Bol. Soc. Paran. de Mat. (3s.) **v. 27** 2 (2009): 15–22.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/spm doi:10.5269/bspm.v27i2.10206

$(1,2)^*$-Strongly Semi-Pre-$T_{1/2}$ Spaces

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ABSTRACT: In this paper, we introduce the concepts of $(1,2)^*$-strongly generalized semi-preopen sets and $(1,2)^*$-strongly semi-pre-$T_{1/2}$ spaces in bitopological spaces which are stronger forms of $(1,2)^*$-generalized semi-preopen sets and $(1,2)^*$- semi-pre-$T_{1/2}$ spaces. Further, we study some of their properties.

Key Words: $(1,2)^*$-strongly generalized semi-preopen sets and $(1,2)^*$-strongly semi-pre-$T_{1/2}$ bitopological spaces.

1. Introduction

Levine introduced generalized closed sets and studied their properties. Thivagar et al have introduced the concepts of $(1,2)^*$-semi-open sets, $(1,2)^*$-generalized-closed sets, $(1,2)^*$-semi-generalized closed sets in bitopological spaces.
In this paper we introduce the concept of a new class of sets, namely $(1,2)^*$-strongly generalized semi-preopen sets in bitopological spaces which are stronger forms of $(1,2)^*$-generalized semi-preopen sets. Also we introduce the concept of $(1,2)^*$-strongly semi-pre-$T_{1/2}$ spaces. Further, we study some of their properties.

2. Preliminaries

Throughout this paper (X,τ_1,τ_2) or simply X represents a bitopological space on which no separation axioms are assumed unless otherwise mentioned.

Definition 2.1 [9] *A subset S of a bitopological space (X,τ_1,τ_2) is said to be $\tau_{1,2}$-open if $S = A \cup B$ where $A \in \tau_1$, and $B \in \tau_2$. A subset S of X is said to be $\tau_{1,2}$-closed if the complement of S is $\tau_{1,2}$-open.*

Definition 2.2 [9] *Let S be a subset of X. Then*

(i) The $\tau_{1,2}$-interior of S, denoted by $\tau_{1,2}$-int(S), is defined by $\cup\{G/G \subseteq S$ and G is $\tau_{1,2}$-open $\}$.

(ii) The $\tau_{1,2}$-closure of S, denoted by $\tau_{1,2}$-cl(S), is defined by $\cap\{F/S \subseteq F$ and F is $\tau_{1,2}$-closed $\}$.

Remark 2.1 $\tau_{1,2}$-open sets need not form a topology.
We recall the following definitions which will be useful in the sequel.

Definition 2.3 [10] *A subset A of a bitopological space (X,τ_1,τ_2) is called*

(i) $(1,2)^*$-semi-open if $A \subseteq \tau_{1,2}$-cl$(\tau_{1,2}$-int$(A))$.

2000 *Mathematics Subject Classification:* 54E55, 54D10

(ii) $(1,2)^*$-preopen if $A \subseteq \tau_{1,2}$-int$(\tau_{1,2}$-cl$(A))$.

(iii)$(1,2)^*$-α-open if $A \subseteq \tau_{1,2}$-int$(\tau_{1,2}$-cl$(\tau_{1,2}$-int$(A)))$.

(iv)$(1,2)^*$-semi-preopen or $(1,2)^*$-β-open if $A \subseteq \tau_{1,2}$-cl$(\tau_{1,2}$-int$(\tau_{1,2}$-cl$(A)))$.

(v) $(1,2)^*$-generalized closed (briefly $(1,2)^*$-g-closed) if $\tau_{1,2}$-cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X.

(vi) $(1,2)^*$-semi-generalized closed (briefly $(1,2)^*$-sg-closed) if $(1,2)^*$-$scl(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^*$-semi-open in X.

(vii) $(1,2)^*$-generalized semi-closed (briefly $(1,2)^*$-gs-closed) if $(1,2)^*$-$scl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X.

The complements of the sets mentioned above from (i) to (iv) are their respective closed sets and the complements of the sets mentioned above from (v) to (vii) are their respective open sets.

Let us introduce the following definitions:

Definition 2.4 *Let A be a subset of the bitopological space (X, τ_1, τ_2). Then*

(i) $(1,2)^*$-semi-preclosure of A, denoted by $(1,2)^*$-spcl(A), is defined as the intersection of all $(1,2)^*$-semi-preclosed sets containing A.

(ii) $(1,2)^*$-semi-preinterior of A, denoted by $(1,2)^*$-spint(A), is defined as the union of all $(1,2)^*$-semi-preopen sets contained in A.

Definition 2.5 *A subset A of a bitopological space (X, τ_1, τ_2) is called*

(i) $(1,2)^*$-generalized semi-preclosed (briefly $(1,2)^*$-gsp-closed) if $(1,2)^*$-spcl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X.

(ii) $(1,2)^*$-generalized semi-preopen (briefly $(1,2)^*$-gsp-open) if A^c is $(1,2)^*$-gsp-closed.

Remark 2.2 $((1,2)^*\text{-spcl}(A))^c = (1,2)^*\text{-spint}(A^c)$

We introduce the following definitions:

Definition 2.6 *A bitopological space (X, τ_1, τ_2) is called a*

(i) $(1,2)^*$-$T_{1/2}$-space if every $(1,2)^*$-g-closed set in X is $\tau_{1,2}$-closed in X.

(ii) $(1,2)^*$-semi-$T_{1/2}$-space if every $(1,2)^*$-sg-closed set in X is $(1,2)^*$-semi-closed in X.

(iii) $(1,2)^*$-semi-pre-$T_{1/2}$-space if every $(1,2)^*$-gsp-closed set in X is $(1,2)^*$-semi-preclosed in X.

(iv) $(1,2)^*$-*α-space if every* $(1,2)^*$-*α-closed set in X is $\tau_{1,2}$-closed in X.*

(v) $(1,2)^*$-*T_d-space if every* $(1,2)^*$-*gs-closed set in X is* $(1,2)^*$-*g-closed in X.*

Let us recall the following definition:

Definition 2.7 [4] *A subset A of a bitopological space X is called* $(1,2)^*$-*strongly generalized semi-preclosed (briefly* $(1,2)^*$-*strongly gsp-closed) if* $(1,2)^*$-*spcl(A)* $\subseteq$ *G, whenever $A \subseteq G$ and G is* $(1,2)^*$-*g-open in X.*

3. $(1,2)^*$-Strongly Generalized Semi-preopen Sets

In this section, we introduce $(1,2)^*$-strongly generalized semi-preopen sets and study some of their properties.

Definition 3.1 *A subset A of a bitopological space X is called* $(1,2)^*$-*strongly generalized semi-preopen (briefly* $(1,2)^*$-*strongly gsp-open) if A^c is* $(1,2)^*$-*strongly gsp-closed.*

Theorem 3.1 *A subset A of a bitopological space X is* $(1,2)^*$-*strongly gsp-open if and only if $F \subseteq$* $(1,2)^*$-*spint(A) whenever F is* $(1,2)^*$-*g-closed and $F \subseteq A$.*

Proof: Assume that A is $(1,2)^*$-strongly gsp-open in X. Let F be $(1,2)^*$-g-closed and $F \subseteq A$. This implies F^c is $(1,2)^*$-g-open and $A^c \subseteq F^c$. Since A^c is $(1,2)^*$-strongly gsp-closed, $(1,2)^*$-spcl$(A^c) \subseteq F^c$. Since $(1,2)^*$-spcl$(A^c) = ((1,2)^*$-spint$(A))^c$, $((1,2)^*$-spint$(A))^c \subseteq F^c$. Therefore $F \subseteq (1,2)^*$-spint(A).
Conversely, assume that $F \subseteq (1,2)^*$-spint(A) whenever F is $(1,2)^*$-g-closed and $F \subseteq A$. Let G be a $(1,2)^*$-g-open set in X containing A^c. Therefore G^c is a $(1,2)^*$-g-closed set contained in A. By hypothesis, $G^c \subseteq (1,2)^*$-spint(A). Taking complements, $G \supseteq (1,2)^*$-spcl(A^c). Therefore A^c is $(1,2)^*$-strongly gsp-closed in X. Hence A is $(1,2)^*$-strongly gsp-open in X. □

Remark 3.1 Intersection of two $(1,2)^*$-strongly gsp-open sets need not be a $(1,2)^*$-strongly gsp-open set.

Example 3.1 *Let $X = \{a,b,c,d\}$; $\tau_1 = \{\phi, \{a,b\}, X\}$; $\tau_2 = \{\phi, \{a,c\}, X\}$;*
$\tau_{1,2}$-open sets $= \{\phi, \{a,b\}, \{a,c\}, \{a,b,c\}, X\}$.
The sets $A = \{a,b,d\}$ and $B = \{b,c,d\}$ are $(1,2)^*$-*strongly gsp-open but their intersection $\{b,d\}$ is not* $(1,2)^*$-*strongly gsp-open.*

Theorem 3.2 *If a set A is* $(1,2)^*$-*strongly gsp-closed, then* $(1,2)^*$-*spcl(A)* $- A$ *is* $(1,2)^*$-*strongly gsp-open.*

Proof: If A is $(1,2)^*$-strongly gsp-closed, by Theorem 3.9 [4], $(1,2)^*$-spcl$(A) - A$ contains no nonempty $(1,2)^*$-g-closed set. Therefore, by Theorem 3.1, $(1,2)^*$-spcl$(A) - A$ is $(1,2)^*$-strongly gsp-open. □

Theorem 3.3 *If a set A is $(1,2)^*$-strongly gsp-open in X, then $G = X$ whenever G is $(1,2)^*$-g-open and $(1,2)^*$-spint(A) $\cup A^c \subseteq G$.*

Proof: Suppose that G is $(1,2)^*$-g-open and $(1,2)^*$-spint$(A) \cup A^c \subseteq G$. Now $G^c \subseteq (1,2)^*$-spcl$(A^c) \cap A = (1,2)^*$-spcl$(A^c) - A^c$. Since G^c is $(1,2)^*$-g-closed and A^c is $(1,2)^*$-strongly gsp-closed, by Theorem 3.9 [4], $G^c = \phi$ and hence $G = X$. □

Theorem 3.4 *For each $x \in X$, $\{x\}$ is $(1,2)^*$-g-closed or $(1,2)^*$-strongly gsp-open.*

Proof: If $\{x\}$ is not $(1,2)^*$-g-closed, then the only $(1,2)^*$-g-open set containing $X - \{x\}$ is X. Thus $X - \{x\}$ is $(1,2)^*$-strongly gsp-closed and $\{x\}$ is $(1,2)^*$-strongly gsp-open. □

4. $(1,2)^*$-Strongly Semi-pre-$T_{1/2}$ Spaces

In this section we introduce $(1,2)^*$-strongly semi-pre-$T_{1/2}$ spaces and study some of their properties.

Definition 4.1 *A bitopological space (X, τ_1, τ_2) is called $(1,2)^*$-strongly semi-pre-$T_{1/2}$ if every $(1,2)^*$-gsp-closed set is $(1,2)^*$-strongly gsp-closed in X.*

Theorem 4.1 *Every $(1,2)^*$-semi-pre-$T_{1/2}$ space is $(1,2)^*$-strongly semi-pre-$T_{1/2}$.*

Proof: Let (X, τ_1, τ_2) be $(1,2)^*$-semi-pre-$T_{1/2}$. Let A be a $(1,2)^*$-gsp-closed set in X. Since X is $(1,2)^*$-semi-pre-$T_{1/2}$, A is $(1,2)^*$-semi-preclosed in X. Then it follows from Definition 2.7 that A is a $(1,2)^*$-strongly gsp-closed set. Hence X is $(1,2)^*$-strongly semi-pre-$T_{1/2}$. □

The converse of Theorem 4.1 need not be true. The bitopological space given in Example 3.1 is $(1,2)^*$-strongly semi-pre-$T_{1/2}$ but not $(1,2)^*$-semi-pre-$T_{1/2}$.

Lemma 4.1 *If a set A in a biopological space X is $(1,2)^*$-gsp-closed then $(1,2)^*$-spcl(A) $- A$ does not contain non-empty $\tau_{1,2}$-closed set.*

Proof: Let F be a $\tau_{1,2}$-closed subset of $(1,2)^*$-spcl(A) - A. Then $A \subseteq X - F$ where A is $(1,2)^*$-gsp-closed and $X - F$ is $\tau_{1,2}$-open. Therefore $(1,2)^*$-spcl$(A) \subseteq X - F$ or equivalently $F \subseteq X - (1,2)^*$-spcl(A). Thus $F \subseteq (1,2)^*$-spcl$(A) \cap ((1,2)^*$-spcl$(A))^c$ $= \phi$ or $F = \phi$. □

Lemma 4.2 *If every singleton subset in a biopological space X is $\tau_{1,2}$-closed or $(1,2)^*$-preopen then X is $(1,2)^*$-semi-pre-$T_{1/2}$.*

Proof: Let A be $(1,2)^*$-gsp-closed. We need to show that A is $(1,2)^*$-semi-preclosed or equivalently $(1,2)^*$-spcl$(A) = A$. The inclusion $A \subseteq (1,2)^*$-spcl(A) is trivial. To prove the other inclusion, let $x \in (1,2)^*$-spcl(A).
Case (i). $\{x\}$ is $\tau_{1,2}$-closed. By Lemma 4.1, $(1,2)^*$-spcl$(A) - A$ does not contain $\{x\}$. Hence $x \in A$. Case (ii). $\{x\}$ is $(1,2)^*$-preopen. Then $\{x\}$ is $(1,2)^*$-semi-preopen and since $x \in (1,2)^*$-spcl(A), $\{x\} \cap A \neq \phi$. This implies $x \in A$. Thus in both cases $x \in A$ or equivalently $(1,2)^*$-spcl$(A \subseteq A$. □

Theorem 4.2 *Every $(1,2)^*$-$T_{1/2}$bitopological space is $(1,2)^*$-semi-pre-$T_{1/2}$.*

Proof: Let (X, τ_1, τ_2) be a $(1,2)^*$-$T_{1/2}$ bitopological space. First let us prove that for each $x \in X$, $\{x\}$ is $\tau_{1,2}$-open or $\tau_{1,2}$-closed. Let $x \in X$. If $\{x\}$ is not $\tau_{1,2}$-closed, then $X - \{x\}$ is not $\tau_{1,2}$-open. Therefore the only $\tau_{1,2}$-open set containing $X - \{x\}$ is X and hence $X - \{x\}$ is $(1,2)^*$-g-closed. Since X is $(1,2)^*$-$T_{1/2}$, $X - \{x\}$ is $\tau_{1,2}$-closed or $\{x\}$ is $\tau_{1,2}$-open. Thus for each $x \in X$, $\{x\}$ is $\tau_{1,2}$-open or $\tau_{1,2}$-closed. This implies for each $x \in X$, $\{x\}$ is $(1,2)^*$-preopen or $\tau_{1,2}$-closed. Then by Lemma 4.2, X is $(1,2)^*$-semi-pre-$T_{1/2}$. □

Theorem 4.3 *Every $(1,2)^*$-$T_{1/2}$ bitopological space is $(1,2)^*$-strongly semi-pre-$T_{1/2}$ but not conversely.*

Proof: Let (X, τ_1, τ_2) be a $(1,2)^*$-$T_{1/2}$-space. Then by Theorem 4.2, X is $(1,2)^*$-semi-pre-$T_{1/2}$. Let A be a $(1,2)^*$-gsp-closed set in X. Then A is $(1,2)^*$-semi-preclosed in X. By the definition of $(1,2)^*$-strongly gsp-closed set, A is a $(1,2)^*$-strongly gsp-closed set. Hence X is $(1,2)^*$-strongly semi-pre-$T_{1/2}$.
The bitopological space given in Example 3.1 is $(1,2)^*$-strongly semi-pre-$T_{1/2}$ but not $(1,2)^*$-$T_{1/2}$. □

Theorem 4.4 *Every $(1,2)^*$-α-space is $(1,2)^*$-strongly semi-pre-$T_{1/2}$.*

Proof: Let X be a $(1,2)^*$-α-space. By the Lemma 3.13 [3], for each $x \in X$, $\{x\}$ is $(1,2)^*$-nowhere dense or $(1,2)^*$-preopen. Every $(1,2)^*$-nowhere dense subset of X is $(1,2)^*$-α-closed. Since X is a $(1,2)^*$-α-space, it follows that every $(1,2)^*$-nowhere dense subset of X is $\tau_{1,2}$-closed. Thus for each $x \in X$, $\{x\}$ is $\tau_{1,2}$-closed or $(1,2)^*$-preopen. Now it follows from Lemma 4.2, that X is $(1,2)^*$-semi-pre-$T_{1/2}$ and from Theorem 4.1 that X is $(1,2)^*$-strongly semi-pre-$T_{1/2}$. □

The converse of Theorem 4.4 need not be true. The bitopological space given in Example 3.1 is $(1,2)^*$-strongly semi-pre-$T_{1/2}$ but not a $(1,2)^*$-α-space.

Theorem 4.5 *Every singleton set in a $(1,2)^*$-strongly semi-pre-$T_{1/2}$ bitopological space X is $\tau_{1,2}$-closed or $(1,2)^*$-strongly gsp-open.*

Proof: Let $x \in X$. If the set $\{x\}$ is not $\tau_{1,2}$-closed, then the only $\tau_{1,2}$-open set containing $\{x\}^c$ is X. Hence $\{x\}^c$ is $(1,2)^*$-gsp-closed. Since X is $(1,2)^*$-strongly semi-pre-$T_{1/2}$, $\{x\}^c$ is $(1,2)^*$-strongly gsp-closed. Therefore $\{x\}$ is $(1,2)^*$-strongly gsp-open. □

Corollary 4.5A *Every singleton set in a $(1,2)^*$-strongly semi-pre-$T_{1/2}$ space X is $\tau_{1,2}$-closed or $(1,2)^*$-gsp-open.*

The proof follows from the proof of Theorem 4.5.

Remark 4.1 $(1,2)^*$-strongly semi-pre-$T_{1/2}$ bitopological space and $(1,2)^*$-semi-$T_{1/2}$ bitoplogical space are independent.

Example 4.1 *Let $X = \{a,b,c,d\}$; $\tau_1 = \{\phi, \{a,b\}, X\}$; $\tau_2 = \{\phi, \{a,c\}, X\}$;*
$\tau_{1,2}$-open sets $= \{\phi, \{a,b\}, \{a,c\}, \{a,b,c\}, X\}$.
This bitopological space (X, τ_1, τ_2) is $(1,2)^*$-strongly semi-pre-$T_{1/2}$ but not $(1,2)^*$-semi-$T_{1/2}$.

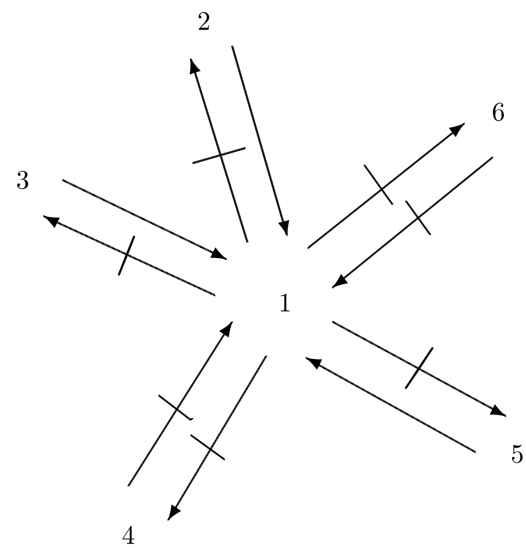
Example 4.2 *Let $X = \{a,b,c,d\}$; $\tau_1 = \{\phi, \{a\}, \{b,c,d\}, X\}$;*
$\tau_2 = \{\phi, \{c\}, \{a,b,d\}, X\}$; $\tau_{1,2}$-open sets $= \{\phi, \{a\}, \{c\}, \{a,c\}, \{b,c,d\}, \{a,b,d\}, X\}$
This bitopological space (X, τ_1, τ_2) is $(1,2)^*$-semi-$T_{1/2}$ but not $(1,2)^*$-strongly semi-pre-$T_{1/2}$.

Remark 4.2 $(1,2)^*$-strongly semi-pre-$T_{1/2}$ bitopological space and $(1,2)^*$-T_d bitoplogical space are independent.

The bitopological space given in Example 4.2 is $(1,2)^*$-T_d but not $(1,2)^*$-strongly semi-pre-$T_{1/2}$.

Example 4.3 *Let $X = \{a,b,c,d\}$;*
$\tau_1 = \{\phi, \{b\}, \{a,b\}, \{a,b,d\}, X\}$; $\tau_2 = \{\phi, \{b,c\}, \{b,c,d\}, X\}$;
$\tau_{1,2}$-open sets $= \{\phi, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, X\}$. This bitopological space is $(1,2)^*$-strongly semi-pre-$T_{1/2}$ but not $(1,2)^*$-T_d.

From the above results we have the following diagram where
$1 = (1,2)^*$-strongly semi-pre-$T_{1/2}$ bitopological space,
$2 = (1,2)^*$-semi-pre-$T_{1/2}$ bitopological space,
$3 = (1,2)^*$-$T_{1/2}$ bitopological space,
$4 = (1,2)^*$-semi-$T_{1/2}$ bitopological space,
$5 = (1,2)^*$-α bitopological space and $6 = (1,2)^*$-T_d bitoplogical space.



Acknowledgments

We are thankful to the referee for giving suggestions to improve the presentation of the paper.

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