Bol. Soc. Paran. de Mat. (3s.) **v. 27** 2 (2009): 43–59.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/spm doi:10.5269/bspm.v27i2.10210

Rapid pointwise stabilization of vibrating strings and beams

Alia BARHOUMI

ABSTRACT: Applying a general construction and using former results on the observability we prove, under rather general assumptions, a rapid pointwise stabilization of vibrating strings and beams.

Key Words: Stabilization, observability, systems theory, feedback control.

Contents

1. Introduction

This paper is devoted to the study of pointwise observability, controllability and uniform stabilization of vibrating systems. It was pointed out earlier by Haraux and Jaffard [6], [7], [8] that the observability and controllability properties depend heavily on the location of the observation or control point. For the stabilization another difficulty appears because the suitable function spaces, as we will show, are not Sobolev spaces. In order to establish satisfactory stabilization theorems we will introduce functions spaces depending on the arithmetical properties of the stabilization point. Working in this framework, we will be able to adapt a method developed by Komornik [11] to vibrating strings, beams and also to a coupled string-beam system; as a result we will construct pointwise feedbacks leading to arbitrarily large prescribed decay rates.

Many works were devoted to the construction of explicit feedback laws and to the proof of exponential decay by different methods; see, e.g., [1], [2], [3], [4], [10], [15], [14], [18]. It is known that this type of feedback does not yield arbitrarily large decay rates.

2000 *Mathematics Subject Classification:* 93D15, 93B07, 93B52

First we study a vibrating string with pointwise control, modeled by the following system:

$$\begin{cases} y_{tt} - y_{xx} = v(t)\delta_\xi & \text{in } \mathbb{R} \times (0,\pi), \\ y(t,0) = y(t,\pi) = 0 & \text{for } t \in \mathbb{R}, \\ y(0,x) = y_0(x) \text{ and } y_t(0,x) = y_1(x) & \text{for } x \in (0,\pi) \end{cases} \tag{1.1}$$

where δ_ξ denotes the Dirac mass at some given point $\xi \in (0,\pi)$ and $v(t)$ is a control function in $L^2_{loc}(\mathbb{R})$.

In order to formulate our result, we assume that ξ/π is irrational, so that $\sin k\xi$, doesn't vanish for any $k = 1, 2, \ldots$, we denote by Z the linear hull of the functions $w_k(x) := \sqrt{2/\pi}\sin kx,\ k = 1, 2, \ldots$, and we denote by D^α_ξ and $(D^\alpha_\xi)'$ for every $\alpha \in \mathbb{R}$ the Hilbert spaces obtained by completing Z with respect to norms given by the following formulae:

$$\left\|\sum a_k w_k\right\|^2_{D^\alpha_\xi} := \sum k^{2\alpha}\sin^2(k\xi)|a_k|^2,$$

$$\left\|\sum a_k w_k\right\|^2_{(D^\alpha_\xi)'} := \sum k^{-2\alpha}\sin^{-2}(k\xi)|a_k|^2.$$

If we identify $L^2(0,\pi)$ with its dual and take into account that

$$\|\sum a_k w_k\|^2_{L^2(0,\pi)} := \sum |a_k|^2,$$

then $(D^\alpha_\xi)'$ is the dual space of D^α_ξ.

Now we may state our first result:

Theorem 1.1 *Fix $\xi \in (0,\pi)$ such that ξ/π is irrational and introduce the Hilbert space $\mathcal{H}_\xi := (D^{-1}_\xi)' \times (D^0_\xi)'$. Given an arbitrarily large positive number ω, there exist two linear operators*

$$(P,Q) : \mathcal{H}_\xi \to D^{-1}_\xi$$

and a positive constant M such that setting

$$v(t) := (Py_t + Qy)(t,\xi)$$

the problem (1.1) is well posed in $\mathcal{H}_\xi$ and its solutions satisfy the inequality

$$\|(y,y_t)\|_{\mathcal{H}_\xi} \le Me^{-\omega t}\|(y_0,y_1)\|_{\mathcal{H}_\xi} \tag{1.2}$$

for all $(y_0,y_1) \in \mathcal{H}_\xi$ and $t \ge 0$.

Next we consider a beam model:

$$\begin{cases} y_{tt} + y_{xxxx} = v(t)\delta_\eta & \text{in } \mathbb{R} \times (0,\pi), \\ y(t,0) = y(t,\pi) = y_{xx}(t,0) = y_{xx}(t,\pi) = 0 & \text{for } t \in \mathbb{R}, \\ y(0,x) = y_0(x) \text{ and } y_t(0,x) = y_1(x) & \text{for } x \in (0,\pi). \end{cases} \tag{1.3}$$

Using the same notation as above, we will prove the following stabilization theorem:

Theorem 1.2 *Fix $\eta \in (0,\pi)$ such that η/π is irrational and introduce the Hilbert space $\mathcal{H}_\eta := (D_\eta^{-2})' \times (D_\eta^0)'$. Given an arbitrarily large positive number ω, there exist two linear operators*

$$(P,Q) : \mathcal{H}_\eta \to D_\eta^{-2}$$

and a positive constant M such that setting

$$v(t) := (Py_t + Qy)(t,\eta)$$

the problem (1.3) is well posed in $\mathcal{H}_\eta$ and its solutions satisfy the inequality

$$\|(y,y_t)\|_{\mathcal{H}_\eta} \le Me^{-\omega t}\|(y_0,y_1)\|_{\mathcal{H}_\eta} \tag{1.4}$$

for all $(y_0,y_1) \in \mathcal{H}_\eta$ and $t \ge 0$.

Finally we consider the coupled string-beam system

$$\begin{cases} y_{1,tt} - y_{1,xx} + Ay_1 + Cy_2 = v_1(t)\delta_\xi & \text{in } \mathbb{R} \times (0,\pi), \\ y_{2,tt} + y_{2,xxxx} + By_1 + Dy_2 = v_2(t)\delta_\eta & \text{in } \mathbb{R} \times (0,\pi), \\ y_1(t,0) = y_1(t,\pi) = 0 & \text{for } t \in \mathbb{R}, \\ y_2(t,0) = y_2(t,\pi) = 0 & \text{for } t \in \mathbb{R}, \\ y_{2,xx}(t,0) = y_{2,xx}(t,\pi) = 0 & \text{for } t \in \mathbb{R}, \\ y_1(0,x) = y_{10}(x) \text{ and } y_{1,t}(0,x) = y_{11}(x) & \text{for } x \in (0,\pi), \\ y_2(0,x) = y_{20}(x) \text{ and } y_{2,t}(0,x) = y_{21}(x) & \text{for } x \in (0,\pi) \end{cases} \tag{1.5}$$

where A, B, C, D are given real numbers, $\xi, \eta \in (0,\pi)$ are given points and $v_1(t), v_2(t)$ are the control functions. The following theorem holds:

Theorem 1.3 *Fix $\xi, \eta \in (0,\pi)$ such that ξ/π and η/π are irrational and introduce the Hilbert space*

$$\mathcal{H}_{\xi,\eta} := (D_\xi^0 \times D_\xi^{-1})' \times (D_\eta^0 \times D_\eta^{-2})'.$$

For almost all choices of $(A,B,C,D) \in \mathbb{R}^4$ and for every positive number ω there exist two linear operators

$$(P_1,Q_1,P_2,Q_2) : \mathcal{H}_{\xi,\eta} \to D_\xi^{-1} \times D_\eta^{-2},$$

and a positive constant M such that setting

$$(v_1,v_2)(t) := ((P_1y_{1t} + Q_1y_1)(t,\xi), (P_2y_{2t} + Q_2y_2)(t,\eta)),$$

the problem (1.5) is well posed in $\mathcal{H}_{\xi,\eta}$ and its solutions satisfy the inequality

$$\|(y_1,y_{1t},y_2,y_{2t})\|_{\mathcal{H}_{\xi,\eta}} \le Me^{-\omega t}\|(y_{10},y_{11},y_{20},y_{21})\|_{\mathcal{H}_{\xi,\eta}} \tag{1.6}$$

for all $(y_{10},y_{11},y_{20},y_{21}) \in \mathcal{H}_{\xi,\eta}$ and $t \ge 0$.

Remark 1.4 *It follows from some results of Komornik and Loreti that the system (1.5) can not be exactly controllable for some exceptional choices of the parameters A, B, C, D: see [12] and [13] for explicit counter examples concerning an equivalent observability problem.*

Since the Hilbert spaces in the above three theorems were defined in an abstract way, let us mention the following inclusions between these spaces and usual Sobolev spaces:

Proposition 1.5
(a) If ξ/π is irrational, then

$$\begin{aligned}
&(D_\xi^0)' \subset L^2(0,\pi),\\
&(D_\xi^{-1})' \subset H_0^1(0,\pi),\\
&(D_\xi^{-2})' \subset H^2(0,\pi)\cap H_0^1(0,\pi).
\end{aligned}$$

(b) Furthermore, for almost every irrational ξ/π, we have for every $\varepsilon>0$ the following inclusions:

$$\begin{aligned}
&H_0^{1+\varepsilon}(0,\pi) \subset (D_\xi^0)',\\
&H^{2+\varepsilon}(0,\pi)\cap H_0^1(0,\pi) \subset (D_\xi^{-1})',\\
&\{z\in H^{3+\varepsilon}(0,\pi)\cap H_0^1(0,\pi) \ :\ z_{xx}\in H_0^1(0,\pi)\} \subset (D_\xi^{-2})'.
\end{aligned}$$

(c) Moreover, if ξ/π is a quadratic irrational number, then

$$\begin{aligned}
&H_0^{1}(0,\pi) \subset (D_\xi^0)',\\
&H^{2}(0,\pi)\cap H_0^1(0,\pi) \subset (D_\xi^{-1})',\\
&\{z\in H^{3}(0,\pi)\cap H_0^1(0,\pi) \ :\ z_{xx}\in H_0^1(0,\pi)\} \subset (D_\xi^{-2})'.
\end{aligned}$$

The plan of the paper is the following. In the next section we recall some results concerning a method analogous to HUM developed in [11] which useful for the proof of our results. The following three sections contain the proofs of Theorems 1.1, 1.2 and 1.3, respectively. In Section 6 we also state and prove two new observability and controllability theorems. Proposition 1.5 is proved in the last section 7.

2. Review of some abstract results

We consider the abstract observability problem

$$U' = \mathcal{A}U, \quad U(0)=U_0, \quad \psi=\mathcal{B}U \tag{2.1}$$

in a complex Hilbert space $\mathcal{H}$, where $\mathcal{A}$ is a (bounded or unbounded) linear operator defined on some linear subspace $D(\mathcal{A})$ of $\mathcal{H}$, with values in $\mathcal{H}$ and $\mathcal{B}$ is a linear operator, defined on some linear subspace $D(\mathcal{B})$ of $\mathcal{H}$ with values in another Hilbert space $\mathcal{G}$. We make the following four assumptions:

(H1) The operator $\mathcal{A}$ generates a *strongly continuous group* of automorphisms $e^{t\mathcal{A}}$ in $\mathcal{H}$.

(H2) $D(\mathcal{A}) \subset D(\mathcal{B})$, and there exists a constant c such that $\|\mathcal{B}U_0\|_{\mathcal{G}} \le c\|\mathcal{A}U_0\|_{\mathcal{H}}$ for all $U_0 \in D(\mathcal{A})$.

(H3) There exist a non degenerate bounded interval I and a constant c_I such that the solutions of (2.1) satisfy the inequality

$$\|\mathcal{B}U\|_{L^2(I;\mathcal{G})} \le c_I\|U_0\|_{\mathcal{H}}$$

for all $U_0 \in D(\mathcal{A})$.

(H4) There exists a bounded interval I' and a positive number c' such that the solutions of (2.1) satisfy the inequality

$$\|U_0\|_{\mathcal{H}} \le c'\|\mathcal{B}U\|_{L^2(I';\mathcal{G})}$$

for all $U_0 \in D(\mathcal{A})$.

Fix two numbers $T > |I'|$, $\omega > 0$, set $T_\omega = T + (2\omega)^{-1}$, define

$$e_\omega(s) = \begin{cases} e^{-2\omega s} & \text{if } 0 \le s \le T, \\ 2\omega e^{-2\omega T}(T_\omega - s) & \text{if } T \le s \le T_\omega, \end{cases}$$

and set

$$\langle \Lambda_\omega U_0, \tilde{U}_0\rangle_{\mathcal{H}',\mathcal{H}} := \int_0^{T_\omega} e_\omega(s)(\mathcal{B}e^{s\mathcal{A}}U_0, \mathcal{B}e^{s\mathcal{A}}\tilde{U}_0)_{\mathcal{G}}\ ds.$$

Then Λ_ω is a self-adjoint, positive definite isomorphism $\Lambda_\omega \in L(\mathcal{H}, \mathcal{H})')$. Let us denote by $J : \mathcal{G} \to \mathcal{G}'$ the canonical Riesz anti-isomorphism.

The following result is a special case of a theorem obtained in [11].

Theorem 2.1 *Assume (H1)-(H4) and fix $\omega > 0$ arbitrarily. Then the problem*

$$v' = (-\mathcal{A}^* - \mathcal{B}^* J\mathcal{B}\Lambda_\omega^{-1})v, \qquad v(0) = v_0, \tag{2.2}$$

is well-posed in $\mathcal{H}'$. Furthermore, there exists a constant M such that the solutions of (2.2) satisfy the estimates

$$\|v(t)\|_{\mathcal{H}'} \le M\|v_0\|_{\mathcal{H}'}e^{-\omega t} \tag{2.3}$$

for all $v_0 \in \mathcal{H}'$ and for all $t \ge 0$.

In other words, this theorem asserts that the *feedback law*

$$W = -J\mathcal{B}\Lambda_\omega^{-1}v$$

uniformly stabilizes the control problem

$$v' = -\mathcal{A}^*v + \mathcal{B}^*W, \quad v(0) = v_0$$

with a decay rate at least equal to ω.

The well-posedness means here that (2.2) has a unique solution $v \in C(\mathbb{R}; \mathcal{H}')$ for every $v_0 \in \mathcal{H}'$.

3. Proof of Theorem 1.1

We consider the following system:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \pi), \\ u(t, 0) = u(t, \pi) = 0 & \text{for } t \in \mathbb{R}, \\ u(0, x) = u_0(x) \text{ and } u_t(0, x) = u_1(x) & \text{for } x \in (0, \pi), \\ \psi(t) = u(t, \xi) & \text{for } t \in \mathbb{R}. \end{cases} \tag{3.1}$$

If the initial data are given by the formulae

$$u_0(x) = \sum_{k=1}^{\infty} a_k \sin kx \quad \text{and} \quad u_1(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

with only finitely many nonvanishing coefficients a_k and b_k, then a simple computation shows that the solution is given by the formula

$$u(t, x) = \sum_{k=1}^{\infty} (a_k \cos kt + \frac{b_k}{k} \sin kt) \sin kx.$$

In the sequel we write $A \asymp B$ if there exist two positive constants c_1, c_2 satisfying $c_1 A \leq B \leq c_2 A$. The constants are assumed to be independent of the particular choice of the initial data or of the parameter k.

If $T \geq 2\pi$, then using Parseval's equality it follows that

$$\int_0^T |u(t, \xi)|^2 dt \asymp \sum_{k=1}^{\infty} \left(|a_k|^2 + k^{-2}|b_k|^2\right) \sin^2 k\xi.$$

It can be rewritten in the form

$$\int_0^T |u(t, \xi)|^2 dt \asymp \|u_0\|^2_{D^0_\xi} + \|u_1\|^2_{D^{-1}_\xi}. \tag{3.2}$$

We rewrite (3.1) as a first-order system

$$U' = AU, \quad U(0) = U_0, \quad \psi = BU$$

in the usual way, by setting

$$U := (u, u'), \quad U_0 := (u_0, u_1), \quad \mathcal{A}(u, v) := (v, \Delta u) \quad \text{and} \quad \mathcal{B}(u, v) := u(\xi).$$

We are going to verify the hypotheses (H1)-(H4) of Theorem 2.1 for the Hilbert spaces $\mathcal{H} := D^0_\xi \times D^{-1}_\xi$, $\mathcal{G} := \mathbb{R}$ if we define the domain of definition of the linear operators $\mathcal{A}$ and $\mathcal{B}$ by

$$D(\mathcal{A}) = D(\mathcal{B}) := D^1_\xi \times D^0_\xi.$$

Proof. In order to obtain hypothesis $(H1)$, we show that if $(u_0, u_1) \in \mathcal{H}$ then $(u(s), u_t(s)) \in \mathcal{H}$ for all $s \in \mathbb{R}$. This follows from Parseval's equality:

$$\|(u(s), u_t(s))\|^2_{\mathcal{H}} = \int_s^{s+T} |u(t,\xi)|^2 \asymp \sum_{k \in \mathbb{Z}^*} (|a_k|^2 + |\frac{b_k}{k}|^2) \sin^2 k\xi$$

and from the observation that the last expression does not depend on $s \in \mathbb{R}$.

It follow by a straightforward computation that for all $(u, v) \in D^1_\xi \times D^0_\xi$ we have the estimate

$$|u(\xi)| \le c \left(\|v\|_{D^0_\xi} + \|u_{xx}\|_{D^{-1}_\xi} \right)$$

then the hypothesis $(H2)$ is satisfied.

In deed, using the norms of D^0_ξ and D^{-1}_ξ and writing

$$u(x) = \sum_{k=1}^{\infty} a_k \sin kx$$

we have

$$u_{xx}(x) = \sum_{k=1}^{\infty} -k^2 a_k \sin kx$$

and therefore, using the Cauchy-Schwarz inequality at the first step,

$$\begin{aligned}
|u(\xi)|^2 &\le \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right) \cdot \sum_{k=1}^{\infty} k^2 |a_k|^2 \sin^2(k\xi) \\
&= \frac{\pi^2}{6} \sum_{k=1}^{\infty} k^2 |a_k|^2 \sin^2(k\xi) \\
&= \frac{\pi^2}{6} \|u_{xx}\|^2_{D^{-1}_\xi}.
\end{aligned}$$

It follow from the equality (3.2) that we have (H3)-(H4).

We may now apply Theorem 2.1 and we have (1.2). In order to write down explicitly the stabilization result, we multiply the equation (1.1) by u and we integrate by parts as follows (we use all conditions in (1.1) and (3.1)):

$$\begin{aligned}
0 &= \int_0^T \int_0^\pi (y_{tt} - y_{xx} - v(t)\delta_\xi) u \; dx \; dt \\
&= \left[\int_0^\pi y_t u - y u_t \; dx \right]_0^T + \int_0^T \int_0^\pi y(u_{tt} - u_{xx}) \; dx \; dt \\
&\qquad - \int_0^T v(t) u(t,\xi) \; dt \\
&= \left[\int_0^\pi y_t u - y u_t \; dx \right]_0^T - \int_0^T v(t) u(t,\xi) \; dt.
\end{aligned}$$

This shows that if we write (3.1) in the form (2.1), then its dual (2.2) corresponds to (1.1). Furthermore, writing the operator

$$\Lambda_\omega^{-1} : (D_\xi^0)' \times (D_\xi^{-1})' \to D_\xi^{-1} \times D_\xi^0$$

in the matrix form

$$\Lambda_\omega^{-1} = \begin{pmatrix} -P & Q \\ R & -S \end{pmatrix},$$

we have

$$v(t) = -(Py_t + Qy)(t, \xi). \tag{3.3}$$

4. Proof of Theorem 1.2

We consider the following system:

$$\begin{cases} u_{tt} + u_{xxxx} = 0 & \text{in } \mathbb{R} \times (0, \pi), \\ u(t,0) = u(t,\pi) = \Delta u(t,0) = \Delta u(t,\pi) = 0 & \text{for } t \in \mathbb{R}, \\ u(0,x) = u_0(x) \text{ and } u_t(0,x) = u_1(x) & \text{for } x \in (0,\pi), \\ \psi(t) = u(t,\eta) & \text{for } t \in \mathbb{R} \end{cases} \tag{4.1}$$

where $\Delta u = u_{xx}$. If the initial data are given by the formulae

$$u_0(x) = \sum_{k=1}^{\infty} a_k \sin kx \quad \text{and} \quad u_1(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

with only finitely many nonvanishing coefficients a_k and b_k, then a simple computation shows that the solution is given by the formula

$$u(t,x) = \sum_{k=1}^{\infty} (a_k \cos k^2 t + \frac{b_k}{k^2} \sin k^2 t) \sin kx.$$

If $T > 0$, then using a result of Haraux in [7] it follows that

$$\int_0^T |u(t,\eta)|^2 dt \asymp \sum_{k=1}^{\infty} \left(|a_k|^2 + k^{-4}|b_k|^2\right) \sin^2 k\eta.$$

It can be rewritten in the form

$$\int_0^T |u(t,\eta)|^2 dt \asymp \|u_0\|^2_{D_\eta^0} + \|u_1\|^2_{D_\eta^{-2}}. \tag{4.2}$$

We rewrite (4.1) as a first-order system

$$U' = AU, \quad U(0) = U_0, \quad \psi = BU$$

in the usual way, by setting

$$U := (u, u'), \quad U_0 := (u_0, u_1), \quad \mathcal{A}(u,v) := (v, -\Delta^2 u) \quad \text{and} \quad \mathcal{B}(u,v) := u(\eta)$$

where $\Delta^2 u = u_{xxxx}$.

Furthermore, we introduce the Hilbert spaces $\mathcal{H} := D^0_\eta \times D^{-2}_\eta$, $\mathcal{G} := \mathbb{R}$ and we define the domain of definition of the linear operators $\mathcal{A}$ and $\mathcal{B}$ by

$$D(\mathcal{A}) = D(\mathcal{B}) := D^2_\eta \times D^0_\eta.$$

Let us show that hypotheses (H1)-(H4) of Theorem 2.1 are satisfied.

Proof. We show that if $(u_0, u_1) \in \mathcal{H}$ then $(u(s), u_t(s)) \in \mathcal{H}$ for all $s \in \mathbb{R}$. This follows by applying a result of Haraux in [7] as follows:

$$\|(u(s), u_t(s))\|^2_{\mathcal{H}} = \int_s^{s+T} |u(t,\eta)|^2 \asymp \sum_{k\in\mathbb{Z}^*} (|a_k|^2 + |\frac{b_k}{k^2}|^2) \sin^2 k\eta$$

and by observing that the last expression does not depend on $s \in \mathbb{R}$. Then hypothesis $(H1)$ is verified.

In order to obtain $(H2)$ it suffices to establish for all $(u, v) \in D^0_\eta \times D^{-2}_\eta$ the estimate

$$|u(\eta)| \le c \left(\|v\|_{D^0_\eta} + \|u_{xx}\|_{D^{-2}_\eta} \right).$$

This follows by a straightforward computation, using the norms of D^0_η and D^{-2}_η: writing

$$u(x) = \sum_{k=1}^\infty a_k \sin kx$$

we have

$$u_{xxxx}(x) = \sum_{k=1}^\infty k^4 a_k \sin kx$$

and therefore, using the Cauchy-Schwarz inequality at the first step,

$$\begin{aligned} |u(\xi)|^2 &\le \left(\sum_{k=1}^\infty \frac{1}{k^4} \right) \cdot \sum_{k=1}^\infty k^4 |a_k|^2 \sin^2(k\eta) \\ &= \frac{\pi^4}{90} \sum_{k=1}^\infty k^4 |a_k|^2 \sin^2(k\eta) \\ &= \frac{\pi^4}{90} \|u_{xxxx}\|^2_{D^{-2}_\eta}. \end{aligned}$$

It follow from the equality (4.2) that hypotheses (H3)-(H4) are verified .

We may now apply Theorem 2.1 and we obtain (1.4). In order to write down explicitly the stabilization result, we multiply the equation (1.3) by u and we integrate by parts as follows (we use all conditions in (1.3) and (4.1)). This shows that if we write (4.1) in the form (2.1), then its dual (2.2) corresponds to (1.3). Furthermore, writing the operator

$$\Lambda^{-1}_\omega : (D^0_\eta)' \times (D^{-2}_\eta)' \to D^{-2}_\eta \times D^0_\eta$$

in the matrix form

$$\Lambda_\omega^{-1} = \begin{pmatrix} -P & Q \\ R & -S \end{pmatrix},$$

we have

$$v(t) = -(Py_t + Qy)(t, \eta).$$

5. Proof of Theorem 1.3

We consider the following system:

$$\begin{cases} u_{1tt} - u_{1xx} + Au_1 + Cu_2 = 0 & \text{in } \mathbb{R} \times (0, \pi), \\ u_{2tt} + u_{2xxxx} + Bu_1 + Du_2 = 0 & \text{in } \mathbb{R} \times (0, \pi), \\ u_1(t, 0) = u_1(t, \pi) = 0 & \text{for } t \in \mathbb{R}, \\ u_2(t, 0) = u_2(t, \pi) = 0 & \text{for } t \in \mathbb{R}, \\ u_{2xx}(t, 0) = u_{2xx}(t, \pi) = 0 & \text{for } t \in \mathbb{R}, \\ u_1(0, x) = u_{10}(x) \text{ and } u_{1t}(0, x) = u_{11}(x) & \text{for } x \in (0, \pi), \\ u_2(0, x) = u_{20}(x) \text{ and } u_{2t}(0, x) = u_{21}(x) & \text{for } x \in (0, \pi), \\ \psi(t) = u_1(t, \xi) + u_2(t, \xi) & \text{for } t \in \mathbb{R}. \end{cases} \tag{5.1}$$

If the initial data are given by the formula

$$u_{10}(x) = \sum_{k=1}^{\infty} a_k \sin kx, \qquad u_{11}(x) = \sum_{k=1}^{\infty} b_k \sin kx,$$

and

$$u_{20}(x) = \sum_{k=1}^{\infty} \alpha_k \sin kx, \qquad u_{21}(x) = \sum_{k=1}^{\infty} \beta_k \sin kx$$

with only finitely many non vanishing coefficients a_k b_k, α_k β_k, then a simple computation shows that

$$u_1(t, x) = \sum_{k=1}^{\infty} (c_k e^{ikt} + c_{-k} e^{-ikt}) \sin kx$$

and

$$u_2(t, x) = \sum_{k=1}^{\infty} (d_k e^{ik^2 t} + d_{-k} e^{-ik^2 t}) \sin kx$$

with

$$c_k = \frac{1}{2}(a_k - i\frac{b_k}{k}), \qquad c_{-k} = \frac{1}{2}(a_k + i\frac{b_k}{k}),$$

and

$$d_k = \frac{1}{2}(\alpha_k - i\frac{\beta_k}{k^2}), \qquad\qquad d_{-k} = \frac{1}{2}(\alpha_k + i\frac{\beta_k}{k^2}).$$

If $T > 2\pi$, then using Parseval's equality and a result of Haraux [7] it follows that

$$\int_0^T (|u_1(t,\xi)|^2 + |u_2(t,\eta)|^2)dt \asymp$$

$$\sum_{k=1}^{\infty} \left(|a_k|^2 + k^{-2}|b_k|^2\right)\sin^2 k\xi + \left(|\alpha_k|^2 + k^{-4}|\beta_k|^2\right)\sin^2 k\eta.$$

It can be rewritten in the form

$$\int_0^T |u_1(t,\xi)|^2 + |u_2(t,\eta)|^2 dt \asymp \|u_{10}\|^2_{D^0_\xi} + \|u_{20}\|^2_{D^0_\eta} + \|u_{11}\|_{(D^{-1}_\xi)} + \|u_{21}\|^2_{D^{-2}_\eta}. \quad (5.2)$$

We rewrite (5.1) as a first-order system

$$U' = AU, \quad U(0) = U_0, \quad \psi = BU$$

by setting

$$U := (u_1, u_2, u_{1t}, u_{2t}),$$
$$U_0 := (u_{10}, u_{20}, u_{11}, u_{21}),$$
$$\mathcal{A}(u_1, u_2, v_1, v_2) := (v_1, v_2, \Delta u_1 - Au_1 - Cu_2, -\Delta^2 u_2 - Bu_1 - Du_2)$$

and

$$\mathcal{B}(u_1, u_2, v_1, v_2) := (u_1(\xi), u_2(\eta)).$$

We introduce the Hilbert spaces $\mathcal{H} := D^0_\xi \times D^0_\eta \times D^{-1}_\xi \times D^{-2}_\eta$, $\mathcal{G} := \mathbb{R}^2$ and we define the domain of definition of the linear operators $\mathcal{A}$ and $\mathcal{B}$ by

$$D(\mathcal{A}) = D(\mathcal{B}) = D^1_\xi \times D^2_\eta \times D^0_\xi \times D^0_\eta.$$

We are going to check the validity of hypotheses (H1)-(H4) of Theorem 2.1.

Proof. We have to show that if

$$(u_{10}, u_{20}, u_{11}, u_{21}) \in \mathcal{H},$$

then

$$(u_1(s), u_2(s), u_{1t}(s), u_{2t}(s)) \in \mathcal{H}$$

for all $s \in \mathbb{R}$. This follows by applying Parseval's inequality and Haraux's result as in the preceding cases:

$$\|(u_1, u_2, u_{1t}, u_{2t})(s)\|_{\mathcal{H}}^2 = \int_s^{s+T} |u_1(t,\xi)|^2 + |u_2(t,\eta)|^2 dt \\ \asymp \sum_{k \in \mathbb{Z}^*} |c_k|^2 \sin^2 k\xi + |d_k|^2 \sin^2 k\eta$$

and by observing that the last expression does not depend on $s \in \mathbb{R}$. Then we have $(H1)$.

In order to verify hypothesis $(H2)$ it suffices to establish the estimate

$$|u_1(\xi)|^2 + |u_2(\eta)|^2 \le c\|\mathcal{A}(u_1, u_2, v_1, v_2)\|_{\mathcal{H}}^2$$

for all $(u_1, u_2, v_1, v_2) \in \mathcal{H}$. Writing

$$u_1(x) = \sum_{k=1}^{\infty} A_k \sin kx \quad \text{and} \quad u_2(x) = \sum_{k=1}^{\infty} B_k \sin kx$$

we have

$$\begin{aligned} |u_1(\xi)|^2 + |u_2(\eta)|^2 &= \left|\sum_{k=1}^{\infty} A_k \sin k\xi\right|^2 + \left|\sum_{k=1}^{\infty} B_k \sin(k\eta)\right|^2 \\ &\le \left(\sum_{k}^{\infty} \frac{1}{k^2}\right) \sum_{k=1}^{\infty} k^2 |A_k|^2 \sin^2 k\xi \\ &\qquad + \left(\sum_{k}^{\infty} \frac{1}{k^4}\right) \sum_{k=1}^{\infty} k^4 |B_k|^2 \sin^2 k\eta \\ &\asymp \sum_{k}^{\infty} k^2 |A_k|^2 \sin^2 k\xi + k^4 |B_k|^2 \sin^2 k\eta, \end{aligned}$$

$$\|\Delta u_1 - Au_1 - Cu_2\|_{D_\xi^{-1}}^2 = \sum_{k=1}^{\infty} |(k^2 + A)A_k + CB_k|^2 k^{-2} \sin^2 k\xi$$

and

$$\left\|-\Delta^2 u_2 - Bu_1 - Du_2\right\|_{D_\eta^{-2}}^2 = \sum_{k=1}^{\infty} |(k^4 + D)B_k + BA_k|^2 k^{-4} \sin^2 k\eta.$$

so that the desired estimate is equivalent to the inequalities

$$\begin{aligned} &k^2 |A_k|^2 \sin^2 k\xi + k^4 |B_k|^2 \sin^2 k\eta \\ &\qquad \le c\Big(|(k^2 + A)A_k + CB_k|^2 k^{-2} \sin^2 k\xi \\ &\qquad\qquad + |(k^4 + D)B_k + BA_k|^2 k^{-4} \sin^2 k\eta\Big) \end{aligned} \tag{5.3}$$

uniformly in k.

Fix a real number $1 < p < 3$ and choose two numbers $\xi, \eta \in (0,\pi)$ such that ξ/π, η/π are irrational, and

$$\sin^{-2} k\xi = O\left(k^{-2p}\right), \quad \sin^{-2} k\eta = O\left(k^{-2p}\right)$$

for $k \to \infty$. It is well-known from the theory of Diophantine approximation (see, e.g., [5]) that almost all numbers $\xi, \eta \in (0,\pi)$ have this property. It follows from this choice that

$$\frac{\sin^2 k\xi}{\sin^2 k\eta} = o\bigl(k^6\bigr) \quad \text{and} \quad \frac{\sin^2 k\eta}{\sin^2 k\xi} = o\bigl(k^6\bigr). \tag{5.4}$$

We are going to prove that there exists a positive integer k_0 such that

$$\begin{aligned} k^2|A_k|^2 \sin^2 k\xi + k^4|B_k|^2 \sin^2 k\eta \\ \le 4\Bigl(|(k^2+A)A_k + CB_k|^2 k^{-2} \sin^2 k\xi \\ + |(k^4+D)B_k + BA_k|^2 k^{-4} \sin^2 k\eta\Bigr) \end{aligned} \tag{5.5}$$

for all $k > k_0$.

Using the elementary inequality $|x+y+z|^2 \le 3\bigl(|x|^2+|y|^2+|z|^2\bigr)$ and (5.4) we have

$$\begin{aligned} &|(k^2+A)A_k + CB_k|^2 k^{-2} \sin^2 k\xi \\ &\quad = |kA_k + k^{-1}AA_k + k^{-1}CB_k|^2 \sin^2 k\xi \\ &\quad \ge \left(\frac{1}{3}|kA_k|^2 - |k^{-1}AA_k|^2 - |k^{-1}CB_k|^2\right) \sin^2 k\xi \\ &\quad = \left(\frac{1}{3} - |k^{-2}A|^2\right) k^2|A_k|^2 \sin^2 k\xi - \left(k^{-6}|C|^2 \frac{\sin^2 k\xi}{\sin^2 k\eta}\right) k^4|B_k|^2 \sin^2 k\eta \\ &\quad = \left(\frac{1}{3} - o(1)\right) k^2|A_k|^2 \sin^2 k\xi - o(1) k^4|B_k|^2 \sin^2 k\eta \end{aligned}$$

and

$$\begin{aligned} &|(k^4+D)B_k + BA_k|^2 k^{-4} \sin^2 k\eta \\ &\quad = |k^2B_k + k^{-2}DB_k + k^{-2}BA_k|^2 \sin^2 k\eta \\ &\quad \ge \left(\frac{1}{3}|k^2B_k|^2 - |k^{-2}DB_k|^2 - |k^{-2}BA_k|^2\right) \sin^2 k\eta \\ &\quad = \left(\frac{1}{3} - |k^{-4}D|^2\right) k^4|B_k|^2 \sin^2 k\eta - \left(k^{-6}|B|^2 \frac{\sin^2 k\eta}{\sin^2 k\xi}\right) k^2|A_k|^2 \sin^2 k\xi \\ &\quad = \left(\frac{1}{3} - o(1)\right) k^4|B_k|^2 \sin^2 k\eta - o(1) k^2|A_k|^2 \sin^2 k\xi, \end{aligned}$$

so that

$$\begin{aligned} &|(k^2+A)A_k + CB_k|^2 k^{-2} \sin^2 k\xi + |(k^4+D)B_k + BA_k|^2 k^{-4} \sin^2 k\eta \\ &\quad \ge \left(\frac{1}{3} - o(1)\right) k^2|A_k|^2 \sin^2 k\xi + \left(\frac{1}{3} - o(1)\right) k^4|B_k|^2 \sin^2 k\eta \end{aligned}$$

Choosing a sufficiently large k_0 hence (5.5) follows.

It remains to ensure the validity of (5.3) for each $k = 1, \dots, k_0$. It suffices to choose the parameters (A, B, C, D) so that none of the determinants

$$\begin{vmatrix} -k^2 - A & -C \\ -B & k^4 - D \end{vmatrix} = -(k^2 + A)(k^4 - D) - BC$$

vanishes. Indeed, then we have for each k the implication

$$(-k^2 - A)A_k - CB_k = DA_k - (k^4 - B)B_k \Longleftrightarrow A_k = B_k = 0, \quad k = 1, \dots, k_0;$$

since all norms are equivalent on finite-dimensional vector spaces, this implies that both

$$\sum_{k=1}^{k_0} k^2 |A_k|^2 \sin^2 k\xi + k^4 |B_k|^2 \sin^2 k\eta$$

and

$$\sum_{k=1}^{k_0} |(k^2 + A)A_k + CB_k|^2 k^{-2} \sin^2 k\xi + |(k^4 + D)B_k + BA_k|^2 k^{-4} \sin^2 k\eta$$

are equivalent to

$$\sum_{k=1}^{k_0} (|A_k|^2 + |B_k|^2)$$

and therefore

$$\begin{aligned} k^2 |A_k|^2 \sin^2 k\xi + k^4 |B_k|^2 \sin^2 k\eta & \\ \le c\Big(|(k^2 + A)A_k + CB_k|^2 k^{-2} \sin^2 k\xi & \\ + |(k^4 + D)B_k + BA_k|^2 k^{-4} \sin^2 k\eta \Big) & \qquad (5.6) \end{aligned}$$

for every $k = 1, \dots, k_0$ with a suitable constant c.

The exceptional parameters (A, B, C, D) for which one of these determinants vanishes form a finite number of three-dimensional manifolds in $\mathbb{R}^4$, hence a set of Lebesgue measure zero. We conclude that for almost all choices of $\xi, \eta \in (0, \pi)$, hypothesis $(H2)$ holds for almost all choices of $(A, B, C, D) \in \mathbb{R}^4$.

It follows from (5.2) that hypotheses (H3)-(H4) are satisfied. Theorem 1.3 now follows by applying Theorem 2.1. In order to write down explicitly the stabilization result, we multiply the equation (1.5) by u and we integrate by parts as follows (we use all conditions in (1.5) and (5.1)). This shows that if we write (5.1) in the form (2.1), then its dual (2.2) corresponds to (1.5). Furthermore, writing the operator

$$\Lambda_\omega^{-1} : (D_\xi^0)' \times (D_\xi^{-1})' \times (D_\eta^0)' \times (D_\eta^{-2})' \to D_\xi^{-1} \times D_\xi^0 \times D_\eta^{-2} \times D_\eta^0$$

in the matrix form

$$\Lambda_\omega^{-1} = \begin{pmatrix} -P_1 & Q_1 & \Lambda_{13} & \Lambda_{14} \\ \Lambda_{21} & \Lambda_{22} & -P_2 & Q_2 \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44} \end{pmatrix},$$

we have

$$(v_1(t), v_2(t)) = -((P_1 y_{1t} + Q_1 y_1)(t, \xi), (P_2 y_{2t} + Q_2 y_2)(t, \eta)). \tag{5.7}$$

6. Observability and controllability results of the coupled string-beam system

In this section we state and prove two new observability and controlability theorems concerning the coupled string-beam system.

Theorem 6.1 *Fix $\xi,\ \eta \in (0, \pi)$ such that ξ/π and η/π are irrational and introduce the space*

$$(\mathcal{H}_{\xi,\eta})' = D_\xi^0 \times D_\xi^{-1} \times D_\eta^0 \times D_\eta^{-2}.$$

Let $(u_{10}, u_{11}, u_{20}, u_{21}) \in (\mathcal{H}_{\xi,\eta})'$, for almost all choices of $(A, B, C, D) \in \mathbb{R}^4$, if $T > 2\pi$ there exist a positive constante $C_1 > 0$ such that the following inequality is verified

$$C_1 \|(u_{10}, u_{11}, u_{20}, u_{21})\|^2_{(\mathcal{H}_{\xi,\eta})'} \leq \int_0^T (|u_1(t, \xi)|^2 + |u_2(t, \eta)|^2) dt \tag{6.1}$$

for any $(u_{10}, u_{11}, u_{20}, u_{21}) \in (\mathcal{H}_{\xi,\eta})'$.

Proof. It follow from hypothesis $(H4)$ of the abstract result and (5.2).

Definition 6.2 *Fix $\xi,\ \eta \in (0, \pi)$ such that ξ/π and η/π are irrational. the system (1.5) is exactly controllable if for any given initial and final data*

$$(y_{10}, y_{11}, y_{20}, y_{21}) \in \mathcal{H}_{\xi,\eta}$$

and

$$(z_{10}, z_{11}, z_{20}, z_{21}) \in \mathcal{H}_{\xi,\eta}$$

there exist control functions

$$v_1, v_2 \in L^2(0, T; \mathcal{H}_{\xi,\eta})$$

such that the corresponding solution of (1.5) satisfies the final condition

$$(y_1, y_{1t}, y_2, y_{2t})(T) = (z_{10}, z_{11}, z_{20}, z_{21}).$$

Remark 6.3 *Inequality (6.1) is called observability inequality, which plays a fundamental role in control theory, it is not only sufficient but also necessary condition of exact controllability.*

Theorem 6.4 *Assume* $(H1)-(H4)$. *If* $T>2\pi$, *then the system* (1.5) *is exactly controllable for almost all choices of* $(A,B,C,D)\in\mathbb{R}^4$.

Proof. Theorem 6.4 is an abstract form of the Hilbert Uniqueness Method of Lions [16], [17], see also [18] for a general duality principle between observability and controllability.

7. Proof of Proposition 1.5

By the definition of the norms the three inclusions in part (a) are equivalent to the inequalities

$$\sum|a_k|^2\le\sum\sin^{-2}(k\xi)|a_k|^2,$$
$$\sum k^2|a_k|^2\le\sum k^2\sin^{-2}(k\xi)|a_k|^2,$$
$$\sum k^4|a_k|^2\le\sum k^4\sin^{-2}(k\xi)|a_k|^2.$$

Turning to (b), as we have already mentioned in the preceding section, by a theorem on Diophantine approximation almost every $\xi\in(0,\pi)$ satisfies for all $\varepsilon>0$ the estimates

$$|\sin(k\xi)|\ge\frac{c_\varepsilon}{k^{1+\varepsilon}}$$

with suitable positive constants c_ε, independent of $k=1,2,\ldots$.. It follows that

$$\sum\sin^{-2}(k\xi)|a_k|^2\le c_\varepsilon^{-2}\sum k^{2+2\varepsilon}|a_k|^2,$$
$$\sum k^2\sin^{-2}(k\xi)|a_k|^2\le c_\varepsilon^{-2}\sum k^{4+2\varepsilon}|a_k|^2,$$
$$\sum k^4\sin^{-2}(k\xi)|a_k|^2\le c_\varepsilon^{-2}\sum k^{6+2\varepsilon}|a_k|^2,$$

and they are equivalent to the stated inclusions.

Finally, the inclusions of part (c) are obtained by establish the same proof of (b) which $\varepsilon=0$.

References

1. K. Ammari, M. Tucsnak, *Stabilisation of second order evolution equations by a class of unbounded feedbacks*, ESAIM, 6, (2001) 361–386.

2. K. Ammari and M. Tucsnak, Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force, *SIAM J. Control. Optim.,* 39, (2000), 1160-1181.

3. K. Ammari, A. Henrot and M. Tucsnak, Asymptotic behaviour of the solutions and optimal location of the actuator for the pointwise stabilization of a string, *Asymptotic Analysis,* 28, (2001), 215-240.

4. K. Ammari, Z. Liu and M. Tucsnak, Decay rates for a beam with pointwise force and moment feedback, *Mathematics of Control, Signals, and Systems,* 15, (2002), 229-255.

5. J.W.S. Cassels, *An Introduction to Diophantine Approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45. Cambridge University Press, New York, 1957.

6. A. Haraux, *Quelques méthodes et résultats récents en théorie de la contrôlabilité exacte*, Rapport de recherche No. 1317, INRIA Rocquencourt, Octobre 1990.

7. A. Haraux, *Quelques propriétés des séries lacunaires utiles dans l'étude des systèmes élastiques*, Nonlinear partial differential equations and their applications. Collège de France Seminar,XII (Paris, 1991–1993), 113–124 Pitman Res. Notes Math.Ser., 302, Longman Sci.Tech.Harlow, 1994.

8. A. Haraux, S. Jaffard, *Pointwise and spectral control of plate vibrations*, Rev. Mat. Iberoamericana 7 (1) (1991), 1âĂŞ-24.

9. A. Haraux, *Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps*, Portugal. Math. 46 (1989), no. 3, 245-258.

10. V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, Masson, Paris, and John Wiley & Sons, Chicester, 1994.

11. V. Komornik, *Rapid boundary stabilization of linear distributed systems*, SIAM J. Control Optim. 35 (1997), 1591–1613.

12. V. Komornik, P. Loreti *Ingham type theorems for vector-valued functions and observability of coupled linear systems*, SIAM J. Control Optim. 37 (1998), 461–485.

13. V. Komornik, P. Loreti, *Fourier Series in Control Theory*, Springer-Verlag, New York, 2005.

14. J.E. Lagnese, *Boundary Stabilization of Thin Plates*, SIAM Studies in Appl. Math., Philadelphia, 1989.

15. I. Lasiecka and R. Triggiani, *Control theory for partial differential equations: continuous and approximation theories I-II*, Encyclopedia of Mathematics and Its Applications, 74–75. Cambridge University Press, 2000.

16. J.-L. Lions, *Exact controllability, stabilizability, and perturbations for distributed systems*, Siam Rev. 30 (1988), 1–68.

17. J.-L. Lions, *Contrôlabilité exacte et stabilisation de systèmes distribués I-II* , Masson, Paris, 1988.

18. D.L. Russell, *Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions*, SIAM Rev. 20 (1978), 639–739.

Alia BARHOUMI
Faculté des Sciences de Monastir, 5019 Monastir, Tunisie.
Université de Monastir
Tunisie
Alia.Barhoumi@isimm.rnu.tn