Bol. Soc. Paran. de Mat. (3s.) **v. 27** 2 (2009): 9–14.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/spm doi:10.5269/bspm.v27i2.10205

On (δ, p)-continuous functions and (δ, p)-closed graphs

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abstract: It is the object of this paper to introduce the notions of (δ, p) - continuity and (δ, p)-closed graphs by utilizing the notion of (δ, p)-open sets and investigate the fundamental properties of (δ, p)-continuous functions and also present some properties of functions with (δ, p)-closed graphs.

Key Words: Topological spaces, (δ, p)-open set, (δ, p)-closed graph, (δ, p)-T_1, (δ, p)-continuous. (δ, p)-W-continuous.

Contents

1. Introduction

In this paper X and Y denote the topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$ respectively. Jafari [2] introduced the notion of pre-regular p-open sets and further investigated its fundamental properties in [3]. A subset A of a topological space (X, τ) is called a *pre-regular p-open* [2] if $A = pInt(pCl(A))$. Now we recall the following notions from [1] which will be used in the sequel: A point $x \in X$ is called the (δ, p)-*cluster point* of A if $A \cap U \neq \emptyset$ for every pre-regular p-open set U of X containing x. The set of all (δ, p)-cluster points of A is called the (δ, p)-*closure* of A, denoted by $\delta Cl_p(A)$. If $\delta Cl_p(A) = A$, then A is called (δ, p)-*closed*. The complement of a (δ, p)-closed set is called (δ, p)-*open*. We say that a set U in a topological space (X, τ) is a (δ, p)-*neighborhood* of a point x if U contains a (δ, p)-open set to which x belongs. We denote the collection of all (δ, p)-open (respectively (δ, p)-closed) sets by $\delta PO(X, \tau)$ (respectively $\delta PC(X, \tau)$).

In this paper we offer a new class of functions called (δ, p)-continuous functions and a new notion of the graph of a function called a (δ, p)-closed graph. We also investigate some of their fundamental properties.

2. Some properties

Definition 2.1 *A function $f : X \to Y$ is said to be (δ, p)-continuous if for every open set V of Y, $f^{-1}(V)$ is (δ, p)-open in X.*

2000 *Mathematics Subject Classification:* 54C10, 54D10

Theorem 2.1 *The following are equivalent for a function $f : X \to Y$:*

(1) f is (δ, p)-continuous,

(2) The inverse image of every closed set in Y is (δ, p)-closed in X,

(3) For each subset A of X, $f(\delta Cl_p(A)) \subset Cl(f(A))$,

(4) For each subset B of Y, $\delta Cl_p(f^{-1}(B)) \subset f^{-1}(Cl(B))$.

Proof. (1) $\Leftrightarrow$ (2) : Obvious.

(3) $\Leftrightarrow$ (4) : Let B is any subset of Y. Then by (3), we have $f(\delta Cl_p(f^{-1}(B))) \subset Cl(f(f^{-1}(B))) \subset Cl(B)$. This implies $\delta Cl_p(f^{-1}(B)) \subset f^{-1}(f(\delta Cl_p(f^{-1}(B)))) \subset f^{-1}(Cl(B))$.

Conversely, let $B = f(A)$ where A is a subset of X. Then, by (4), we have, $\delta Cl_p(A) \subset \delta Cl_p(f^{-1}(f(A))) \subset f^{-1}(Cl(f(A)))$. Thus, $f(\delta Cl_p(A)) \subset Cl(f(A))$.

(2) $\Rightarrow$ (4) : Let $B \subset Y$. Since $f^{-1}(Cl(B))$ is (δ, p)-closed and $f^{-1}(B) \subset f^{-1}(Cl(B))$, then $\delta Cl_p(f^{-1}(B)) \subset f^{-1}(Cl(B))$.

(4) $\Rightarrow$ (2) : Let $K \subset Y$ be a closed set. By (4), $\delta Cl_p(f^{-1}(K)) \subset f^{-1}(Cl(K)) = f^{-1}(K)$. Thus, $f^{-1}(K)$ is (δ, p)-closed.

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) \mid x \in X\}$ of the product space $X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 2.2 *For a function $f : X \to Y$, the graph $G(f) = \{(x, f(x)) \mid x \in X\}$ is said to be (δ, p)-closed if for each $(x, y) \in X \times Y \setminus G(f)$, there exist $U \in \delta PO(X, x)$ and an open set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 2.1 *Let $f : X \to Y$ be a function. Then the graph $G(f)$ is (δ, p)-closed in $X \times Y$ if and only if for each point $(x, y) \in X \times Y \setminus G(f)$, there exist a (δ, p)-open set U and an open set V containing x and y, respectively, such that $f(U) \cap V = \emptyset$.*

Proof. It follows readily from the above definition.

Definition 2.3 *A space X is said to be (δ, p)-T_1 [1] if for each pair of distinct points x and y of X, there exist a (δ, p)-open set U containing x but not y and a (δ, p)-open set V containing y but not x.*

Theorem 2.2 *If $f : X \to Y$ is an injective function with the (δ, p)-closed graph, then X is (δ, p)-T_1.*

Proof. Let x and y be two distinct points of X. Then $f(x) \neq f(y)$. Thus there exist a (δ, p)-open set U and an open set V containing x and $f(y)$, respectively, such that $f(U) \cap V = \emptyset$. Therefore $y \notin U$ and it follows that X is (δ, p)-T_1.

Recall that a space X is said to be T_1 if for each pair of distinct points x and y of X, there exist an open set U containing x but not y and an open set V containing y but not x.

Theorem 2.3 *If $f : X \to Y$ is a surjective function with the (δ, p)-closed graph, then Y is T_1.*

Proof. Let y_1 and y_2 be two distinct points of Y. Since f is surjective, there exists a point x in X such that $f(x) = y_2$. Therefore $(x, y_1) \notin G(f)$. By Lemma 2.1, there exist a (δ, p)-open set U and an open set V containing x and y_1, respectively, such that $f(U) \cap V = \emptyset$. It follows that $y_2 \notin V$. Hence Y is T_1.

Definition 2.4 *A function $f : X \to Y$ is said to be (δ, p)-W-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a (δ, p)-open set U in X containing x such that $f(U) \subset Cl(V)$.*

Theorem 2.4 *If $f : X \to Y$ is (δ, p)-W-continuous and Y is Hausdorff, then $G(f)$ is (δ, p)-closed.*

Proof. Suppose that $(x, y) \notin G(f)$, then $f(x) \neq y$. By the fact that Y is Hausdorff, there exist open sets W and V such that $f(x) \in W$, $y \in V$ and $V \cap W = \emptyset$. It follows that $Cl(W) \cap V = \emptyset$. Since f is (δ, p)-W-continuous, there exists $U \in \delta PO(X, x)$ such that $f(U) \subset Cl(W)$. Hence, we have $f(U) \cap V = \emptyset$. This means that $G(f)$ is (δ, p)-closed.

Corollary 2.4A *If $f : X \to Y$ is (δ, p)-continuous and Y is Hausdorff, then $G(f)$ is (δ, p)-closed in $X \times Y$.*

Definition 2.5 *A subset A of a space X is said to be (δ, p)-compact relative to X if every cover of A by (δ, p)-open sets of X has a finite subcover.*

Theorem 2.5 *Let $f : X \to Y$ have a (δ, p)-closed graph. If K is (δ, p)-compact relative to X, then $f(K)$ is closed in Y.*

Proof. Suppose $y \notin f(K)$. For each $x \in K$, $f(x) \neq y$. By Lemma 2.1, there exist $U_x \in \delta PO(X, x)$ and an open neighbourhood V_x of y such that $f(U_x) \cap V_x = \emptyset$. The family $\{U_x \mid x \in K\}$ is a cover of K by (δ, p)-open sets of X and there exists a finite subset K_0 of K such that $K \subset \bigcup\{U_x \mid x \in K_0\}$. Put $V = \bigcap\{V_x \mid x \in K_0\}$. Then V is an open neighbourhood of y and $f(K) \cap V = \emptyset$. This means that $f(K)$ is closed in Y.

Definition 2.6 *A function $f : X \to Y$ is called perfectly continuous [4] if for each open set $A \subset Y$, $f^{-1}(A)$ is open and closed in X.*

Lemma 2.2 *([3]) If A and B are pre-regular p-open sets of the spaces X and Y, respectively, then $A \times B$ is a pre-regular p-open set of $X \times Y$.*

Theorem 2.6 *If $f : X \to Z$ has a (δ, p)-closed graph $G(f)$ and $g : Y \to Z$ is a perfectly continuous function, then the set $\{(x, y) : f(x) = g(y)\}$ is (δ, p)-closed in $X \times Y$.*

Proof. Let $A = \{(x, y) : f(x) = g(y)\}$ and $(x, y) \in X \backslash A$. We have $f(x) \neq g(y)$ and then $(x, g(y)) \in (X \times Z) \backslash G(f)$. Since f has a (δ, p)-closed graph $G(f)$, there exist a (δ, p)-open set U and an open set V containing x and $g(y)$, respectively such that $f(U) \cap V = \varnothing$. This implies that there exists a pre-regular p-open set N containing x such that $N \subset U$ and $f(N) \cap V = \varnothing$. Since g is a perfectly continuous function, then there exist an open and closed set G containing y such that $g(G) \subset V$. We have $f(U) \cap g(G) = \varnothing$. This implies that $(N \times G) \cap A = \varnothing$. Since $N \times G$ is pre-regular p-open, then $(x, y) \notin \delta Cl_p(A)$. Thus, E is (δ, p)-closed in $X \times Y$.

Corollary 2.6B *If $f : X \to Z$ is a (δ, p)-continuous function and $g : Y \to Z$ is a perfectly continuous function and Z is Hausdorff, then the set $\{(x, y) : f(x) = g(y)\}$ is (δ, p)-closed in $X \times Y$.*

Proof. It follows from Corollary 2.6A and Theorem 2.6.

Theorem 2.7 *If $f : X \to Y$ is a (δ, p)-continuous function and Y is Hausdorff, then the set $\{(x, y) \in X \times X : f(x) = f(y)\}$ is (δ, p)-closed in $X \times X$.*

Proof. Let $A = \{(x, y) : f(x) = f(y)\}$ and let $(x, y) \in (X \times X)\backslash A$. It follows that $f(x) \neq f(y)$. Since Y is Hausdorff, there exist open sets U and V containing $f(x)$ and $f(y)$, respectively, such that $U \cap V = \emptyset$. Since f is (δ, p)-continuous, there exist pre-regular p-open sets H and G in X containing x and y, respectively, such that $f(H) \subset U$ and $f(G) \subset V$. This implies $(H \times G) \cap A = \emptyset$. We have $H \times G$ is a pre-regular p-open set in $X \times X$ containing (x, y). Hence, A is (δ, p)-closed in $X \times X$.

Definition 2.7 *A function $f : X \to Y$ is called contra (δ, p)-open if the image of every (δ, p)-open set in X is closed in Y.*

Theorem 2.8 *If $f : X \to Y$ is a contra (δ, p)-open function such that inverse image of each point of Y is (δ, p)-closed, then f has a (δ, p)-closed graph $G(f)$.*

Proof. Let $(x, y) \in X\backslash G(f)$. We have $x \notin f^{-1}(y)$. Since $f^{-1}(y)$ is (δ, p)-closed, there exists a pre-regular p-open set A containing x such that $A \cap f^{-1}(y) = \varnothing$. Since f is contra (δ, p)-open, then $f(A)$ is closed. This implies that there exist an open set B in Y containing y such that $f(A) \cap B = \varnothing$. Hence, f has a (δ, p)-closed graph $G(f)$.

Theorem 2.9 *If $f : (X, \tau) \to (Y, \sigma)$ has a (δ, p)-closed graph $G(f)$, then for each $x \in X$, $\{f(x)\} = \bigcap_{x \in A \in \delta PO(X,\tau)} Cl(f(A))$.*

Proof. Suppose that $y \neq f(x)$ and $y \in \bigcap_{x \in A \in \delta PO(X,\tau)} Cl(f(A))$. Then $y \in Cl(f(A))$ for each $x \in A \in \delta PO(X, \tau)$. This implies that for each open set B containing y, $B \cap f(A) \neq \varnothing$. Since $(x, y) \notin G(f)$ and $G(f)$ is a (δ, p)-closed graph, this is a contradiction.

Definition 2.8 *A space X is said to be (δ, p)-T_2 if for each pair of distinct points x and y in X, there exist disjoint (δ, p)-open sets A and B in X such that $x \in A$ and $y \in B$.*

Definition 2.9 *A function $f : X \to Y$ is called (δ, p)-open if the image of every (δ, p)-open set in X is open in Y.*

Theorem 2.10 *If $f : X \to Y$ is a surjective (δ, p)-open function with a (δ, p)-closed graph $G(f)$, then Y is T_2 .*

Proof. Let y_1 and y_2 be any distinct points of Y. Since f is surjective $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y)\backslash G(f)$. This implies that there exist a (δ, p)-open set A of X and an open set B of Y such that $(x, y_2) \in A \times B$ and $(A \times B) \cap G(f) = \emptyset$. We have $f(A) \cap B = \emptyset$. Since f is (δ, p)-open, then $f(A)$ is open such that $f(x) = y_1 \in f(A)$. Thus, Y is T_2.

Theorem 2.11 *If $f : X \to Y$ is a (δ, p)-continuous injection and Y is T_2, then X is (δ, p)-T_2.*

Proof. Let x and y in X be any pair of distinct points. Then there exist disjoint open sets A and B in Y such that $f(x) \in A$ and $f(y) \in B$. Since f is (δ, p)-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ is (δ, p)-open in X containing x and y respectively. We have $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Thus, X is (δ, p)-T_2.

Lemma 2.3 *([3]) If a space X is submaximal, then any finite intersection of pre-regular p-open sets is pre-regular p-open.*

Theorem 2.12 *If f, $g : X \to Y$ are (δ, p)-continuous functions, X is submaximal and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is (δ, p)-closed in X.*

Proof. Let $A = \{x \in X : f(x) = g(x)\}$. Take $x \in X\backslash A$. We have $f(x) \neq g(x)$. Since Y is Hausdorff, then there exist open sets U and V in Y containing $f(x)$ and $g(x)$, respectively, such that $U \cap V = \emptyset$. Since f and g are (δ, p)-continuous, then $f^{-1}(U)$ and $g^{-1}(V)$ are (δ, p)-open in X with $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$. Then there exist pre-regular p-open sets G and H such that $x \in G \subset f^{-1}(U)$ and $x \in H \subset g^{-1}(V)$. Take $K = G \cap H$. By Lemma 2.3, K is pre-regular p-open. Thus, $f(K) \cap g(K) = \emptyset$ and hence $x \notin \delta Cl_p(A)$. This shows that A is (δ, p)-closed in X.

Acknowledgments

S. P. Moshokoa acknowledges the support by the South African National Research Foundation under Grant number 2053847. Parts of this paper were written while S. Jafari was a visitor to the Department of Mathematical Sciences, University of South Africa during October 2005.

References

1. M. Caldas, S. Jafari, T. Noiri and M. Sarsak, Weak separatin axioms via pre-regular p-open sets, Istitute of Adv. Sci. Research, Pure Math., 2(2) 2010- 1-13.

2. S. Jafari, Pre-rarely-p-continuous functions, Far East J. Math. Sci. (FJMS) Special Vol. (2000), Part I (Geometry and Topology), 87-96.

3. S. Jafari, On certain types of notions via preopen sets, Tamkang J. Math. 37(4)(2006), 391-398.

4. T. Noiri, Super-continuity and some strong forms of continuity, Indian J. Pure Appl. Math., 15 (1984), 17-22.

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