Bol. Soc. Paran. Mat. (3s.) **v. 28** 2 (2010): 15–24.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/spm doi:10.5269/bspm.v28i2.10727

A unified theory for R_0 , R_1 and certain other separation properties and their variant forms

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abstract: The purpose of the present paper is towards working out a unified version of the study of certain separation axioms and their neighbouring forms, as are already available in the literature. In terms of an operation, as initiated by Á. Császár, we introduce unified definitions of R_0, R_1, T_0 and T_1 spaces and derive results concerning them from which many of the existing results follow as special cases.

Key Words: operation, ψ-open set, ψ-closure, ψ-kernel, ψ-R_0 space, ψ-R_1 space.

Contents

1. Introduction

It is observed from literature that there has been a considerable work on different relatively weak forms of separation axioms, like R_0 and R_1 axioms in particular; several other neighbouring forms of them have also been studied in many papers. For instance, semi-R_0, pre-R_0, δ-pre-R_0, α-R_0, β-R_0 are some of the variant forms of R_0-property, that have been investigated by different researchers as separate entities. Similar observation applies to R_1, T_0 and T_1 axioms. As can be observed, all these variations have been effected by using different types of operators like int, $intcl$, $intcl_\delta$, $clint_\delta$, $clint$, $intclint$, $clintcl$, where int and cl respectively stand for interior and closure operators, and cl_δ denotes the δ-closure operator. Now, the concept of a generalized type of operator, called operation on the power set $\mathcal{P}(X)$ of a topological space (X,τ), was introduced by [5]. It turns out from the investigations here that by judicious use of the notion of 'operation', one can give generalized definitions of R_0, R_1, T_0, T_1 axioms from which the definitions of different varied forms of such properties and many known results thereon follow as particular consequences. These known definitions and theorems, encompassed by our unified study, are appended mainly at the end of the paper in two tables.

[1] The first author acknowledges the financial support from UGC, New Delhi.
2000 *Mathematics Subject Classification:* 54A05, 54D10

2. R_0 spaces

We now begin by recalling a few definitions and observe that many of the existing relevant definitions considered in various papers turn out to be special cases of the ones given below.

Definition 2.1. [5] Let (X, τ) be a topological space. A mapping $\psi : \mathcal{P}(X) \to \mathcal{P}(X)$ is called an operation on $\mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes as usual the power set of X, if for each $A \in \mathcal{P}(X) \setminus \{\varnothing\}$, $intA \subseteq \psi(A)$ and $\psi(\varnothing) = \varnothing$.

The set of all operations on a space X will be denoted by $\mathcal{O}(X)$

Observation 2.2. It is easy to check that some examples of operations on a space X are the well known operators viz. *int*, *intcl*, $intcl_\delta$, *clint*, *intclint*, *clintcl*.

Definition 2.3. [5] Let ψ denote an operation on a space (X, τ). Then a subset A of X is called ψ-open if $A \subseteq \psi(A)$. Complements of ψ-open sets will be called ψ-closed sets. The family of all ψ-open (resp. ψ-closed) subsets of X is denoted by $\psi\mathcal{O}(X)$ (resp. $\psi\mathcal{C}(X)$)

Observation 2.4. It is clear that if ψ stands for any of the operators *int*, *intcl*, $intcl_\delta$, *clint*, *intclint*, *clintcl*, then ψ-openness of a subset A of X coincides with respectively the openness, preopenness, δ-preopenness, semi-openness, α-openness and β-openness of A (see [10,20,27,28,11,1])

Definition 2.5. [5] Let (X, τ) be a topological space, $\psi \in \mathcal{O}(X)$ and $A \subseteq X$. Then the intersection of all ψ-closed sets containing A is called the ψ-closure of A, denoted by $\psi\text{-}clA$; alternately, $\psi\text{-}clA$ is the smallest ψ-closed set containing A.

It is known [12] that $x \in \psi\text{-}clA$ iff $A \cap U \neq \varnothing$, for all U with $x \in U \in \psi\mathcal{O}(X)$.

Observation 2.6. Obviously if one takes interior as the operation ψ, then ψ-closure becomes equivalent to the usual closure. Similarly, ψ-closure becomes *pcl*, pcl_δ, *scl*, α-*cl*, β-*cl*, if ψ is taken to stand for the operators *intcl*, $intcl_\delta$, *clint*, *intclint* and *clintcl* respectively (see [10,20,27,28,11,1] for details).

Definition 2.7. For any $\psi \in \mathcal{O}(X)$ and any subset A of a space (X, τ), the ψ-kernel of A, denoted by $\psi\text{-}ker(A)$, is defined by the relation : $\psi\text{-}ker(A) = \cap\{G \in \psi\mathcal{O}(X) : A \subseteq G\}$.

Observation 2.8. For $\psi \in \mathcal{O}(X)$, if we take $\psi = int$ (resp. *intcl*, *clint*, $intcl_\delta$, *intclint*, *clintcl*), then $\psi\text{-}ker(A)$ becomes $ker(A)$ [6,25] (resp. $pker(A)$ [2], $pker_\delta(A)$ [3], $ker_\alpha(A)$ [4], $\beta\text{-}ker(A)$ [29]).

A new expression for $\psi\text{-}ker(A)$ is given by the following

Lemma 2.9. Let X be a space, $\psi \in \mathcal{O}(X)$ and $A \subseteq X$. Then
$\psi\text{-}ker(A) = \{x \in X : \psi\text{-}cl(\{x\}) \cap A \neq \varnothing\}$.

Proof. Let $x \in \psi\text{-}ker(A)$ and $\psi\text{-}ker(\{x\}) \cap A = \varnothing$. Then $x \notin X \setminus \psi\text{-}cl(\{x\})$, which is a ψ-open set containing A. Thus $x \notin \psi\text{-}ker(A)$, a contradiction.

Conversely, let $x \in X$ be such that $\psi\text{-}ker(\{x\}) \cap A \neq \varnothing$. If possible, let $x \notin \psi\text{-}ker(A)$. Then $\exists\, G \in \psi\mathcal{O}(X)$ such that $x \notin G$ and $A \subseteq G$. Let $y \in \psi(\{x\}) \cap A$. Then $y \in \psi\text{-}cl(\{x\})$ and $y \in G$, which gives $x \in G$, a contradiction.

Definition 2.10. [12] Corresponding to a $\psi \in \mathcal{O}(X)$, a topological space (X, τ) is called ψ-R_0 if ($U \in \psi\mathcal{O}(X)$ and $x \in U \Rightarrow \psi\text{-}cl(\{x\}) \subseteq U$).

Observation 2.11. R_0 space, pre-R_0 space, semi-R_0 space, δ-pre-R_0 space, α-R_0 space and β-R_0 space have been defined and studied in [6,22], [2], [8], [3],

[4] and [29] respectively. The above definition gives a unified version of all these definitions if the role of ψ is taken respectively by the operators *int*, *intcl*, *clint*, $intcl_\delta$, *intclint* and *clintcl*.

Theorem 2.12. For any topological space X and any $\psi \in \mathcal{O}(X)$, the following are equivalent :

(a) X is ψ-R_0.

(b) $F \in \psi\mathcal{C}(X)$ and $x \notin F \Rightarrow F \subseteq U$ and $x \notin U$ for some $U \in \psi\mathcal{O}(X)$.

(c) $F \in \psi\mathcal{C}(X)$ and $x \notin F \Rightarrow F \cap \psi\text{-}cl(\{x\}) = \varnothing$.

(d) For any two distinct points x, y of X, either $\psi\text{-}cl(\{x\}) = \psi\text{-}cl(\{y\})$ or $\psi\text{-}cl(\{x\}) \cap \psi\text{-}cl(\{y\}) = \varnothing$.

Proof. **(a) $\Rightarrow$ (b) :** $F \in \psi\mathcal{C}(X)$ and $x \notin F \Rightarrow x \in X \setminus F \in \psi\mathcal{O}(X) \Rightarrow \psi\text{-}cl(\{x\}) \subseteq X \setminus F$ (by (a)). Put $U = X \setminus \psi\text{-}cl(\{x\})$. Then $x \notin U \in \psi\mathcal{O}(X)$ and $F \subseteq U$.

(b) $\Rightarrow$ (c) : $F \in \psi\mathcal{C}(X)$ and $x \notin F \Rightarrow \exists\, U \in \psi\mathcal{O}(X)$ such that $x \notin U$ and $F \subseteq U$ (by (b)) $\Rightarrow U \cap \psi\text{-}cl(\{x\}) = \varnothing \Rightarrow F \cap \psi\text{-}cl(\{x\}) = \varnothing$.

(c) $\Rightarrow$ (d) : Suppose that for any two distinct points x, y of X, $\psi\text{-}cl(\{x\}) \neq \psi\text{-}cl(\{y\})$. Then suppose without any loss of generality that there exists some $z \in \psi\text{-}cl(\{x\})$ such that $z \notin \psi\text{-}cl(\{y\})$. Thus $\exists\, V \in \psi\mathcal{O}(X)$ such that $z \in V$ and $y \notin V$ but $x \in V$. Thus $x \notin \psi\text{-}cl(\{y\})$. Hence by (c), $\psi\text{-}cl(\{x\}) \cap \psi\text{-}cl(\{y\}) = \varnothing$.

(d) $\Rightarrow$ (a) : Let $U \in \psi\mathcal{O}(X)$ and $x \in U$. Then for each $y \notin U$, $x \notin \psi\text{-}cl(\{y\})$. Thus $\psi\text{-}cl(\{x\}) \neq \psi\text{-}cl(\{y\})$. Hence by (d), $\psi\text{-}cl(\{x\}) \cap \psi\text{-}cl(\{y\}) = \varnothing$, for each $y \in X \setminus U$. So $\psi\text{-}cl(\{x\}) \cap [\cup\{\psi\text{-}cl(\{y\}) : y \in X \setminus U\}] = \varnothing$ ... (i).

Now, $U \in \psi\mathcal{O}(X)$ and $y \in X \setminus U \Rightarrow \{y\} \subseteq \psi\text{-}cl(\{y\}) \subseteq \psi\text{-}cl(X \setminus U) = X \setminus U$. Thus $X \setminus U = \cup\{\psi\text{-}cl(\{y\}) : y \in X \setminus U\}$. Hence from (i), $\psi\text{-}cl(\{x\}) \cap (X \setminus U) = \varnothing \Rightarrow \psi\text{-}cl(\{x\}) \subseteq U$, showing that (X, τ) is ψ-R_0.

Lemma 2.13. Let ψ be an operation on a topological space (X, τ). Then $y \in \psi\text{-}ker(\{x\})$ iff $x \in \psi\text{-}cl(\{y\})$.

Proof. $y \notin \psi\text{-}ker(\{x\}) \Rightarrow \exists\, V \in \psi\mathcal{O}(X)$ containing x such that $y \notin V \Rightarrow x \notin \psi\text{-}cl(\{y\})$. The converse part can be proved in a similar way.

Theorem 2.14. Let ψ be an operation on a topological space (X, τ). Then for any two points $x, y \in X$, the following are equivalent :

(a) $\psi\text{-}ker(\{x\}) \neq \psi\text{-}ker(\{y\})$.

(b) $\psi\text{-}cl(\{x\}) \neq \psi\text{-}cl(\{y\})$.

Proof. **(a) $\Rightarrow$ (b) :** $\psi\text{-}ker(\{x\}) \neq \psi\text{-}ker(\{y\}) \Rightarrow \exists\, z \in \psi\text{-}ker(\{x\})$ such that $z \notin \psi\text{-}ker(\{y\})$ (say). Now, $z \in \psi\text{-}ker(\{x\}) \Leftrightarrow x \in \psi\text{-}cl(\{z\})$; and $z \notin \psi\text{-}ker(\{y\}) \Leftrightarrow y \notin \psi\text{-}cl(\{z\})$. As $\psi\text{-}cl(\{x\}) \subseteq \psi\text{-}cl(\{z\})$, we have $y \notin \psi\text{-}cl(\{x\})$. Hence $\psi\text{-}cl(\{x\}) \neq \psi\text{-}cl(\{y\})$.

(b) $\Rightarrow$ (a) : $\psi\text{-}cl(\{x\}) \neq \psi\text{-}cl(\{y\}) \Rightarrow \exists\, z \in X$ such that $z \in \psi\text{-}cl(\{x\})$ and $z \notin \psi\text{-}cl(\{y\})$ (say) $\Rightarrow \exists\, U \in \psi\mathcal{O}(X)$ such that $z \in U$, $y \notin U$ and $x \in U$ $\Rightarrow y \notin \psi\text{-}ker(\{x\})$. Thus $\psi\text{-}ker(\{x\}) \neq \psi\text{-}ker(\{y\})$.

Theorem 2.15. Let ψ be an operation on a topological space (X, τ). Then (X, τ) is ψ-R_0 iff for any two points $x, y \in X$, $\psi\text{-}ker(\{x\}) \neq \psi\text{-}ker(\{y\})$ implies $\psi\text{-}ker(\{x\}) \cap \psi\text{-}ker(\{y\}) = \varnothing$.

Proof. Let x, y be any two points in a ψ-R_0 space X such that $\psi\text{-}ker(\{x\}) \neq \psi\text{-}ker(\{y\})$. Hence by Theorem 2.14, $\psi\text{-}cl(\{x\}) \neq \psi\text{-}cl(\{y\})$. We show that

ψ-$ker(\{x\}) \cap \psi$-$ker(\{y\}) = \varnothing$. In fact, $z \in \psi$-$ker(\{x\}) \cap \psi$-$ker(\{y\}) \Rightarrow x, y \in \psi$-$cl(\{z\})$ (by Lemma 2.13) $\Rightarrow \psi$-$cl(\{x\}) = \psi$-$cl(\{z\}) = \psi$-$cl(\{y\})$ (using Theorem 2.12).

Conversely, let for any points $x, y \in X$, ψ-$cl(\{x\}) \neq \psi$-$cl(\{y\})$. Then by Theorem 2.14, ψ-$ker(\{x\}) \neq \psi$-$ker(\{y\})$. Thus ψ-$ker(\{x\}) \cap \psi$-$ker(\{y\}) = \varnothing$. Hence ψ-$cl(\{x\}) \cap \psi$-$cl(\{y\}) = \varnothing$. In fact, $z \in \psi$-$cl(\{x\}) \cap \psi$-$cl(\{y\}) \Rightarrow x, y \in \psi$-$ker(\{z\})$. Thus ψ-$cl(\{x\}) \cap \psi$-$cl(\{z\}) \neq \varnothing$. Hence by hypothesis, ψ-$ker(\{x\}) = \psi$-$ker(\{z\})$. By similar way it follows that ψ-$ker(\{y\}) = \psi$-$ker(\{z\})$. Thus ψ-$ker(\{x\}) = \psi$-$ker(\{y\})$ which is a contradiction. Hence ψ-$cl(\{x\}) \cap \psi$-$cl(\{y\}) = \varnothing$ and then by Theorem 2.12, the space X becomes ψ-R_0.

Theorem 2.16. For any $\psi \in \mathcal{O}(X)$ on a space X the following statements are equivalent :
(a) X is a ψ-R_0 space.
(b) For any non-empty set A and any $G \in \psi\mathcal{O}(X)$ such that $A \cap G \neq \varnothing$, there exists $F \in \psi\mathcal{C}(X)$ such that $A \cap F \neq \varnothing$ and $F \subseteq G$.
(c) For any $G \in \psi\mathcal{O}(X)$, $G = \cup\{F \in \psi\mathcal{C}(X) : F \subseteq G\}$.
(d) For any $F \in \psi\mathcal{C}(X)$, $F = \cap\{G \in \psi\mathcal{O}(X) : F \subseteq G\}$.
(e) For any $x \in X$, ψ-$cl(\{x\}) \subseteq \psi$-$ker(\{x\})$.

Proof. **(a) $\Rightarrow$ (b) :** Let A be a non-empty subset of X and $G \in \psi\mathcal{O}(X)$ such that $A \cap G \neq \varnothing$. Let $x \in A \cap G$. Then as $x \in G \in \psi\mathcal{O}(X)$, we have by (a), ψ-$cl(\{x\}) \subseteq G$. Put $F = \psi$-$cl(\{x\})$. Then $F \in \psi\mathcal{C}(X)$, $F \subseteq G$ and $A \cap F \neq \varnothing$.
(b) $\Rightarrow$ (c) : Let $G \in \psi\mathcal{O}(X)$. Then $G \supseteq \cup\{F \in \psi\mathcal{C}(X) : F \subseteq G\}$. Let $x \in G$. Then $\exists\ F \in \psi\mathcal{C}(X)$ such that $x \in F$ and $F \subseteq G$. Thus $x \in F \subseteq \cup\{R \in \psi\mathcal{C}(X) : R \subseteq G\}$. Hence (c) follows.
(c) $\Rightarrow$ (d) : It is trivial.
(d) $\Rightarrow$ (e) : Let $x \in X$. Now, $y \notin \psi$-$ker(\{x\} \Rightarrow \exists\ V \in \psi\mathcal{O}(X)$ such that $x \in V$ and $y \notin V \Rightarrow \psi$-$cl(\{y\}) \cap V = \varnothing \Rightarrow [\cap\{G \in \psi\mathcal{O}(X) : \psi$-$cl(\{y\}) \subseteq G\}] \cap V = \varnothing$ (by (d)) $\Rightarrow \exists\ G \in \psi\mathcal{O}(X)$ such that $x \notin G$ and ψ-$cl(\{y\}) \subseteq G \Rightarrow y \notin \psi$-$cl(\{x\})$.
(e) $\Rightarrow$ (a) : Let $G \in \psi\mathcal{O}(X)$ and $x \in G$. Let $y \in \psi$-$ker(\{x\})$. Then $x \in \psi$-$cl(\{y\})$ and hence $y \in G$. This implies that ψ-$ker(\{x\}) \subseteq G$. Thus $x \in \psi$-$cl(\{x\}) \subseteq \psi$-$ker(\{x\}) \subseteq G \Rightarrow X$ is ψ-R_0.

Corollary 2.17. Let ψ be an operation on a space X. Then X is ψ-R_0 iff ψ-$cl(\{x\}) = \psi$-$ker(\{x\})$, $\forall\ x \in X$.

Proof. Suppose X is ψ-R_0. By Theorem 2.16, ψ-$cl(\{x\}) \subseteq \psi$-$ker(\{x\})$ for each $x \in X$. Let $y \in \psi$-$ker(\{x\})$. Then $x \in \psi$-$cl(\{y\})$ (by Lemma 2.13), and hence by Theorem 2.12, ψ-$cl(\{x\}) = \psi$-$cl(\{y\})$. Thus $y \in \psi$-$cl(\{x\})$ and hence ψ-$ker(\{x\}) \subseteq \psi$-$cl(\{x\})$. Thus ψ-$cl(\{x\}) = \psi$-$ker(\{x\})$.

The converse is obvious in view of Theorem 2.16.

Theorem 2.18. A topological space X is ψ-R_0, for a given $\psi \in \mathcal{O}(X)$, iff for any $x, y \in X$ ($x \in \psi$-$cl(\{y\})$ iff $y \in \psi$-$cl(\{x\})$).

Proof. First suppose that the space X is ψ-R_0. Let $x \in \psi$-$cl(\{y\})$ and $U \in \psi\mathcal{O}(X)$ with $y \in U$. Then ψ-$cl(\{x\}) \subseteq U$ so that $x \in U$. Thus $y \in \psi$-$cl(\{x\})$.

Conversely, let $U \in \psi\mathcal{O}(X)$ and $x \in U$. If $y \notin U$, then $x \notin \psi$-$cl(\{y\})$ and hence by hypothesis, $y \notin \psi$-$cl(\{x\})$. Thus ψ-$cl(\{x\}) \subseteq U$. Consequently, X is ψ-R_0.

Theorem 2.19. Let X be a topological space and $\psi \in \mathcal{O}(X)$. Then the following are equivalent :
(a) X is ψ-R_0.
(b) If $F \in \psi\mathcal{C}(X)$, then $F = \psi\text{-}ker(F)$.
(c) If $F \in \psi\mathcal{C}(X)$ and $x \in F$, then $\psi\text{-}ker(\{x\}) \subseteq F$.
(d) If $x \in X$, then $\psi\text{-}ker(\{x\}) \subseteq \psi\text{-}cl(\{x\})$.
Proof. '(a) $\Rightarrow$ (b)' is obvious by virtue of Theorem 2.16, whereas **'(b) $\Rightarrow$ (c)'** follows from the fact that $\{x\} \subseteq F \Rightarrow \psi\text{-}ker(\{x\}) \subseteq \psi\text{-}ker(F) = F$. Again, since $x \in \psi\text{-}cl(\{x\}) \in \psi\mathcal{C}(X)$ we have by (c), $\psi\text{-}ker(\{x\}) \subseteq \psi\text{-}cl(\{x\})$ and (d) follows.
(d) $\Rightarrow$ (a) : Let $x \in \psi\text{-}cl(\{y\})$. Then by Lemma 2.13, $y \in \psi\text{-}ker(\{x\})$. Hence by (d) we have $y \in \psi\text{-}cl(\{x\})$. Thus $x \in \psi\text{-}cl(\{y\}) \Rightarrow y \in \psi\text{-}cl(\{x\})$. The reverse implication follows similarly. Hence by Theorem 2.18, X is ψ-R_0.
Definition 2.20. Let ψ be an operation on a topological space (X, τ). Then the space X is said to be a
(i) ψ-T_0 space [12] if for any two distinct points x and y in X, there exists a ψ-open set containing one of x, y and not the other;
(ii) ψ-T_1 space [12] if for any two distinct points x and y of X, there exist ψ-open set U, V such that $x \in U$, $y \notin U$, $x \notin V$, $y \in V$;
(iii) ψ-T_2 space if for any two distinct points x and y of X, there exist two disjoint ψ-open sets containing x and y respectively.
Observation 2.21. In a topological space X if we take $\psi = int$ (resp. $intcl$, $clint$, $intcl_\delta$, $intclint$, $clintcl$) then the concept of ψ-T_i property coincides with that of T_i (resp. *pre*-T_i [24], *semi*-T_i [18], (δ, p)-T_i [3], α-T_i [19,16] and β-T_i [15]) property, for $i = 0, 1, 2$.
Theorem 2.22. For any $\psi \in \mathcal{O}(X)$ on a space X, the following are equivalent:
(a) X is ψ-T_1.
(ii) $\psi\text{-}cl(\{x\}) = \{x\}$, for all $x \in X$.
(iii) X is ψ-R_0 and ψ-T_0.
Proof. (a) $\Rightarrow$ (b) : $y \notin \{x\} \Rightarrow \exists\, U \in \psi\mathcal{O}(X)$ such that $y \in U$, $x \notin U \Rightarrow U \cap \{x\} = \varnothing \Rightarrow y \notin \psi\text{-}cl(\{x\})$.
(b) $\Rightarrow$ (c) : Let $x, y \in X$ with $x \neq y$. Then $\{x\}$ and $\{y\}$ are ψ-closed and hence $X \setminus \{x\}$ is a ψ-open set containing y but not x showing X to be ψ-T_0.

Again, $x, y \in X$ with $x \neq y \Rightarrow \psi\text{-}cl(\{x\}) \neq \psi\text{-}cl(\{y\})$. Also, $\psi\text{-}cl(\{x\}) \cap \psi\text{-}cl(\{y\}) = \varnothing$. Thus by Theorem 2.12, X is ψ-R_0.
(c) $\Rightarrow$ (a) : $x, y \in X$ with $x \neq y \Rightarrow \exists\, U \in \psi\mathcal{O}(X)$ such that $x \in U$ and $y \notin U$ (say) $\Rightarrow \psi\text{-}cl(\{x\}) \subseteq U$ (as X is ψ-R_0) and so $y \notin \psi\text{-}cl(\{x\})$. Hence $x \in U \in \psi\mathcal{O}(X)$, $y \notin U$ and $y \in X \setminus \psi\text{-}cl(\{x\}) \in \psi\mathcal{O}(X)$, $x \notin X \setminus \psi\text{-}cl(\{x\})$. So X is a ψ-T_1 space.
Note 2.23. We observe that for any point x in any topological space X, $\{x\}$ is either preopen or preclosed [17] (resp. δ-preclosed or δ-preopen [3], β-open or β-closed [23]) so that every topological space is *pre*-T_0. Thus in view of Theorem 2.22, we arrive at the result, that a topological space is *pre*-R_0 iff it is *pre*-T_1 [4] (resp. (δ, p)-R_0 iff (δ, p)-T_1 [3], β-R_0 iff β-T_1 [29]).
Definition 2.24. [13] Let ψ be an operation on a topological space X. Then a net $\{x_\alpha\}_{\alpha \in D}$ in X is said to ψ-converge to a point x of X if the net is eventually in every ψ-open set containing x.

Observation 2.25. In the above definition if we take $\psi = int$ ((resp. $intcl$, $clint$, $intcl_\delta$, $intclint$, $clintcl$) then ψ-convergence reduces to the usual convergence (resp. preconvergence [21], semi-convergence [28], δ-pre-convergence [26], α-convergence [11] and β-convergence [29]) of nets.

Lemma 2.26. Let x, y be two points in a space X and $\psi \in \mathcal{O}(X)$. If every net in X which ψ-converges to y, also ψ-converges to x, then $x \in \psi\text{-}cl(\{y\})$.

Proof. Let us consider the net $x_n = y$ for each $n \in \mathbb{N}$ ($\mathbb{N}$ denoting the set of natural numbers). Clearly the net ψ-converges to y, and hence ψ-converges to x. Thus if $U \in \psi\mathcal{O}(X)$ with $x \in U$, then $\{x_n\}_{n\in\mathbb{N}}$ is eventually in U, which implies $y \in U$. Thus $x \in \psi\text{-}cl(\{y\})$.

Theorem 2.27. Let ψ be an operation on a topological space X. Then X is $\psi\text{-}R_0$ iff for any $x, y \in X$, [$y \in \psi\text{-}cl(\{x\}) \Leftrightarrow$ every net in X ψ-converging to y also ψ-converges to x].

Proof. Let X be $\psi\text{-}R_0$. Suppose $y \in \psi\text{-}cl(\{x\})$ for some $x, y \in X$ and let $\{x_\alpha\}_{\alpha\in D}$ be a net in X ψ-converging to y. Since $y \in \psi\text{-}cl(\{x\})$, by Theorem 2.12, $\psi\text{-}cl(\{x\}) = \psi\text{-}cl(\{y\})$. Let $U \in \psi\mathcal{O}(X)$ such that $x \in U$. Then $y \in U$ (as $x \in \psi\text{-}cl(\{y\})$) and hence $\exists\ \alpha_0 \in D$ such that if $\alpha \geq \alpha_0$ then $x_\alpha \in U$. Thus $\{x_\alpha\}_{\alpha\in D}$ ψ-converges to x. On the other hand, suppose that every net in X ψ-converging to y, ψ-converges to x. Then by Lemma 2.26, $x \in \psi\text{-}cl(\{y\})$. Thus by Theorem 2.12, $\psi\text{-}cl(\{x\}) = \psi\text{-}cl(\{y\})$ and hence $y \in \psi\text{-}cl(\{x\})$.

Conversely, to prove X to be $\psi\text{-}R_0$, let U be a ψ-open set and $x \in U$. Let $y \in X \setminus U$. For each $n \in \mathbb{N}$, let $x_n = y$. Then the net $\{x_n\}_{n\in\mathbb{N}}$ ψ-converges to y, but $\{x_n\}$ is not ψ-convergent to x. Thus $y \notin \psi\text{-}cl(\{x\})$ (by hypothesis). Hence $\psi\text{-}cl(\{x\}) \subseteq U$, proving X to be $\psi\text{-}R_0$.

3. R_1 spaces

Definition 3.1. Let ψ be an operation on a topological space X. The space X is said to be $\psi\text{-}R_1$ if for $x, y \in X$ with $\psi\text{-}cl(\{x\}) \neq \psi\text{-}cl(\{y\})$, there exist disjoint ψ-open sets U and V such that $\psi\text{-}cl(\{x\}) \subseteq U$ and $\psi\text{-}cl(\{y\}) \subseteq V$.

Observation 3.2. It is easy to check that the above definition of a $\psi\text{-}R_1$ space unifies the existing definitions of R_1, pre-R_1, semi-R_1, δ-pre-R_1, α-R_1, and β-R_1 spaces if the operators int, $intcl$, $clint$, $intcl_\delta$, $intclint$, $clintcl$ respectively take the role of ψ in the above definition (refer to the papers [22], [2], [7], [3], [4], [29] respectively).

Theorem 3.3. If a space X is $\psi\text{-}R_1$, for some $\psi \in \mathcal{O}(X)$, then X is $\psi\text{-}R_0$.

Proof. Let $U \in \psi\mathcal{O}(X)$ and $x \in U$. If $y \notin U$ then $\psi\text{-}cl(\{x\}) \neq \psi\text{-}cl(\{y\})$ (as $x \notin \psi\text{-}cl(\{y\})$). Hence $\exists\ V \in \psi\mathcal{O}(X)$ such that $\psi\text{-}cl(\{y\}) \subseteq V$ and $x \notin V$. This gives $y \notin \psi\text{-}cl(\{x\})$, proving that $\psi\text{-}cl(\{x\}) \subseteq U$. So X is a $\psi\text{-}R_0$ space.

Theorem 3.4. Let ψ be an operation on a topological space X. Then the following are equivalent :

(a) X is $\psi\text{-}T_2$.

(b) X is $\psi\text{-}R_1$ and $\psi\text{-}T_1$.

(c) X is $\psi\text{-}R_1$ and $\psi\text{-}T_0$.

Proof. **(a) $\Rightarrow$ (b) :** Let X be $\psi\text{-}T_2$. Then X is clearly $\psi\text{-}T_1$. Now if $x, y \in X$ with $\psi\text{-}cl(\{x\}) \neq \psi\text{-}cl(\{y\})$, then $x \neq y$ and so $\exists\ U, V \in \psi\mathcal{O}(X)$ such that $x \in U$,

$y \in V$ and $U \cap V = \varnothing$. Hence by Theorem 2.22, ψ-$cl(\{x\}) = \{x\} \subseteq U$ and ψ-$cl(\{y\}) = \{y\} \subseteq V$ and $U \cap V = \varnothing$. Thus X is ψ-R_1.

(b) $\Rightarrow$ (c) : It is obvious.

(c) $\Rightarrow$ (a) : Let X be ψ-R_1 and ψ-T_0. Then by Theorem 3.3, X is ψ-R_0 and ψ-T_0. Thus X is ψ-T_1 (by Theorem 2.22). Let $x, y \in X$ with $x \neq y$. Then ψ-$cl(\{x\}) = \{x\} \neq \{y\} = \psi$-$cl(\{y\})$. As X is ψ-R_1, there exist $U, V \in \psi O(X)$ such that ψ-$cl(\{x)\}) = \{x\} \subseteq U$, ψ-$cl(\{y\}) = \{y\} \subseteq V$ and $U \cap V = \varnothing$. Thus X is ψ-T_2.

Theorem 3.5. Let ψ be an operation on a topological space X. Then the following are equivalent :
(a) X is ψ-R_1
(b) For any $x, y \in X$, one of the following holds :
(i) for $U \in \psi\mathcal{O}(X)$, $x \in U$ iff $y \in U$;
(ii) $\exists$ disjoint ψ-open sets U and V such that $x \in U$, $y \in V$.
(c) If $x, y \in X$ such that ψ-$cl(\{x\}) \neq \psi$-$cl(\{y\})$, then $\exists$ ψ-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

Proof. (a) $\Rightarrow$ (b) : Let $x, y \in X$. Then ψ-$cl(\{x\}) = \psi$-$cl(\{y\})$ or ψ-$cl(\{x\}) \neq \psi$-$cl(\{y\})$. If ψ-$cl(\{x\}) = \psi$-$cl(\{y\})$ and $U \in \psi\mathcal{O}(X)$, then $x \in U \Rightarrow y \in \psi$-$cl(\{y\}) = \psi$-$cl(\{x\}) \subseteq U$ (as X is ψ-R_0). If ψ-$cl(\{x\}) \neq \psi$-$cl(\{y\})$, then $\exists$ $U, V \in \psi\mathcal{O}(X)$ such that $x \in \psi$-$cl(\{x\}) \subseteq U$, $y \in \psi$-$cl(\{y\}) \subseteq V$ and $U \cap V = \varnothing$.

(b) $\Rightarrow$ (c) : Let $x, y \in X$ such that ψ-$cl(\{x\}) \neq \psi$-$cl(\{y\})$. Then $x \notin \psi$-$cl(\{y\})$, so that $\exists$ $G \in \psi\mathcal{O}(X)$ such that $x \in G$ and $y \notin G$. Thus by (b), $\exists$ disjoint ψ-open sets U and V such that $x \in U$, $y \in V$. Put $F_1 = X \setminus V$ and $F_2 = X \setminus U$. Then $F_1, F_2 \in \psi\mathcal{C}(X)$, $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

(c) $\Rightarrow$ (a) : Let $U \in \psi\mathcal{O}(X)$ and $x \in U$. Then ψ-$cl(\{x\}) \subseteq U$. In fact, otherwise $\exists$ $y \in \psi$-$cl(\{x\}) \cap (X \setminus U)$. Then ψ-$cl(\{x\}) \neq \psi$-$cl(\{y\})$ (as $x \notin \psi$-$cl(\{y\})$) and so by (c), $\exists$ $F_1, F_2 \in \psi\mathcal{C}(X)$ such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$. Then $y \in F_2 \setminus F_1 = X \setminus F_1$ and $x \notin X \setminus F_1$, where $X \setminus F_1 \in \psi\mathcal{O}(X)$, which is a contradiction to the fact that $y \in \psi$-$cl(\{x\})$. Hence ψ-$cl(\{x\}) \subseteq U$. Thus X is ψ-R_0. To show X to be ψ-R_1 assume that $a, b \in X$ with ψ-$cl(\{a\}) \neq \psi$-$cl(\{b\})$. Then as above, $\exists$ $P_1, P_2 \in \psi\mathcal{C}(X)$ such that $a \in P_1$, $b \notin P_1$, $b \in P_2$, $a \notin P_2$ and $X = P_1 \cup P_2$. Thus $a \in P_1 \setminus P_2 \in \psi\mathcal{O}(X)$, $b \in P_2 \setminus P_1 \in \psi\mathcal{O}(X)$. So ψ-$cl(\{a\}) \subseteq P_1 \setminus P_2$, ψ-$cl(\{b\}) \subseteq P_2 \setminus P_1$. Thus X is ψ-R_1.

In view of Theorem 3.4 and 3.5, it now follows that

Theorem 3.6. Let ψ be an operation on a space X. Then X is ψ-T_2 iff for $x, y \in X$ with $x \neq y$, there exist ψ-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

Theorem 3.7. Let ψ be an operation on a topological space (X, τ). Then X is ψ-R_1 iff for $x, y \in X$, with ψ-$ker(\{x\}) \neq \psi$-$ker(\{y\})$, there exist disjoint ψ-open sets U and V such that ψ-$cl(\{x\}) \subseteq U$ and ψ-$cl(\{y\}) \subseteq V$.

Proof. Follows from Theorem 2.14 and Definition 3.1.

Conclusion : As the concluding observations we append below two tables; the first one shows the existing definitions that have been unified by our introduced ones while the second table demonstrates the different results obtained in different existing papers and which follow as special cases of their unified versions obtained in this paper.

ψ	τ [7,9]	$PO(X)$ [2]	$SO(X)$ [8]	δ-$PO(X)$ [3]	$\beta(X)$ [29]	$\alpha(X)$ [4]
Def. 2.7	Def. 1.2	Def. 2		Def. 10	Def. 4.1	Def. 10
Def. 2.10	Def. 1.1	Def. 3	Def. 1.6	Def. 12	Def. 4.4	Def. 12
Def. 2.20	Well-known	Page 19	Def. 1.4 and 1.5	Def. 6	Preliminaries	Preliminaries
Def. 2.24	Def. 1.2	Def. 4	Def. 1.8		Page 60	Page 11
Def. 3.1	Def. 1.3	Def. 5	Def. 1.6	Def. 13	Def. 4.17	Def. 13

Table - 1

ψ	τ [7,9]	$PO(X)$ [2]	$SO(X)$ [8]	δ-$PO(X)$ [3]	$\beta(X)$ [29]	$\alpha(X)$ [4]
Lem. 2.9		Lem. 3.2		Lem. 4.1	Lem. 4.3	Lem. 4.1
Th. 2.12		Prop. 3.4			Th. 4.9	
Lem. 2.13		Lem. 3.1			Lem. 4.7	Lem. 4.6
Th. 2.14	Lem. 2.1	Lem. 3.6			Lem. 4.8	Lem. 4.7
Th. 2.15	Th. 2.1	Th. 3.7			Th. 4.10	Th. 4.9
Th. 2.16	Th. 2.2	Th. 3.8		Th. 5.5	Th. 4.11	Th. 4.10
Cor. 2.17		Cor. 3.9			Cor. 4.12	Cor. 4.11
Th. 2.18		Th. 3.12			Th. 4.13	Th. 4.12
Th. 2.19		Th. 3.13		Th. 5.6		Th. 4.13
Th. 2.22	Cor.(of page 889) [6]	Th. 3.11	Th. 2.2	Cor. 5.3		
Lem. 2.26	Lem. 2.1	Lem. 3.14	Lem. 3.1		Lem. 4.14	Lem. 4.14
Th. 2.27	Th. 2.7	Th. 3.15	Th. 3.1		Th. 4.15	Th. 4.15
Th. 3.3		Prop. 4.1	Th. 2.1	Cor. 6.3	Th. 4.16	Page 12
Th. 3.4		Th. 4.2	Th. 2.2	Th. 6.2		
Th. 3.5	Th. 3.1	Th. 4.3	Th. 2.3	Th. 6.4		
Th. 3.6			Th. 2.4			
Th. 3.7		Th. 4.4			Th. 4.19	Th. 4.17

Table - 2

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