a lacunary sequence and f be an unbounded modulus function. If

$$\lim_{r\to\infty} \inf\ q_r > 1 \text{ and } \lim_{v\to\infty} \frac{f(v)}{v} > 0,$$

then $S_f(Z) \subset S_f^\theta(Z)$ such that $f(xy) \geq cf(x)f(y)$, for $x \geq 0, y \geq 0$.

Proof. Suppose that $\lim\ \inf_r q_r > 1$, then there exist a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficently r, which implies that

$$\frac{h_r}{k_r} = \frac{k_r - k_{r-1}}{k_r} = 1 - (q_r)^{-1} \geq 1 - (1 + \delta)^{-1} = \frac{\delta}{1 + \delta}.$$

If $stat(Z, f) - \lim_{k\to\infty} z(k) = l$, the for each neighbourhood U of 0 and for sufficiently large r, we have

$$\begin{aligned}
\frac{1}{f(k_r)} f(|\{k \leq k_r: z(k) - l \notin U\}|) &\geq \frac{1}{f(k_r)} f(|\{k \in I_r: z(k) - l \notin U\}|) \\
&= \frac{f(h_r)}{f(k_r)} \frac{1}{f(h_r)} f(|\{k \in I_r: z(k) - l \notin U\}|) \\
&= \frac{f(h_r)}{h_r} \frac{k_r}{f(k_r)} \frac{h_r}{k_r} \frac{f(|\{k \in I_r: z(k) - l \notin U\}|)}{f(h_r)} \\
&\geq \frac{f(h_r)}{h_r} \frac{k_r}{f(k_r)} \frac{\delta}{1 + \delta} \frac{f(|\{k \in I_r: z(k) - l \notin U\}|)}{f(h_r)}.
\end{aligned}$$

Taking limit as $r \to \infty$ and using, we get

$$stat(Z, f) - \lim_{k\to\infty} z(k) \Rightarrow stat_\theta(Z, f) - \lim_{k\to\infty} z(k).$$

Theorem 3. Let $\theta = (k_r)$ be a lacunary sequence and f be an unbounded modulus function. If

$$\lim_{r\to\infty} \sup\ q_r < \infty \text{ and } \lim_{v\to\infty} \frac{f(v)}{v} > 0,$$

then $S_f^\theta(Z) \subset S_f(Z)$ such that $f(xy) \geq cf(x)f(y)$, for $x \geq 0, y \geq 0$.

Proof. Suppose that $\lim\ \sup_r q_r < \infty$, then there exist a $\beta > 0$ such that $q_r < \beta$ for sufficently r. Let $stat_\theta(Z, f) - \lim_{k\to\infty} z(k) = l$, the for each neighbourhood U of 0 and $\lim_{r\to\infty} \frac{f(h_r)}{h_r} = l'$. Therefore, for given $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$

$$\frac{f(h_r)}{h_r} < l' + \varepsilon,\ \frac{1}{f(h_r)} f(|\{k \in I_r: z(k) - l \notin U\}|) < \varepsilon.$$

Let $M_r = |\{k \in I_r: z(k) - l \notin U\}|$. Using this notation, we have

$$\frac{f(M_r)}{f(h_r)} < \varepsilon \text{ for all } r > r_0.$$

Now let $N = \max\{f(M_1), f(M_2), \dots, f(M_{r_0})\}$ and choose n such that $k_{r-1} < n \leq k_r$. Then we have

$$\begin{aligned}\frac{1}{f(n)}f(|\{k\le n: z(k)-l\notin U\}|) &\le \frac{1}{f(k_{r-1})}f(|\{k\le k_r: z(k)-l\notin U\}|)\\ &= \frac{1}{f(k_{r-1})}f\left(M_1+M_2+\cdots+M_{r_0}+M_{r_0+1}+\cdots+M_r\right)\\ &\le \frac{1}{f(k_{r-1})}\left(f(M_1)+f(M_2)+\cdots+f\left(M_{r_0}\right)+f\left(M_{r_0+1}\right)+\cdots+f(M_r)\right)\\ &\le \frac{N}{f(k_{r-1})}r_0+\frac{1}{f(k_{r-1})}\left[f\left(M_{r_0+1}\right)+\cdots+f(M_r)\right]\\ &= \frac{N}{f(k_{r-1})}r_0+\frac{1}{f(k_{r-1})}\left[\frac{f\left(h_{r_0+1}\right)}{h_{r_0+1}}\frac{f\left(M_{r_0+1}\right)}{f\left(h_{r_0+1}\right)}h_{r_0+1}+\cdots+\frac{f(h_r)}{h_r}\frac{f(M_r)}{f(h_r)}h_r\right]\\ &< \frac{N}{f(k_{r-1})}r_0+\frac{1}{f(k_{r-1})}\left[\left((l'+\varepsilon)\varepsilon h_{r_0+1}+\cdots+(l'+\varepsilon)\varepsilon h_r\right)\right]\\ &= \frac{N}{f(k_{r-1})}r_0+\frac{1}{f(k_{r-1})}\varepsilon(l'+\varepsilon)\left[h_{r_0+1}+\cdots+h_r\right]\\ &= \frac{N}{f(k_{r-1})}r_0+\frac{1}{f(k_{r-1})}\varepsilon(l'+\varepsilon)\left[k_r-k_{r_0}\right]\\ &< \frac{N}{f(k_{r-1})}r_0+\varepsilon(l'+\varepsilon)\left[\frac{k_r}{f(k_{r-1})}\right]\\ &= \frac{N}{f(k_{r-1})}r_0+\varepsilon(l'+\varepsilon)\frac{k_{r-1}}{f(k_{r-1})}\frac{k_r}{k_{r-1}}\\ &= \frac{N}{f(k_{r-1})}r_0+\varepsilon(l'+\varepsilon)q_r\frac{k_{r-1}}{f(k_{r-1})}\\ &< \frac{N}{f(k_{r-1})}r_0+\varepsilon(l'+\varepsilon)\beta\frac{k_{r-1}}{f(k_{r-1})}\end{aligned}$$

and the result follows immediately, in view of the fact that $\lim_{r\to\infty}\frac{k_{r-1}}{f(k_{r-1})}>0$.

Now as a result of Theorem 2 and Theorem 3 we can give the following result.

Corollary 1. Let $\theta=(k_r)$ be a lacunary sequence and f be an unbounded modulus function. If

$$1<\liminf_{r\to\infty} q_r\le\limsup_{r\to\infty} q_r<\infty \text{ and } \lim_{v\to\infty}\frac{f(v)}{v}>0,$$

then $stat(Z,f)-\lim_{k\to\infty}z(k)\to l=stat_\theta(Z,f)-\lim_{k\to\infty}z(k)\to l$ such that $f(xy)\ge cf(x)f(y)$, for $x\ge 0$, $y\geqslant 0$.

Theorem 4. Let $\theta=(k_r)$ be a lacunary sequence and f be an unbounded modulus function. If

$$z(k)\in S_f(Z)\cap S_f^\theta(Z),$$

then $stat(Z,f)-\lim_{k\to\infty}z(k)=stat_\theta(Z,f)-\lim_{k\to\infty}z(k)$ such that $|f(x)-f(y)|=f(|x-y|)$, for $x\ge 0$, $y\ge 0$.

Proof. Take any $z(k)\in S_f(Z)\cap S_f^\theta(Z)$ and $stat(Z,f)-\lim_{k\to\infty}z(k)=l_1$, $stat_\theta(Z,f)-\lim_{k\to\infty}z(k)=l_2$, say. Suppose that $l_1\ne l_2$. Since Z is Hausdorff space, there exist a symmetric neighbourhood U of 0 such that $l_1-l_2\notin U$. Then we may choose a symmetric neighbourhood V of 0 such that $V+V\subset U$. Then we obtain the following inequality

$$\frac{1}{f(n)}f(|\{k\le n: y(k)\notin U\}|)\le\frac{1}{f(n)}f(|\{k\le n: z(k)-l_1\notin V\}|)+\frac{1}{f(n)}f(|\{k\le n: l_2-z(k)\notin V\}|)$$

where $y(k)=l_2-l_1$ for all $k\in\mathbb{N}$. It follows from this inequality that

$$1\le\frac{1}{f(n)}f(|\{k\le n: z(k)-l_1\notin V\}|)+\frac{1}{f(n)}f(|\{k\le n: l_2-z(k)\notin V\}|).$$

The second term on the right side of this inequality tends to 0 as $n\to\infty$, we get

$$1\le 0+\lim_{n\to\infty}\frac{1}{f(n)}f(|\{k\le n: l_2-z(k)\notin V\}|)\le 1$$

and so

$$\lim_{n\to\infty}\frac{1}{f(n)}f(|\{k\le n: l_2-z(k)\notin V\}|)=1.$$

We consider the subsequence

$$\frac{1}{f(k_m)}f(|\{k\le k_m: l_2-z(k)\notin V\}|)$$

of sequence

$$\frac{1}{f(n)}f(|\{k \leq n: l_2 - z(k) \notin V\}|).$$

Then

$$\begin{aligned}
\frac{1}{f(k_m)}f(|\{k \leq k_m: l_2 - z(k) \notin V\}|) &= \frac{1}{f(k_m)}f(|\{k \in \cup_{r=1}^{m} I_r : l_2 - z(k) \notin V\}|) \\
&= \frac{1}{f(k_m)}f(\textstyle\sum_{r=1}^{m}|\{k \in I_r: l_2 - z(k) \notin V\}|) \\
&\leq \frac{1}{f(k_m)}\textstyle\sum_{r=1}^{m} f(|\{k \in I_r: l_2 - z(k) \notin V\}|) \\
&= \frac{1}{f(k_m)}\textstyle\sum_{r=1}^{m} f(h_r)\frac{1}{f(h_r)}f(|\{k \in I_r: l_2 - z(k) \notin V\}|)
\end{aligned} \quad (1)$$

and

$$\begin{aligned}
\textstyle\sum_{r=1}^{m} f(h_r) &= f(h_1) + f(h_2) + \cdots + f(h_m) \\
&= f(k_1 - k_0) + f(k_2 - k_1) + \cdots + f(k_m - k_{m-1}) \\
&= f(|k_1 - k_0|) + f(|k_2 - k_1|) + \cdots + f(|k_m - k_{m-1}|) \\
&= |f(k_1 - k_0)| + |f(k_2 - k_1)| + \cdots + |f(k_m - k_{m-1})| \\
&= f(k_1) - f(k_0) + f(k_2) - f(k_1) + \cdots + f(k_m) - f(k_{m-1}) \\
&= f(k_m)\,.
\end{aligned} \quad (2)$$

Using (2) in (1), we take

$$\frac{1}{f(k_m)}f(|\{k \leq k_m: l_2 - z(k) \notin V\}|) \leq \frac{\sum_{r=1}^{m} f(h_r)}{\sum_{r=1}^{m} f(h_r)}\frac{1}{f(h_r)}f(|\{k \in I_r: l_2 - z(k) \notin V\}|)$$

so

$$\frac{1}{f(k_m)}f(|\{k \leq k_m: l_2 - z(k) \notin V\}|) \to 0$$

but this is contradiction to

$$\lim_{n\to\infty}\frac{1}{f(n)}f(|\{k \leq n: l_2 - z(k) \notin V\}|) = 1.$$

Therefore $l_1 = l_2$.

Now as a result of Theorem 4 we have the following Corollary 2.

Corollary 2. Let $\theta = (k_r)$ and $\theta' = (t_r)$ be two lacunary sequences and f be an unbounded modulus function. If

$$z(k) \in S_f(Z) \cap \left(S_f^{\theta}(Z) \cap S_f^{\theta'}(Z)\right),$$

then $stat_{\theta}(Z, f) - \lim_{k\to\infty} z(k) = stat_{\theta'}(Z, f) - \lim_{k\to\infty} z(k)$.

Theorem 5. Let $\theta = (k_r)$ and $\theta' = (t_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and f be an unbounded modulus function. If

$$\lim_{r\to\infty}\inf \frac{f(h_r)}{f(l_r)} > 0,$$

where $I_r = (k_{r-1}, k_r]$, $h_r = k_r - k_{r-1}$ and $J_r = (t_{r-1}, t_r]$, $l_r = t_r - t_{r-1}$, then $S_f^{\theta'}(Z) \subset S_f^{\theta}(Z)$.

Proof. Let $z(k) \in S_f^{\theta'}(Z)$. We can write

$$\begin{aligned}
\frac{1}{f(l_r)}f(|\{k \in J_r: z(k) - l \notin U\}|) &= \frac{1}{f(l_r)}f(|\{k \in J_r - I_r: z(k) - l \notin U\}|) + \frac{1}{f(l_r)}f(|\{k \in I_r: z(k) - l \notin U\}|) \\
&\geq \frac{1}{f(l_r)}f(|\{k \in I_r: z(k) - l \notin U\}|) \\
&= \frac{f(h_r)}{f(l_r)}\frac{1}{f(h_r)}f(|\{k \in I_r: z(k) - l \notin U\}|)\,.
\end{aligned}$$

Since $\lim_{r\to\infty}\inf\frac{f(h_r)}{f(l_r)} > 0$, we can say that if $z(k) \in S_f^{\theta'}(Z)$, then $z(k) \in S_f^{\theta}(Z)$. So $S_f^{\theta'}(Z) \subset S_f^{\theta}(Z)$.

Theorem 6. Let $\theta = (k_r)$ and $\theta' = (t_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and f be an unbounded modulus function. If

$$\lim_{r\to\infty} \frac{f(l_r)}{f(h_r)} = 1,$$

where $I_r = (k_{r-1}, k_r]$, $h_r = k_r - k_{r-1}$ and $J_r = (t_{r-1}, t_r]$, $l_r = t_r - t_{r-1}$, then $S_f^{\theta}(Z) \subset S_f^{\theta'}(Z)$ such that $|f(x) - f(y)| = f(|x-y|)$, for $x \geq 0, y \geq 0$.

Proof. Let $z(k) \in S_f^{\theta}(Z)$. We can write

$$\begin{aligned}
\frac{1}{f(l_r)} f(|\{k \in J_r : z(k) - l \notin U\}|) &= \frac{1}{f(l_r)} f(|\{k \in J_r - I_r : z(k) - l \notin U\}|) + \frac{1}{f(l_r)} f(|\{k \in I_r : z(k) - l \notin U\}|) \\
&\leq \frac{f(l_r - h_r)}{f(l_r)} + \frac{1}{f(l_r)} f(|\{k \in I_r : z(k) - l \notin U\}|) \\
&= \frac{f(|l_r - h_r|)}{f(l_r)} + \frac{1}{f(l_r)} f(|\{k \in I_r : z(k) - l \notin U\}|) \\
&= \frac{|f(l_r) - f(h_r)|}{f(l_r)} + \frac{1}{f(l_r)} f(|\{k \in I_r : z(k) - l \notin U\}|) \\
&= \frac{f(l_r) - f(h_r)}{f(l_r)} + \frac{1}{f(l_r)} f(|\{k \in I_r : z(k) - l \notin U\}|) \\
&\leq \frac{f(l_r) - f(h_r)}{f(h_r)} + \frac{1}{f(h_r)} f(|\{k \in I_r : z(k) - l \notin U\}|) \\
&= \left(\frac{f(l_r)}{f(h_r)} - 1\right) + \frac{1}{f(h_r)} f(|\{k \in I_r : z(k) - l \notin U\}|).
\end{aligned}$$

Since $\lim_{r\to\infty} \frac{f(l_r)}{f(h_r)} = 1$, we can say that if $z(k) \in S_f^{\theta}(Z)$, then $z(k) \in S_f^{\theta'}(Z)$. So $S_f^{\theta}(Z) \subset S_f^{\theta'}(Z)$.

Definition 3. Let $\theta = (k_r)$ be a lacunary sequence and $z = (z(k))$ be a sequence of real numbers. The sequence $z = (z(k))$ is said to be $S_f^{\theta}(Z)$-Cauchy sequence if there is a subsequence $\{z(k'(r))\}$ of z such that $k'(r) \in I_r$ for each r, $\lim_{r\to\infty} z(k'(r)) = l$ and for each neighbourhood U of 0

$$\lim_{r\to\infty} \frac{1}{f(h_r)} f\left(\left|\{k \in I_r : z(k) - z(k'(r)) \notin U\}\right|\right) = 0,$$

where f is unbounded modulus function.

We see that a sequence is $S_f^{\theta}(Z)$-convergent if and only if it is an $S_f^{\theta}(Z)$-Cauchy sequence without the completeness assumption.

Theorem 7. Let $\theta = (k_r)$ be a lacunary sequence and f be an unbounded modulus function. The sequence $(z(k))$ is $S_f^{\theta}(Z)$-Cauchy sequence, then $(z(k))$ is $S_f^{\theta}(Z)$-convergent.

Proof. Suppose that $(z(k))$ is an $S_f^{\theta}(Z)$-Cauchy sequence. Let U be any neighbourhood of 0. Then we may choose a neighbourhood V of 0 such that $V + V \subset U$. Thus

$$\frac{1}{f(h_r)} f(|\{k \in I_r : z(k) - l \notin U\}|) \leq \frac{1}{f(h_r)} f\left(\left|\{k \in I_r : z(k) - z(k'(r)) \notin V\}\right|\right)$$
$$+ \frac{1}{f(h_r)} f\left(\left|\{k \in I_r : z(k'(r)) - l \notin V\}\right|\right).$$

Since $\lim_{r\to\infty} \frac{1}{f(h_r)} f(|\{k \in I_r : z(k) - z(k'(r)) \notin U\}|) = 0$ and $\lim_{r\to\infty} z(k'(r)) = l$, by the assumption, it follows from the last inequality that $stat_{\theta}(Z, f) - \lim_{k\to\infty} z(k) = l$.

Theorem 8. Let $\theta = (k_r)$ be a lacunary sequence and f and g be two unbounded modulus functions. Then

(i) if $\lim_{t\to\infty} \frac{f(t)}{g(t)} > 0$, then a sequence $(z(k))$ is f-lacunary statistically convergent in Z if it is g-lacunary statistically convergent in Z, that is $S_g^{\theta}(Z) \subset S_f^{\theta}(Z)$,

(ii) if $\lim_{t\to\infty} \frac{f(t)}{g(t)} < \infty$, then a sequence $(z(k))$ is f-lacunary statistically convergent in Z if and only if it is g-lacunary statistically convergent in Z, that is $S_g^{\theta}(Z) = S_f^{\theta}(Z)$.

Proof. Let f and g be two unbounded modulus functions.

(i) Suppose $\lim_{t\to\infty} \frac{f(t)}{g(t)} = \beta > 0$, then $\lim_{r\to\infty} \frac{f(h_r)}{g(h_r)} = \beta > 0$ and $(z(k))$ is g-lacunary statistically convergent to l, that is $stat_\theta(Z,g)-\lim_{k\to\infty} z(k)=l$. Then given any $\varepsilon>0$ there exist a real number r_0 such that $(\beta-\varepsilon)g(h_r)<f(h_r)<(\beta+\varepsilon)g(h_r)$ if $r>r_0$. Therefore we have the inequality $f(h_r)<2\beta g(h_r)$ if $r>r_0$. Now we may write the inequality the for each neighbourhood U of 0

$$\frac{1}{g(h_r)}g(|\{k\in I_r: z(k)-l\notin U\}|)\geq \frac{1}{2\beta}\frac{f(|\{k\in I_r: z(k)-l\notin U\}|)}{f(h_r)}\frac{f(h_r)}{g(h_r)}$$

if $|\{k\in I_r: z(k)-l\notin U\}|>r_0$. Since $\lim_{r\to\infty}\frac{f(h_r)}{g(h_r)}=\beta>0$ from the above inequality we obtain

$$\lim_{r\to\infty}\frac{1}{f(h_r)}f(|\{k\in I_r: z(k)-l\notin U\}|)=0$$

if

$$\lim_{r\to\infty}\frac{1}{g(h_r)}g(|\{k\in I_r: z(k)-l\notin U\}|)=0.$$

Therefore $S_g^\theta(Z)\subset S_f^\theta(Z)$.

(ii) We may write the following equality the for each neighbourhood U of 0

$$\frac{1}{g(h_r)}g(|\{k\in I_r: z(k)-l\notin U\}|)=\frac{g(|\{k\in I_r: z(k)-l\notin U\}|)}{f(|\{k\in I_r: z(k)-l\notin U\}|)}\frac{f(|\{k\in I_r: z(k)-l\notin U\}|)}{f(h_r)}\frac{f(h_r)}{g(h_r)}.$$

Suppose $0<\lim_{r\to\infty}\frac{f(h_r)}{g(h_r)}=\beta<\infty$ and so that $\lim_{r\to\infty}\frac{g(h_r)}{f(h_r)}=\frac{1}{\beta}$. Using this fact from the above equality we obtain

$$\lim_{r\to\infty}\frac{1}{f(h_r)}f(|\{k\in I_r: z(k)-l\notin U\}|)=0$$

if and only if

$$\lim_{r\to\infty}\frac{1}{g(h_r)}g(|\{k\in I_r: z(k)-l\notin U\}|)=0.$$

Therefore $S_g^\theta(Z)=S_f^\theta(Z)$.

Theorem 9. Let $\theta=(k_r)$ be a lacunary sequence and f and g be two unbounded modulus functions. Then

(i) if $\lim_{t\to\infty}\frac{f(t)}{g(t)}>0$, then a sequence $(z(k))$ is g-lacunary statistically Cauchy sequence in Z if it is f-lacunary statistically Cauchy sequence in Z,

(ii) if $\lim_{t\to\infty}\frac{f(t)}{g(t)}<\infty$, then a sequence $(z(k))$ is f-lacunary statistically Cauchy sequence in Z if and only if it is g-lacunary statistically Cauchy sequence in Z.

Proof. The proof can be similarly to the proof of Theorem 8.

Conclusion

Statistical convergence is a natural generalization of classical convergence and allows for the analysis of the convergence behavior of sequences using a density-based approach. In this context, we examined the concept of lacunary statistical convergence using an unbounded modulus function in topological groups. In the future, researchers can use this work to investigate different types of convergence in topological groups.

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