



Polynomial and Analytic Boundary Feedback Stabilization of Square Plate

Salsabil Nouira

ABSTRACT: We consider a boundary feedback stabilization problem of the plate equation in a square, in the case where the geometric condition of Ammari-Tucsnak [6] is not satisfied. We prove a polynomial decay for regular initial data. Moreover, we prove an exponential stability result for some subspace of the energy space. Finally, we give a precise estimate on the analyticity of reachable functions where we have an exponential stability.

Key Words: Plate equation, polynomial stabilization, analytic stabilization, observability inequality, low and high frequency.

Contents

1 Introduction	23
2 Main results	24
3 Inequality of observability	27
4 Some background on a class of dynamical systems	30
5 Proof of the main results	31
5.1 Proof of the first assertion of the Theorem 2.1	31
5.2 Proof of the second assertion of Theorem 2.1	33
5.3 Proof of Theorem 2.2	33

1. Introduction

Let $\Omega = (0, \pi) \times (0, \pi) \subset \mathbb{R}^2$. We denote by $\partial\Omega$ the boundary of Ω and we assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 = \{(0, y)/y \in (0, \pi)\}$ and $\Gamma_1 = \partial\Omega \setminus \bar{\Gamma}_0$.

We consider the plate equation as follows :

$$\left\{ \begin{array}{ll} \partial_t^2 u + \Delta^2 u = 0 & \Omega \times (0, +\infty), \\ \Delta u = -\frac{\partial}{\partial \nu} [G(\partial_t u)] 1_{|\Gamma_0} & \partial\Omega \times (0, +\infty), \\ u = 0 & \partial\Omega \times (0, +\infty) \\ u(x, 0) = u^0(x), \quad \partial_t u(x, 0) = u^1(x) & \Omega, \end{array} \right. \quad (1)$$

where the operator G is defined as $(-\Delta)^{-1} : H^{-1}(\Omega) \longrightarrow H_0^1(\Omega)$, ν is the unit normal vector of $\partial\Omega$ pointing towards the exterior of Ω and $\Delta^2 : \mathcal{D}(\Delta^2) \longrightarrow H^{-1}(\Omega)$ be a self-adjoint, positive and boundedly invertible operator where

$$\mathcal{D}(\Delta^2) = \{u \in H^{-1}(\Omega) / \Delta^2 u \in H^{-1}(\Omega)\}.$$

2000 *Mathematics Subject Classification*: 35B40, 74K20, 34H05, 34H15, 93D15.

The controllability of the dynamical system modelling the vibrations of the plate with boundary control acting on the moment has been investigated in several works such as Ammari and Khenissi [4]-[10], Ammari and Vodev [9], Krabs, Leugering and Seidman [15], Leugering [20], Lebeau [17], [14] and in [23]. In [16] the exact controllability of the same system has been established under the assumption that Ω is a square and under much weaker assumption on the controlled part of the boundary (Γ_1 is only supposed to contain non-empty vertical and horizontal subsets). The geometric optics condition introduced by Bardos, Lebeau and Rauch in [12] for the wave equation is thus not necessary in this case. In fact, recently, Ammari and Tucsnak (see [6]) have proved that the system is exponentially stable if and only if the controlled part of the boundary contains a vertical and horizontal part of non-zero length.

In this work, we study the polynomial stability for regular initial data and we study the exponential stability for some analytic initial data of a square Euler-Bernoulli plate with feedback. We use the methodology introduced in [5] (see also [13] for the bounded case), where the exponential stability for this problem is reduced to an observability inequality proved by [23] :

$$\int_0^T \left\| \frac{\partial[G(\partial_t \phi)]}{\partial \nu} \right\|_{L^2(\Gamma_0)}^2 dt \geq C \|u\|_{H_0^1(\Omega) \times H^{-1}(\Omega)}^2, \quad \forall (u^0, u^1) \in H_0^1(\Omega) \times H^{-1}(\Omega), \quad (2)$$

where ϕ is the solution of the following undamped system associated to (1) :

$$\begin{cases} \partial_t^2 \phi + \Delta^2 \phi = 0 & \Omega \times (0, +\infty), \\ \phi = 0 & \partial\Omega \times (0, +\infty) \\ \phi(x, 0) = u^0(x), \quad \partial_t \phi(x, 0) = u^1(x) & \Omega. \end{cases} \quad (3)$$

The paper is organized as follows. The statements of the main results are given in the following section. Section 3 is devoted to the observability inequality of high and low frequency. In Section 4, we give some background on a class of dynamical systems. Finally, Section 5 contains the proof of main results.

2. Main results

The system (1) is well-posed for initial condition satisfying $(u^0, u^1) \in E = H_0^1(\Omega) \times H^{-1}(\Omega)$, i.e there exists a unique solution (see [6])

$$u \in C((0, +\infty), H_0^1(\Omega)) \cap C^1((0, +\infty), H^{-1}(\Omega)).$$

The energy $E(t)$ of system (1) is given by the following expression :

$$E(t) = \frac{1}{2} \left(\|u(t)\|_{H_0^1(\Omega)}^2 + \|\partial_t u(t)\|_{H^{-1}(\Omega)}^2 \right).$$

The solution of (1) satisfies the following energy estimate:

$$E(t) - E(0) = - \int_0^t \int_{\Gamma_0} \left| \frac{\partial[G(\partial_t u)]}{\partial \nu} \right|^2 d\Gamma_0 ds, \quad \forall t \geq 0. \quad (4)$$

Let $n_0 \in \mathbb{N}^*$ fixed, we denote by E_{n_0} the following space :

$$\begin{aligned} E_{n_0} &= \left\{ \sum_{(n,k) \in (\mathbb{N}^*)^2} a_{n,k} \varphi_{n,k} \in E, (a_{n,k}) \in l^2, a_{n,k} = 0, \forall k \neq n_0 \right\}; \\ E^1 &= \left\{ \sum_{(n,k) \in (\mathbb{N}^*)^2} a_{n,k} \varphi_{n,k} \in E, (a_{n,k}) \in l^2, a_{n,k} = 0, \forall k > n \right\}; \\ E^2 &= \left\{ \sum_{(n,k) \in (\mathbb{N}^*)^2} a_{n,k} \varphi_{n,k} \in E, (a_{n,k}) \in l^2, a_{n,k} = 0, \forall k \leq n \right\}; \end{aligned}$$

and by $E_n^i = E^i \cap E_n$, $i = 1, 2$. Where

$$\varphi_{n,k} = \frac{2}{\pi} \sqrt{\lambda_{n,k}} (\sin ny \sin kx, i\lambda_{n,k} \sin ny \sin kx), \quad \forall n, k \in \mathbb{N}^*$$

be the eigenfunctions sequence of Δ^2 normalized in $H_0^1(\Omega) \times H^{-1}(\Omega)$ and

$$\lambda_{n,k} = n^2 + k^2, \quad \forall n, k \in \mathbb{N}^*$$

is the eigenvalues sequence of Δ^2 .

For all $(u^0, u^1) \in E$, there exists $(a_{n,k}) \in l^2$ such that

$$(u^0, u^1) = \sum_{n,k \geq 1} a_{n,k} \varphi_{n,k}.$$

Which implies, $E = \bigoplus_{n \in \mathbb{N}^*} E_n$. For all $\alpha \in \mathbb{R}_+^*$, we define the following spaces (for more details, see [18]):

$$\begin{aligned} X_{0,\alpha} &:= \left\{ \sum_{(n,k) \in \mathbb{N}^* \times \mathbb{N}^*} \alpha_{n,k} e^{-\alpha n} \sin kx \sin ny / \sqrt{\lambda_{n,k}} \alpha_{n,k} \in l^2 \right\}; \\ X_{1,\alpha} &:= \left\{ \sum_{(n,k) \in \mathbb{N}^* \times \mathbb{N}^*} \alpha_{n,k} e^{-\alpha n} \sin kx \sin ny / \frac{\alpha_{n,k}}{\sqrt{\lambda_{n,k}}} \in l^2 \right\}; \\ X_\alpha &:= X_{0,\alpha} \times X_{1,\alpha}. \end{aligned}$$

Let $T > 2\pi\sqrt{2}$,

$$S_T = \left\{ (u^0, u^1) \in E / \exists C > 0, \left\| \frac{\partial[G(\partial_t \phi)]}{\partial \nu} \right\|_{L^2(\Gamma_0 \times]0, T])} \geq C \|(u^0, u^1)\|_E \right\}.$$

We notice that if $\alpha' > \alpha$, then $X_{\alpha'} \subset X_\alpha$ and $X_0 = E$.

Let $T > 2\pi\sqrt{2}$ and $\underline{u} := (u^0, u^1) \in E$.

For $n \in \mathbb{N}^*$, if we restrict to E_n , there exists $C(n) > 0$ such that for ϕ solution of (3), it holds :

$$\|(u^0, u^1)\|_E \leq C(n) \left\| \frac{\partial[G(\partial_t \phi)]}{\partial \nu} \right\|_{L^2(\Gamma_0 \times]0, T[)}.$$

We take for the $C(n)$ the smallest constant for which the previous inequality is checked and we denote by

$$\alpha_S(T) := \lim_{n \rightarrow +\infty} \sup \frac{\ln(C(n))}{n}.$$

According to H.U.M method, see [21], and to [3, chapter 1] we have :

$$\begin{cases} \alpha' > \alpha_S(T) \implies X_{\alpha'} \subset S_T, \\ \alpha' < \alpha_S(T) \implies X_{\alpha'} \not\subset S_T. \end{cases} \quad (5)$$

Thus for all $T > 2\pi\sqrt{2}$, $\alpha_S(T) = \inf \{\alpha \in \mathbb{R}_+ / X_\alpha \subset S_T\}$.

We give, now the main results of this paper :

Theorem 2.1 1. For all $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$\alpha_S(T) \leq \frac{C_\delta}{T^{1-\delta}}.$$

2. For $\alpha > \alpha_S(T)$, there exists a constant $C_\alpha, \gamma_\alpha > 0$ such that

$$E(t) \leq C_\alpha e^{-\gamma_\alpha t} E(0), \quad \forall \underline{u} \in X_\alpha, \forall t \geq 0.$$

Remark 2.1

1. We remark that all the elements of X_α can be continued as an holomorphic function over the complex strip $|\Im m(y)| < \alpha$.
2. The first assertion of the previous theorem implies that any analytic initial condition belongs to some S_T for T large enough, i.e., any initial condition whose Fourier coefficients in y decrease like $e^{-\alpha n}$ belongs to S_T if T is larger than $T(\alpha) = \sqrt[1-\delta]{\frac{C_\delta}{\alpha}}$.

Theorem 2.2 The system described by (1) is polynomial stable i.e., for all $\Gamma_0 \neq \emptyset$, there exists a constant $C > 0$ such as for all $(u^0, u^1) \in \mathcal{D}(\mathcal{A})$ we have :

$$E(t) \leq \frac{C}{1+t} \|(u^0, u^1)\|_{\mathcal{D}(\mathcal{A})}^2, \quad \forall t \geq 0, \quad (6)$$

where

$$\mathcal{D}(\mathcal{A}) = \{(u, v) \in H_0^1(\Omega) \times H^{-1}(\Omega); \Delta u \in H_0^1(\Omega), \Delta u = \partial_\nu[G(v)] 1_{|\Gamma_0}\}.$$

3. Inequality of observability

In this section we give the observability inequality at low and high frequency of the solution of (3) has been used for the proof of the main results. We specify the dependence of the constant which occurs in this estimation in function of the frequency of cut n .

Proposition 3.1 (*low frequencies estimate*) *For all $\epsilon > 0$, $\delta > 0$, there exist $T_1(\epsilon, \delta) \leq \frac{C_\delta}{\epsilon^{1+\delta}}$, $C_{\epsilon, \delta}$, $n_1 \in \mathbb{N}^*$ such that for all $n \geq n_1$ and for all $\underline{u} \in E_n$, the solution of problem (3) satisfies*

$$\|\underline{u}\|_{E^1}^2 \leq C_{\epsilon, \delta} e^{2\epsilon n} \int_0^\pi \int_{-T_1(\epsilon, \delta)}^{T_1(\epsilon, \delta)} \left| \frac{\partial G(\phi')}{\partial x}(0, y, t) \right|^2 dt dy.$$

Proof. Since we do not have a uniform gap, we adapt the method proposed by Allibert and Micu in [4], which is a method inspired from the WKB technique. First we need the next technical lemma, for this proof we refer to [4], paragraph 4.3, pages 580-591.

Lemma 3.1 *For all positive and odd integer q and for all $\epsilon > 0$, there exists a positive real number C_q and a real number $T_1(q, \epsilon)$ smaller than $C_{\epsilon, q} \epsilon^{\frac{q+1}{1-q}}$ such that for all $(n, k_0) \in \mathbb{N}^* \times \mathbb{N}^*$, there exists a function $h_{\epsilon, q}^{k_0, n}$ that satisfies :*

1. $\text{supp}(h_{\epsilon, q}^{k_0, n}) \subset [-T_1(q, \epsilon), T_1(q, \epsilon)]$.

2. For (k_0, n) such that $k_0 \leq n$,

$$\|h_{\epsilon, q}^{k_0, n}\|_{L^2} \leq C_{\epsilon, q} e^{2\epsilon n}.$$

3. If $k \neq k_0$, then $\int h_{\epsilon, q}^{k_0, n}(t) e^{i\lambda_{n, k} t} dt = 0$.

4. If $(n, k_0) \in \{(n, k) \in \mathbb{N}^* \times \mathbb{N}^* / k \leq n \text{ and } n \geq n_1(q, \epsilon)\}$, then

$$\left| \int h_{\epsilon, q}^{k_0, n}(t) e^{i\lambda_{n, k_0} t} dt \right| \geq \frac{c}{n^{N_q}}.$$

The two above positive constants C, c depend only on q and ϵ . Moreover it is always possible to choose $h_{\epsilon, q}^{k_0, n}$ even or odd, that we denote by $h_{\epsilon, q}^{k_0, n}$ and $h_{o\epsilon, q}^{k_0, n}$.

The analogue of Proposition 3.1 is proved in [4, Lemme 6]. The proof is quite similar, but for the sake of completeness, let us give the main steps.

Proof of Proposition 3.1. Let $n \in \mathbb{N}^*$ be such that $n \geq n_1(q, \epsilon)$, and let

$$(u^0, u^1) = \sum_{k \in \mathbb{N}^*} a_{n, k} \varphi_{n, k} \in E_n.$$

Then we have

$$\frac{\partial G(\phi')}{\partial x}(0, y, t) = \sum_{k \in \mathbb{N}^*} \frac{2i k a_{n,k}}{\pi \sqrt{\lambda_{n,k}}} e^{i\lambda_{n,k}t} \sin ny.$$

Hence for (k_0, n) such that $k_0 \leq n$ and $L \in \mathbb{N}^*$,

$$\begin{aligned} & \int h_{e_\epsilon, q}^{k_0, n}(t) K \left(\sum_{k \leq L} a_{n,k} \varphi_{n,k} \right) dt = \\ & \sum_{1 \leq k \leq L} \frac{2i k}{\pi \sqrt{\lambda_{n,k}}} (a_{n,k} + a_{n,-k}) \sin ny \int h_{e_\epsilon, q}^{k_0, n}(t) e^{i\lambda_{n,k}t} dt, \end{aligned}$$

where K is the operator defined by

$$\begin{aligned} K : \quad & E_n \rightarrow L^2(0, \pi) \\ & (u^0, u^1) \rightarrow \frac{\partial[G(\partial_t \phi)]}{\partial x}(0, y, t). \end{aligned}$$

If $L \geq k_0$, then by the point 3 of Lemma 3.1 we will have

$$\begin{aligned} & \int h_{e_\epsilon, q}^{k_0, n}(t) K \left(\sum_{k \leq L} a_{n,k} \varphi_{n,k} \right) (y, t) dt = \\ & \frac{2i k}{\pi \sqrt{\lambda_{n,k_0}}} (a_{n,k_0} + a_{n,-k_0}) \sin ny \int h_{e_\epsilon, q}^{k_0, n}(t) e^{i\lambda_{n,k_0}t} dt. \end{aligned}$$

For point 4 of Lemma 3.1 we deduce that there exists a constant $c > 0$ such that

$$\left| \int h_{e_\epsilon, q}^{k_0, n}(t) K \left(\sum_{k \leq L} a_{n,k} \varphi_{n,k} \right) (y, t) dt \right| \geq \frac{c}{n^{N_q+2}} |a_{n,k_0} + a_{n,-k_0}| |\sin ny|.$$

Consequently, if L tends to infinity, we obtain

$$\left| \int h_{e_\epsilon, q}^{k_0, n}(t) K \left(\sum_{k \in \mathbb{N}^*} a_{n,k} \varphi_{n,k} \right) (y, t) dt \right| \geq \frac{c}{n^{N_q+2}} |a_{n,k_0} + a_{n,-k_0}| |\sin ny|.$$

In the same we obtain

$$\left| \int h_{o_\epsilon, q}^{k_0, n}(t) K \left(\sum_{k \in \mathbb{N}^*} a_{n,k} \varphi_{n,k} \right) (y, t) dt \right| \geq \frac{c}{n^{N_q+2}} |a_{n,k_0} - a_{n,-k_0}| |\sin ny|.$$

These two estimates yield

$$|\sin ny| |a_{n,k_0}| \leq \frac{n^{N_q}}{c} \left(\left| \int h_{e_\epsilon, q}^{k_0, n}(t) K \left(\sum_{k \in \mathbb{N}^*} a_{n,k} \varphi_{n,k} \right) (y, t) dt \right| + \right.$$

$$\frac{n^{N_q}}{c} \left| \int h_{o_\epsilon, q}^{k_0, n}(t) K \left(\sum_{k \in \mathbb{N}^*} a_{n, k} \varphi_{n, k} \right) (y, t) dt \right|.$$

Then

$$\begin{aligned} |\sin ny|^2 \|(u_0, u_1)\|_E^2 &\leq \frac{n^{N_q+2}}{c} \left(\sum_{k \leq n} \int |h_{e_\epsilon, q}^{k_0, n}(t)|^2 \int_{T_1(q, \epsilon)}^{T_1(q, \epsilon)} \left| \frac{\partial[G(\phi')]}{\partial x}(0, y, t) \right|^2 dt + \right. \\ &\quad \left. \sum_{k \leq n} \int |h_{o_\epsilon, q}^{k_0, n}(t)|^2 \int_{T_1(q, \epsilon)}^{T_1(q, \epsilon)} \left| \frac{\partial[G(\phi')]}{\partial x}(0, y, t) \right|^2 dt \right). \end{aligned}$$

Integrating this estimate in $y \in (0, \pi)$ and using point 2 of Lemma 3.1, we obtain a constant $c_1 > 0$ such that

$$\|(u_0, u_1)\|_E^2 \leq c_1 e^{2\epsilon n} \int_{T_1(q, \epsilon)}^{T_1(q, \epsilon)} \int_0^\pi \left| \frac{\partial[G(\phi')]}{\partial x}(0, y, t) \right|^2 dy dt.$$

As $T_1(q, \epsilon) \leq C_q \epsilon^{\frac{1+q}{1-q}} = \frac{C_\delta}{\epsilon^{1+\delta}}$ and $\delta \rightarrow 0^+$ for $q \rightarrow +\infty$, this shows Proposition 3.1. \square

Lemma 3.2 (*High frequencies*) *For all $T_2 > 2\pi\sqrt{2}$, there exists a constant $C_{T_2} > 0$ such that for all integer $n > 0$ and initial data \underline{u} in E_n^2 the solution of problem (3) satisfies*

$$\|\underline{u}\|_{E^2}^2 \leq C_{T_2} \int_0^\pi \int_0^{T_2} \left| \frac{\partial[G(\partial_t \phi)]}{\partial x}(0, y, t) \right|^2 dt dy.$$

Proof. For $\underline{u} \in E_n^2$, we have $\partial_t \phi(x, y, t) = \sum_{k > n} \frac{2a_{n, k}}{\pi} \sqrt{\lambda_{n, k}} \sin kx \sin ny e^{it\lambda_{n, k}}$.

Then if we use the Ingham inequality [11], we obtain

$$\begin{aligned} \int_0^\pi \int_0^{T_2} \left| \frac{\partial[G(\partial_t \phi)]}{\partial x}(0, y, t) \right|^2 dt dy &= \sum_{n \in \mathbb{N}^*} \frac{1}{\pi^2} \int_0^{T_2} \left| \sum_{k > n} \frac{ka_{n, k}}{\sqrt{\lambda_{n, k}}} e^{i\lambda_{n, k} t} dt \right|^2 \\ &\geq C_{T_2} \sum_{k > n} \left| \frac{ka_{n, k}}{\sqrt{\lambda_{n, k}}} \right|^2. \end{aligned}$$

Which implies $\int_0^\pi \int_0^{T_2} \left| \frac{\partial[G(\partial_t \phi)]}{\partial x}(0, y, t) \right|^2 dt dy \geq C_{T_2} \|\underline{u}\|_{E^2}^2$. \square

4. Some background on a class of dynamical systems

Let H a Hilbert space with the norm $\|\cdot\|_H$, and let $A_1 : \mathcal{D}(A_1) \rightarrow H$ be a self-adjoint, positive and boundedly invertible operator. For $\alpha \geq 0$, we introduce the scale of Hilbert spaces $H_\alpha = \mathcal{D}(A_1^\alpha)$, with the norm $\|z\|_\alpha = \|A_1^\alpha z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space H as follows : $H_{-\alpha} = H_\alpha^*$ for $\alpha > 0$. The operator A_1 can be extended (or restricted) to each H_α , such that it becomes a bounded operator

$$A_1 : H_\alpha \rightarrow H_{\alpha-1}, \quad \forall \alpha \in \mathbb{R}. \quad (7)$$

The second ingredient needed for our construction is a bounded linear operator $B_1 : U \rightarrow H_{-\frac{1}{2}}$, where U is another Hilbert space which will be identified with its dual.

The system we consider are described by

$$\ddot{w}(t) + A_1 w(t) + B_1 y(t) = 0, \quad w(0) = w_0, \quad \dot{w}(0) = w_1, \quad t \in [0, \infty), \quad (8)$$

$$y(t) = B_1^* \dot{w}(t), \quad t \in [0, \infty). \quad (9)$$

The system (8)-(9) is well-posed :

For $(w_0, w_1) \in H_{\frac{1}{2}} \times H$, the problem (8)-(9) allows a unique solution :

$$w \in C([0, \infty); H_{\frac{1}{2}}) \cap C^1([0, \infty); H)$$

such that $B_1^* w(\cdot) \in H^1(0, T; U)$. Moreover, w satisfies the energy estimate, for all $t \geq 0$:

$$\|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2 - \|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}} \times H}^2 = 2 \int_0^t \left\| \frac{d}{dt} B_1^* w(s) \right\|_U^2 ds. \quad (10)$$

For (10) we remark that the mapping $t \mapsto \|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}} \times H}^2$ is non-increasing.

Consider the initial value problem:

$$\ddot{\varphi}(t) + A_1 \varphi(t) = 0, \quad (11)$$

$$\varphi(0) = w_0, \quad \dot{\varphi}(0) = w_1. \quad (12)$$

It is well known that (11)-(12) is well posed in $H_1 \times H_{\frac{1}{2}}$ and in $H_{\frac{1}{2}} \times H$. Now, we consider the unbounded linear operator

$$\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \rightarrow H_{\frac{1}{2}} \times H, \quad \mathcal{A}_d = \begin{pmatrix} 0 & I \\ -A_1 & -B_1 B_1^* \end{pmatrix}, \quad (13)$$

where

$$\mathcal{D}(\mathcal{A}_d) = \left\{ (u, v) \in H_{\frac{1}{2}} \times H, A_1 u + B_1 B_1^* v \in H, v \in H_{\frac{1}{2}} \right\}.$$

The result below, proved in [5], shows that, under a certain regularity assumption, the exponential stability of (8)-(9) is equivalent to a strong observability inequality for (11)-(12) and the polynomial stability of (8)-(9) is a consequence of weak observability inequality. More precisely, we have :

Theorem 4.1 (Ammari-Tucsnak [5]) *Assume that for any $\gamma > 0$ we have*

$$\sup_{\operatorname{Re} \lambda = \gamma} \|\lambda B_1^* (\lambda^2 I + A_1)^{-1} B_1\|_{\mathcal{L}(U)} < \infty. \quad (14)$$

Then

1. *there exists $C, \delta > 0$ such that for all $t > 0$ and for all $(w^0, w^1) \in H_{\frac{1}{2}} \times H$, we have*

$$\|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}} \times H} \leq C e^{-\delta t} \|(w^0, w^1)\|_{H_{\frac{1}{2}} \times H},$$

if and only if there exists $T, C > 0$ such that : $\forall (w_0, w_1) \in H_1 \times H_{\frac{1}{2}}$, we have

$$\|B_1^* \varphi'(t)\|_{L^2(0,T;U)} \geq C \|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}, \quad (15)$$

where $\varphi(t)$ is the solution of system (11)-(12).

2. *If there exists $T, C > 0$ and $\alpha > -\frac{1}{2}$ such that : $\forall (w_0, w_1) \in H_1 \times H_{\frac{1}{2}}$, we have*

$$\|B_1^* \varphi'(t)\|_{L^2(0,T;U)} \geq C \|(w_0, w_1)\|_{H_{-\alpha} \times H_{-\alpha-\frac{1}{2}}}, \quad (16)$$

where $\varphi(t)$ is the solution of (11)-(12).

Then there exists a constant $C_1 > 0$ such that for all $t > 0$ and for all $(w^0, w^1) \in \mathcal{D}(\mathcal{A}_d)$, we have

$$\|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}} \times H} \leq \frac{C_1}{(1+t)^{\frac{1}{4\alpha+2}}} \|(w^0, w^1)\|_{\mathcal{D}(\mathcal{A}_d)}. \quad (17)$$

5. Proof of the main results

5.1. PROOF OF THE FIRST ASSERTION OF THE THEOREM 2.1. For this proof, we need a result of the following lemma inspired by [19] (see also [7]).

Lemma 5.1 *Let $v \in E$ then $v \in S_T$ if and only if there exists a constant $C_v > 0$ such that for any initial data $\underline{u} \in E$, the solution u of problem (3) satisfies*

$$|\langle \underline{u}, v \rangle_E| \geq C_v \left\| \frac{\partial[G(\partial_t u)]}{\partial \nu}(\cdot, t) \right\|_{L^2(\Gamma_0 \times (0, T))}.$$

Then, let $\delta, \epsilon > 0$ and $v \in X_\epsilon$. We can put $v(x, y) = \sum_{n \in \mathbb{N}^*} e^{-\epsilon n} v^n(x) \sin ny$, with $(\|v^n\|_E)_n \in l^2(\mathbb{N}^*)$. Take $T(\epsilon, \delta) = \sup (T_1(\epsilon, \delta), 2\pi\sqrt{2})$. As $T_1(\epsilon, \delta) \leq \frac{C}{\epsilon^{1+\delta}}$, for small ϵ , then $T(\epsilon, \delta) \leq \frac{C}{\epsilon^{1+\delta}}$. For any $\underline{u} \in E$ and $\underline{v} = (0, v)$, we have

$$|\langle \underline{u}, \underline{v} \rangle_E| = \left| \sum_{n \in \mathbb{N}^*} \langle e^{-\epsilon n} \underline{u}^n, \underline{v}^n \sin ny \rangle_E \right|.$$

We deduce

$$|\langle \underline{u}, \underline{v} \rangle_E| \leq \sum_{n \in \mathbb{N}^*} e^{-\epsilon n} \|\underline{u}^n\|_E \|v^n \sin ny\|_E. \quad (18)$$

As $\forall n \in \mathbb{N}^*$, and $\forall \underline{u}^n \in E_n$, we have

$$\|\underline{u}\|_E^2 = \|\underline{u}\|_{E^1}^2 + \|\underline{u}\|_{E^2}^2.$$

According to Proposition 3.1 and Lemma 3.2

$$\begin{aligned} \|\underline{u}\|_E^2 &\leq C_{\epsilon, \delta} e^{2\epsilon n} \int_0^\pi \int_{-T_1(\epsilon, \delta)}^{T_1(\epsilon, \delta)} \left| \frac{\partial[G(u')]}{\partial x}(0, y, t) \right|^2 dt dy + \\ &C_T \int_0^\pi \int_0^{T(\epsilon, \delta)} \left| \frac{\partial[G(u')]}{\partial x}(0, y, t) \right|^2 dt dy. \end{aligned}$$

Then

$$\begin{aligned} \|\underline{u}\|_E^2 &\leq C'_{\epsilon, \delta} e^{2\epsilon n} \int_0^\pi \int_{-T(\epsilon, \delta)}^{T(\epsilon, \delta)} \left| \frac{\partial[G(u')]}{\partial x}(0, y, t) \right|^2 dy dt + \\ &C_T \int_0^\pi \int_0^{T(\epsilon, \delta)} \left| \frac{\partial[G(u')]}{\partial x}(0, y, t) \right|^2 dt dy + C_T \int_0^\pi \int_0^{T(\epsilon, \delta)} \left| \frac{\partial[G(u^1)]}{\partial x}(0, y, t) \right|^2 dt dy. \end{aligned}$$

Which implies

$$\begin{aligned} \|\underline{u}\|_E^2 &\leq C'_{\epsilon, \delta} e^{2\epsilon n} \int_0^\pi \int_{-T(\epsilon, \delta)}^{T(\epsilon, \delta)} \left| \frac{\partial[G(u')]}{\partial x}(0, y, t) \right|^2 dt dy + \\ &C_T \int_0^\pi \int_0^{T(\epsilon, \delta)} \left| \frac{\partial[G(u')]}{\partial x}(0, y, t) \right|^2 dt dy + C' \|\underline{u}\|_{E^1}^2. \end{aligned}$$

This inequality according to Proposition 3.1 implies

$$\|\underline{u}\|_E^2 \leq C''_{\epsilon, \delta} e^{2\epsilon n} \int_0^\pi \int_{-T(\epsilon, \delta)}^{T(\epsilon, \delta)} \left| \frac{\partial[G(u')]}{\partial x}(0, y, t) \right|^2 dt dy.$$

If we replace this estimate in (18), we obtain

$$\begin{aligned} |\langle \underline{u}, \underline{v} \rangle_E| &\leq C''_{\epsilon, \delta} \sum_{n \in \mathbb{N}^*} \|v^n(x) \sin ny\|_E \sqrt{\int_0^\pi \int_0^{2T(\epsilon, \delta)} \left| \frac{\partial[G(\partial u^n / \partial t)]}{\partial x}(0, y, t) \right|^2 dt dy} \\ &\leq C''_{\epsilon, \delta} \sqrt{\sum_{n \in \mathbb{N}^*} \int_0^\pi \int_0^{2T(\epsilon, \delta)} \left| \frac{\partial[G(\partial u^n / \partial t)]}{\partial x}(0, y, t) \right|^2 dt dy} \sqrt{\sum_{n \in \mathbb{N}^*} \|v^n(x) \sin ny\|_E^2} \\ &\leq C''_{\epsilon, \delta} C_v \left\| \frac{\partial[G(u^n)]}{\partial \nu} \right\|_{L^2(\Gamma_0 \times (0, 2T(\epsilon, \delta)))}. \end{aligned}$$

This implies that $v \in S_T$, i.e $X_\epsilon \subset S_T$, as well as $\alpha_S(T) \leq \frac{C_\delta}{T(1-\delta)}$.

□

5.2. PROOF OF THE SECOND ASSERTION OF THEOREM 2.1. For $\alpha > \alpha_S(T)$ we have $X_\alpha \subset S_T$. Then, there exists a constant $C > 0$ such that the solution ϕ of problem (3) satisfies

$$\int_0^T \int_0^\pi \left| \frac{\partial[G(\partial\phi/\partial t)]}{\partial x} \right| dy dt \geq C(T)E(0), \quad \forall (u^0, u^1) \in X_\alpha.$$

This inequality according to [5, Theorem 2.2] (see also Theorem 4.1) implies the existence of $C_\alpha, \delta_\alpha > 0$ such that

$$E(t) \leq C_\alpha e^{-\gamma_\alpha t} E(0), \quad \forall (u^0, u^1) \in X_\alpha, \forall t \geq 0.$$

5.3. PROOF OF THEOREM 2.2. Let $\underline{u} \in E$, we have

$$\underline{u} = \sum_{n,k \geq 1} a_{n,k} \varphi_{n,k},$$

where $(a_{n,k}) \in l^2$. Then

$$\partial_t \phi(x, y, t) = \sum_{n,k \in \mathbb{Z}^*} \frac{2a_{n,k}}{\pi} \sqrt{\lambda_{n,k}} \sin kx \sin ny e^{it\lambda_{n,k}}.$$

Due to the orthogonality of the family $(\sin ny)$ in $L^2(0, \pi)$, we get for $T > 2\pi\sqrt{2}$

$$\int_0^\pi \int_0^T \left| \frac{\partial[G(\partial_t \phi)]}{\partial x} (0, y, t) \right|^2 dt dy = \sum_{n \in \mathbb{Z}^*} \frac{1}{\pi^2} \int_0^T \left| \sum_{k \in \mathbb{Z}^*} \frac{ka_{n,k}}{\sqrt{\lambda_{n,k}}} e^{i\lambda_{n,k}t} dt \right|^2.$$

According to [22, Theorem 2.1], we obtain the existence of constant $C_2 > 0$ satisfies

$$\int_0^T \int_0^\pi \left| \frac{\partial[G(\partial_t \phi)]}{\partial x} (0, y, t) \right|^2 dy dt \geq C_2 \sum_{n,k \in \mathbb{Z}^*} \left| \frac{a_{n,k}}{\sqrt{\lambda_{n,k}}} \right|^2 \sim C_2 \|\underline{u}\|_{H^{-1}(\Omega) \times [H_0^1(\Omega)]'}^2,$$

where $[H_0^1(\Omega)]'$ is the dual space of $H_0^1(\Omega)$. The duality is respected to the pivot space $H^{-1}(\Omega)$.

Which shows Theorem 2.2 thanks to Theorem 4.1 for $\alpha = 0$.

References

1. Lions, J. L., *Exact Controllability, Stabilizability, and Perturbations for Distributed Systems*, Siam Rev. 30, 1-68, (1988).
2. C. M. Dafermos, C. M., *An abstract Volterra equation with application to linear viscoelasticity*. J. Differential Equations 7, 554-589, (1970).
3. B. Allibert, *Contrôle analytique de l'équation des ondes sur des surfaces de révolution*, Thèse de doctorat, Ecole Polytechnique, (1997).
4. B. Allibert, S. Micu, *Controllability of analytic functions for a wave equation coupled with a beam*, Revista Mathematica Iberoamericana 15, 547-592, (1999).

5. K. Ammari and M. Tucsnak, *Stabilization of second order evolution equations by a class of unbounded feedbacks*, ESAIM : Control, Optimisation and Calculus of Variations 6, 361-386, (2001).
6. K. Ammari, M. Tucsnak and G. Tenenbaum, *A sharp geometric condition for the boundary exponential stabilizability of a square plate by moment feedbacks only*, Internat. Series of Num. Math., Birkhäuser, Basel/Switzerland 155, 1-11, (2007).
7. K. Ammari and S. Nicaise, *Polynomial and analytic stabilization of a wave equation coupled with a Euler-Bernoulli Beam*, Math. Appl. Sci 5, 556-576, (2009).
8. K. Ammari and M. Khenissi, *Decay rates of plates*, Math. Nach. 14, 1647-1658, (2005).
9. K. Ammari and G. Vodev, *Boundary stabilization of the transmission problem for the Bernoulli-Euler plate equation*, Cubo. A Mathematical Journal 11, 39-49, (2009).
10. K. Ammari and M. Khenissi, *Stabilization of the transmission plate equation*, to appear.
11. J. M. Ball and M. Slemrod, *Nonharmonic Fourier series and the stabilization of semilinear control systems*, Comm. Pure and Appl. Math. 32, 555-587, (1979).
12. C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of the waves from the boundary*, SIAM J. Control Optim. 30, 1024-1065, (1992).
13. A. Haraux, *Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire*, J. Math. Pures. Appl. 68, 457-465, (1989).
14. A. Haraux, *Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps*, Port. Math. 46, 245-258, (1989).
15. W. Krabs, G. Leugering and T. Seidman, *On boundary controllability of a vibrating plate*, Appl. Math. Optim. 13, 205-229 (1985).
16. I. Laseicka and R. Triggiani, *Exact controllability and uniform stabilization of Euler-Bernoulli equations with boundary control only in $\Delta\omega/\Sigma$* , Boll. Un. Math. Ital. B, 7, 665-702, (1991).
17. G. Lebeau, *Contrôle de l'équation de Schrodinger*, Journal de Mathématiques Pures et Appliqués 71, 267-291, (1992).
18. G. Lebeau, *Contrôle analytique. I : Estimations à priori*, Duke Math. J. 68, 1-30, (1992).
19. G. Lebeau, *Control for hyperbolic equations*, Journées "Equations aux dérivées partielles" Saint Jean de Monts, Ecole Polytechnique, Palaiseau, (1992).
20. G. Leugering, *Boundary control of a vibrating plate*, in Optimal control of partial differential equations (Oberwolfach, 1982), vol. 68 of Internat. Schriftenreihe Numer. Math., Birkhäuser, Basel, 167-172, (1984).
21. J. L. Lions, *Contrôlabilité exacte perturbation et stabilisation des systèmes distribués*, RMA. 8-9 Masson, Paris, (1988).
22. M. Mehrenberger, *An Ingham type proof for the boundary observability of $N - d$ wave equation*, C. R. Acad. Sci. Paris, Ser. I 347, 63-68, (2009).
23. K. Ramadani, T. Takahashi, G. Tenenbaum and M. Tucsnak, *A spectral approach for the exact observability of infinite dimensional systems with skew-adjoint generator*, J. Funct. Anal. 226, 193-229, (2005).

Salsabil Noura
 Département de Mathématiques,
 Faculté de Sciences de Monastir, 5019 Monastir,
 Tunisia
 selsebila1@yahoo.fr