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On Lie Structure of Prime Rings with Generalized (α, β)-Derivations

Nadeem ur Rehman*, Radwan Mohammed AL-Omary and Claus Haetinger

ABSTRACT: Let R be a ring and α, β be automorphisms of R. An additive mapping $F\colon R \to R$ is called a generalized (α, β)-derivation on R if there exists an (α, β)-derivation $d\colon R \to R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. For any $x, y \in R$, set $[x, y]_{\alpha,\beta} = x\alpha(y) - \beta(y)x$ and $(x \circ y)_{\alpha,\beta} = x\alpha(y) + \beta(y)x$. In the present paper, we shall discuss the commutativity of a prime ring R admitting generalized (α, β)-derivations F and G satisfying any one of the following properties: (i) $F([x, y]) = (x \circ y)_{\alpha,\beta}$, (ii) $F(x \circ y) = [x, y]_{\alpha,\beta}$, (iii) $[F(x), y]_{\alpha,\beta} = (F(x) \circ y)_{\alpha,\beta}$, (iv) $F([x, y]) = [F(x), y]_{\alpha,\beta}$, (v) $F(x \circ y) = (F(x) \circ y)_{\alpha,\beta}$, (vi) $F([x, y] = [\alpha(x), G(y)]$ and (vii) $F(x \circ y) = (\alpha(x) \circ G(y))$ for all x, y in some appropriate subset of R. Finally, obtain some results on semi-projective Morita context with generalized (α, β)-derivations.

Key Words: : Lie ideals, prime rings, (α, β)-derivations and generalized (α, β)-derivations, Morita context, reduced rings.

Contents

1. Introduction

Let R be an associative ring with center $Z(R)$. For each $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$ and the symbol $x \circ y$ stands for the skew-commutator $xy + yx$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. A Lie ideal L is said to be a *square-closed* Lie ideal if $x^2 \in L$, for all $x \in L$. Let α and β be endomorphisms of R. For any $x, y \in R$, set $[x, y]_{\alpha,\beta} = x\alpha(y) - \beta(y)x$ and $(x \circ y)_{\alpha,\beta} = x\alpha(y) + \beta(y)x$.

An additive mapping $F\colon R \to R$ is called a *generalized derivation* associated with a derivation d if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Familiar examples of generalized derivations are derivations and generalized inner derivations, and the latter includes left multipliers. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$, where c is a fixed element of R and d a derivation of R, is a generalized

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derivation; and if R has multiplicative identity 1, then all generalized derivations have this form.

An additive map $d\colon R \to R$ is called an (α, β)-*derivation* if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. An $(1, 1)$-derivation is called simply a *derivation*, where 1 is the identity map on R. For a fixed a, the map $d_a\colon R \to R$ given by $d_a(x) = [a, x]_{\alpha,\beta}$ for all $x \in R$ is an (α, β)-derivation which is said to be an (α, β)-*inner derivation*. An additive mapping $F\colon R \to R$ is called a *generalized* (α, β)-*inner derivation* if $F(x) = a\alpha(x) + \beta(x)b$, for some fixed $a, b \in R$ and for all $x \in R$. A simple computation yields that if F is a generalized (α, β)-inner derivation, then for all $x, y \in R$, we have $F(xy) = F(x)\alpha(y) + \beta(x)d_{-b}(y)$, where d_{-b} is an (α, β)-inner derivation. With this viewpoint, an additive map $F\colon R \to R$ is called a *generalized* (α, β)-*derivation* associated with an (α, β)-derivation $d\colon R \to R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$.

An $(1, 1)$-generalized derivation is called simply a *generalized derivation*.

Over the last three decades, several authors have proved commutativity theorems for prime or semiprime rings admitting automorphisms or derivations which are centralizing or commuting on some appropriate subsets of R (see [3], [4], [5], [6], [10], [11], [19] and [21] where further references can be found). In 2002, Ashraf and Rehman [3] established that if a 2-torsion free prime ring R admits a derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in L$, where L is a Lie ideal of R, then $L \subseteq Z(R)$. Further, the first author [21] extended the mention results for generalized derivations. In the present paper our aim is to prove some results which are independent interest and related to generalized (α, β)-derivation on prime rings. In fact, our theorems extend some known theorems in the setting of generalized (α, β)-derivation in certain classes of rings. We refer the reader to [2] and [13] for an historical account and applications on derivations.

2. Preliminary results

We shall use without explicit mention the following basic identities, that hold for any $x, y, z \in R$:

- $[xy, z]_{\alpha,\beta} = x[y, z]_{\alpha,\beta} + [x, \beta(z)]y = x[y, \alpha(z)] + [x, z]_{\alpha,\beta}y$;
- $[x, yz]_{\alpha,\beta} = \beta(y)[x, z]_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z)$;
- $(x \circ (yz))_{\alpha,\beta} = (x \circ y)_{\alpha,\beta}\alpha(z) - \beta(y)[x, z]_{\alpha,\beta} = \beta(y)(x \circ z)_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z)$;
- $((xy) \circ z)_{\alpha,\beta} = x(y \circ z)_{\alpha,\beta} - [x, \beta(z)]y = (x \circ z)_{\alpha,\beta}y + x[y, \alpha(z)]$.

We begin our discussion with the following results.

Lemma 2.1 [[7], Lemma 4] *If $L \nsubseteq Z(R)$ is a Lie ideal of a 2-torsion-free prime ring R and $a, b \in R$ are such that $aLb = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2 [[21], Lemma 2.6] *Let R be a 2-torsion-free prime ring and L be a nonzero Lie ideal of R. If L is a commutative Lie ideal of R, then $L \subseteq Z(R)$.*

The next three lemmas are essentially proved in [19].

Lemma 2.3 *Let R be a prime ring with char$(R) \neq 2$, and let L be a nonzero square-closed Lie ideal of R. Let α, β be automorphisms of R. If $[x, y]_{\alpha,\beta} = 0$, for all $x, y \in L$, then $L \subseteq Z(R)$.*

Lemma 2.4 *Let R be a 2-torsion-free prime ring and L be a nonzero square-closed Lie ideal of R. Suppose there exists a nonzero (α, β)-derivation d such that $d(x) = 0$ for all $x \in L$. Then $L \subseteq Z(R)$.*

Lemma 2.5 *Let R be a 2-torsion-free prime ring and L be a nonzero square-closed Lie ideal of R. Suppose that α, β are automorphisms of R. If R admits a generalized (α, β)-derivation F with an associated nonzero (α, β)-derivation d such that $[F(x), x]_{\alpha,\beta} = 0$, for all $x \in L$, then $L \subseteq Z(R)$.*

3. Lie Ideals and Generalized (α, β)-Derivations

Theorem 3.1 *Let R be prime ring, char$(R) \neq 2$ and L be a nonzero square-closed Lie ideal of R. Let α, β be automorphisms of R, and suppose R admits a generalized (α, β)-derivation F associated with an (α, β)-derivation d such that*

(i), $F([x, y]) = (x \circ y)_{\alpha,\beta}$, for all $x, y \in L$, or

(ii), $F([x, y]) = -(x \circ y)_{\alpha,\beta}$, for all $x, y \in L$.

If $F = 0$ or $d \neq 0$, then $L \subseteq Z(R)$.

Proof: (i). Let F be a generalized (α, β)-derivation of R such that

$$F([x, y]) = (x \circ y)_{\alpha,\beta} \text{ for all } x, y \in L. \tag{1}$$

If $F = 0$, then $(x \circ y)_{\alpha,\beta} = 0$ for all $x, y \in L$. Replacing y by $2yz$ in the last expression, we get $\beta(y)[x, z]_{\alpha,\beta} = 0$ for all $x, y, z \in L$. For any $r \in R$, now replace y by $[y, r]$ to get $\beta([y, r])[x, z]_{\alpha,\beta} = 0$. Again replacing r by sr we find that $\beta([y, s])R[x, z]_{\alpha,\beta} = \{0\}$ for all $x, y, z \in L$ and $s \in R$. Thus, the primeness of R forces that either $\beta([y, s]) = 0$ or $[x, z]_{\alpha,\beta} = 0$. Hence, if $\beta([y, s]) = 0$ for all $y \in L$ and $s \in R$, then $[y, s] = 0$. This implies that $L \subseteq Z(R)$. On the other hand, if $[x, z]_{\alpha,\beta} = 0$ for all $x, z \in L$, then by Lemma 2.3, we get the required result.

Therefore, we shall assume that $d \neq 0$. Suppose on contrary that $L \not\subseteq Z(R)$. Now, replacing y by $2yx$ in (1) and using the fact that char$(R) \neq 2$, we obtain

$$[x, y]d(x) = -\beta(y)[x, x]_{\alpha,\beta} \text{ for all } x, y \in L. \tag{2}$$

Again replace y by $2zy$ in (2) to get

$$\beta([x, z])\beta(y)d(x) = 0 \text{ for all } x, y, z \in L. \tag{3}$$

This implies that, $[x, z]L\beta^{-1}(d(x)) = \{0\}$ for all $x, z \in L$; and applying Lemma 2.1 and the fact that $(L, +)$ is not the union of two proper subgroups shows that

either $d(L) = \{0\}$ or L is commutative. By Lemmas 2.2 and 2.4 either of theses conditions yields contradiction.
(ii). If $F([x, y]) = -(x \circ y)_{\alpha,\beta}$, then $-F$ satisfies the condition in part (i), hence our result follows. □

Theorem 3.2 *Let R be a prime ring with $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms of R, and suppose R admits a generalized (α, β)-derivation F with an associated (α, β)-derivation d such that*

(i), $F(x \circ y) = [x, y]_{\alpha,\beta}$ for all $x, y \in L$, or

(ii), $F(x \circ y) = -[x, y]_{\alpha,\beta}$ for all $x, y \in L$.

If $F = 0$ or $d \neq 0$, then $L \subseteq Z(R)$.

Proof: (i). It is given that F is a generalized derivation of R such that $F(x \circ y) = [x, y]_{\alpha,\beta}$ for all $x, y \in L$. If $F = 0$, then $[x, y]_{\alpha,\beta} = 0$, for all $x, y \in L$. Thus by Lemma 2.3, we get the required result.

Hence, onward we shall assume that $d \neq 0$. Suppose on contrary that $L \not\subseteq Z(R)$. For any $x, y \in L$, we have

$$F(x \circ y) = [x, y]_{\alpha,\beta} \tag{4}$$

Replacing y by $2yx$ in (4) and using the fact that $char(R) \neq 2$, we get

$$\beta(x \circ y)d(x) = \beta(y)[x, x]_{\alpha,\beta}. \tag{5}$$

Now, replace y by $2xy$ in (5) to get $\beta([x, z])\beta(y)d(x) = 0$ for all $x, y, z \in L$. The last expression is same as the equation (3) and hence the result follows.

(ii). Using the similar technique with necessary variations, the result follows. □

Theorem 3.3 *Let R be a prime ring with $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms, and suppose R admits a generalized (α, β)-derivation F with an associated (α, β)-derivation d such that*

(i), $[F(x), y]_{\alpha,\beta} = (F(x) \circ y)_{\alpha,\beta}$ for all $x, y \in L$, or

(ii), $[F(x), y]_{\alpha,\beta} = -(F(x) \circ y)_{\alpha,\beta}$ for all $x, y \in L$.

Then $L \subseteq Z(R)$.

Proof: (i). Suppose, on the contrary that $L \not\subseteq Z(R)$. For any $x, y \in L$ we have

$$[F(x), y]_{\alpha,\beta} = (F(x) \circ y)_{\alpha,\beta}. \tag{6}$$

Replacing y by $2yx$ in (6) and using the fact that $char(R) \neq 2$, we find that $\beta(y)[F(x), x]_{\alpha,\beta} = 0$ for all $x, y \in L$. This implies that,

$$\beta^{-1}([F(x), x]_{\alpha,\beta})L\beta^{-1}([F(x), x]_{\alpha,\beta}) = \{0\}$$

for all $x \in L$. Thus, by Lemmas 2.1 and 2.5, we get the required result.

(ii). Using similar arguments as above, the result follows. □

Theorem 3.4 *Let R be a prime ring, $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms, and suppose R admits a generalized (α, β)-derivation F with an associated (α, β)-derivation d such that*

(i), $F([x, y]) = [F(x), y]_{\alpha,\beta}$ for all $x, y \in L$, or

(ii), $F([x, y]) = -[F(x), y]_{\alpha,\beta}$ for all $x, y \in L$.

Then $L \subseteq Z(R)$.

Proof: (i). Suppose, on the contrary that $L \nsubseteq Z(R)$. For any $x, y \in L$ we have $F([x, y]) = [F(x), y]_{\alpha,\beta}$. This can be rewritten as

$$F(x)\alpha(y) + \beta(x)d(y) - F(y)\alpha(x) - \beta(y)d(x) = [F(x), y]_{\alpha,\beta}. \tag{7}$$

Replacing y by $2yx$ in (7), we get

$$\beta([x, y])d(x) = \beta(y)[F(x), x]_{\alpha,\beta}. \tag{8}$$

Again replace y by $2zy$ in (8) to get

$$\beta(z)\beta([x, y]d(x) + \beta([x, z])\beta(y)d(x) = \beta(z)\beta(y)[F(x), y]_{\alpha,\beta} \text{ for all } x, y, z \in L. \tag{9}$$

Combining (8) and (9), we find that $\beta([x, z])\beta(y)d(x) = 0$ for all $x, y, z \in L$, that is, $[x, z]L\beta^{-1}(d(x)) = \{0\}$ for all $x, y \in L$. Notice that the arguments given in the proof of Theorem 3.1 are still valid in the present situation and hence repeating the same process we get the required result.

(ii). Using the same technique as above, we get the required result. □

If the commutator is replaced by the anti-commutator in the Theorem 3.4, then we see that the conclusion of this theorem holds good.

Theorem 3.5 *Let R be a prime ring with $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms, and suppose R admits a generalized (α, β)-derivation F with an associated (α, β)-derivation d such that* *(i),* $F(x \circ y) = (F(x) \circ y)_{\alpha,\beta}$ *for all* $x, y \in L$*, or*

(ii), $F(x \circ y) = -(F(x) \circ y)_{\alpha,\beta}$ *for all* $x, y \in L$*.*

Then $L \subseteq Z(R)$*.*

Proof: (i). Suppose, on the contrary that $L \nsubseteq Z(R)$. For any $x, y \in L$ we have $F(x \circ y) = (F(x) \circ y)_{\alpha,\beta}$. This can be rewritten as

$$F(x)\alpha(y) + \beta(x)d(y) + F(y)\alpha(x) + \beta(y)d(x) = (F(x) \circ y)_{\alpha,\beta}. \tag{10}$$

Replacing y by $2yx$ in (10), we get

$$\beta(x \circ y)d(x) = -\beta(y)[F(x), x]_{\alpha,\beta}. \tag{11}$$

Again replace y by $2zy$ in (11) to get $\beta([x, z])\beta(y)d(x) = 0$ for all $x, y, z \in L$, that is, $[x, z]L\beta^{-1}(d(x)) = \{0\}$ for all $x, z \in L$. Now, application of similar arguments as used after (3) in Theorem 3.1 yields the required result.

(ii). By similar arguments as above, we get the required result. □

Theorem 3.6 *Let R be a prime ring with $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms, and suppose R admits generalized (α, β)-derivations F and G with associated (α, β)-derivations $d \neq 0$ and g respectively such that*

(i), $F([x, y]) = [\alpha(y), G(x)]$ *for all* $x, y \in L$*, or*

(ii), $F([x, y]) = -[\alpha(y), G(x)]$ *for all* $x, y \in L$*.*

If $G = 0$ or $g \neq 0$, then $L \subseteq Z(R)$.

Proof: (i). It is given that F and G are generalized (α, β)-derivations of R such that $F([x, y]) = [\alpha(y), G(x)]$. If $G = 0$, then $F([x, y]) = 0$ for all $x, y \in L$, and hence by Theorem 3.2 of [19] we get the required result.

Henceforth, we shall assume that $g \neq 0$. Suppose on the contrary that $L \nsubseteq Z(R)$. For any $x, y \in L$ we have

$$F([x, y]) = [\alpha(y), G(x)]. \tag{12}$$

Replacing y by $2yx$ in (12) and using the fact that $char(R) \neq 2$, we get $\beta([x, y])d(x) = \alpha(y)[\alpha(x), G(x)]$ for all $x, y \in L$. Again replace y by $2zy$ to get $\beta([x, z])\beta(y)d(x) = 0$, that is, $[x, z]L\beta^{-1}(d(x)) = \{0\}$. Thus for each $x \in L$, by Lemma 2.1, we find that either $[x, z] = 0$ or $\beta^{-1}(d(x)) = 0$. Now using similar arguments as used in the proof of Theorem 3.1, we get the required result.

(ii). Using similar arguments as above it follows. □

Theorem 3.7 *Let R be a prime ring with $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms, and suppose R admits generalized (α, β)-derivations F and G with associated (α, β)-derivations $d \neq 0$ and g respectively such that*

(i), $F(x \circ y) = (\alpha(y) \circ G(x))$ for all $x, y \in L$, or

(ii), $F(x \circ y) = -(\alpha(y) \circ G(x))$ for all $x, y \in L$.

If $G = 0$ or $g \neq 0$, then $L \subseteq Z(R)$.

Proof: (i). For any $x, y \in L$, we have

$$F(x \circ y) = (\alpha(y) \circ G(x)). \tag{13}$$

If $G = 0$, then $F(x \circ y) = 0$ for all $x, y \in L$ and hence by Theorem 3.5 of [19] we get the required result.

Therefore, we shall assume that $g \neq 0$. Suppose on contrary that $L \nsubseteq Z(R)$. Replace y by $2yx$ in (13) to get $\beta(x \circ y)d(x) = \alpha(y)[\alpha(x), G(x)]$ for all $x, y \in L$. Again replacing y by $2zy$, we find that $\beta([x, z])\beta(y)d(x) = 0$, and hence $[x, z]L\beta^{-1}(d(x)) = \{0\}$. Now an application of similar arguments as used after (3) in the proof of Theorem 3.1, yields the required result.

(ii). Follows. □

In view of these results we get the following corollaries:

Corollary 3.7A *Let R be a prime ring and I be a nonzero ideal of R. Suppose that α, β are automorphisms of R and R admits a generalized (α, β)-derivation F associated with an (α, β)-derivation d such that*

(i), $F([x, y]) = (x \circ y)_{\alpha,\beta}$ or $F([x, y]) = -(x \circ y)_{\alpha,\beta}$ for all $x, y \in I$,

(ii), $F(x \circ y) = [x, y]_{\alpha,\beta}$ or $F(x \circ y) = -[x, y]_{\alpha,\beta}$ for all $x, y \in I$,

(iii), $[F(x), y]_{\alpha,\beta} = (F(x) \circ y)_{\alpha,\beta}$ or $[F(x), y]_{\alpha,\beta} = -(F(x) \circ y)_{\alpha,\beta}$ for all $x, y \in I$,

(iv), $F([x, y]) = [F(x), y]_{\alpha,\beta}$ or $F([x, y]) = -[F(x), y]_{\alpha,\beta}$ for all $x, y \in I$,

(v), $F(x \circ y) = (F(x) \circ y)_{\alpha,\beta}$ or $F(x \circ y) = -(F(x) \circ y)_{\alpha,\beta}$ for all $x, y \in I$.

If $F = 0$ or $d \neq 0$, then R is commutative.

Corollary 3.7B *Let R be a prime ring and I be a nonzero ideal of R. Suppose that α, β are automorphisms of R and R admits generalized (α, β)-derivations F and G associated with (α, β)-derivations $d \neq 0$ and g respectively such that*

(i), $F([x, y]) = [\alpha(y), G(x)]$, or $F([x, y]) = -[\alpha(y), G(x)]$ for all $x, y \in I$,

(ii), $F(x \circ y) = (\alpha(y) \circ G(x))$, or $F(x \circ y) = -(\alpha(y) \circ G(x))$ for all $x, y \in I$.

If $G = 0$ or $g \neq 0$, then R is commutative.

4. Morita Contexts with Generalized (α, β)-Derivations

Throughout this section we assume that the datum $K(R,S) = \{R, S, M, N, \mu_R, \tau_R\}$ is said to be *Morita context* (or MC) in which R and S are rings, M and N are (S,R) and (R,S)-bimodules respectively, $\mu_R : N \otimes_S M \to R$ and $\tau_S : M \otimes_R N \to S$ are bimodule homomorphisms with associative conditions $m_1\mu_R(n \otimes m) = \tau_S(m_1 \otimes n)m$ and $\mu_R(n \otimes m)n_1 = n\tau_S(m_1 \otimes n_1)$, where μ_R and τ_S are called *Morita's maps* (or MC maps). If both Morita's maps are epimorphism then $K(R,S)$ is said to be a projective Morita context (or PMC). If one of the MC maps is an epimorphism, then $K(R,S)$ is said to be a semi-projective Morita context or semi-PMC.

We begin with some results due N.M. Muthana and S. K. Nauman which will be used extensively to prove our results.

Lemma 4.1 [[20], Theorem 2.1] *Let R and S be rings of semi-PMC $K(R,S)$ in which τ_S is epic. If R is commutative and S is reduced, then S is also commutative.*

Lemma 4.2 [[20], Corollary 2.4] *Let $K(R,S)$ be a PMC of rings in which R is commutative. Then*

(i). If S is a reduced ring, the R is also reduced and $R \cong S$.

(ii). If S is a domain, then both R and S become isomorphic integral domains.

(iii). If S is division rings, then both R and S an isomorphic field.

Theorem 4.1 *Let R be prime ring, I a nonzero ideal of R and $K(R,S)$ be a semi-PMC in which τ_S is epic. Let α, β be automorphisms, and suppose R admits a generalized (α, β)-derivation F with an associated (α, β)-derivation d such that $F([x,y]) = (x \circ y)_{\alpha,\beta}$ or $F([x,y]) = -(x \circ y)_{\alpha,\beta}$, for all $x, y \in I$. If S is reduced, then S is commutative. Moreover, if R and S are Morita similar rings, and S is a division ring, then R and S are isomorphic fields.*

Proof: By Corollary 3.7A (i), R is commutative. Since S is reduced so by Lemma 4.1 S is commutative.

Suppose R and S are Morita similar rings, then by Lemma 4.2 (i) R also reduce and $Z(R) = Z(S)$. Since R and S are commutative, so $R = Z(R)$ and $S = Z(S)$ and hence $R \cong S$. If S is division ring then S is a field. Since S is a commutative division ring. Thus, by Lemma 4.2 (iii) R and S become isomorphic fields. □

Using the same techniques with necessary variations, we can prove the following result.

Theorem 4.2 *Let R be prime ring, I a nonzero ideal of R and $K(R, S)$ be a semi-PMC in which τ_S is epic. Let α, β be automorphisms, and suppose R admits a generalized (α, β)-derivation F with an associated (α, β)-derivation d satisfying any one of the following properties:*

(i). $F(x \circ y) = [x, y]_{\alpha,\beta}$ *or* $F(x \circ y) = -[x, y]_{\alpha,\beta}$;

(ii). $[F(x), y]_{\alpha,\beta} = (F(x) \circ y)_{\alpha,\beta}$ *or* $[F(x), y]_{\alpha,\beta} = -(F(x) \circ y)_{\alpha,\beta}$;

(iii). $F([x, y]) = [F(x), y]_{\alpha,\beta}$ *or* $F([x, y]) = -[F(x), y]_{\alpha,\beta}$;

(iv). $F(x \circ y) = (F(x) \circ y)_{\alpha,\beta}$ *or* $F(x \circ y) = -(F(x) \circ y)_{\alpha,\beta}$.

If S is reduced, then S is commutative. Moreover, if R and S are Morita similar rings, and S is a division ring, then R and S are isomorphic fields.

Theorem 4.3 *Let R be prime ring, I a nonzero ideal of R and $K(R, S)$ be a semi-PMC in which τ_S is epic. Let α, β be automorphisms, and suppose R admits generalized (α, β)-derivations F and G with associated (α, β)-derivations $d \neq 0$ and g respectively satisfying any one of the following properties:*

(i). $F([x, y]) = [\alpha(y), G(x)]$, *or* $F([x, y]) = -[\alpha(y), G(x)]$ *for all* $x, y \in L$,

(ii). $F(x \circ y) = (\alpha(y) \circ G(x))$, *or* $F(x \circ y) = -(\alpha(y) \circ G(x))$ *for all* $x, y \in L$.

If S is reduced, then S is commutative. Moreover, if R and S are Morita similar rings, and S is a division ring, then R and S are isomorphic fields.

Proof: The proof can be followed from Corollary 3.7A and Lemmas 4.1 and 4.2. □

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Nadeem ur Rehman, Radwan Mohammed AL-Omary
Department of Mathematics
Aligarh Muslim University
202002, Aligarh, India
rehman100@gmail.com
email:radwan959@yahoo.com

and

Claus Haetinger
Center of Exact and Technological Sciences
Univates University Center
95900-000, Lajeado-RS, Brazil
haet@univates.br
http://ensino.univates.br/ chaet