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Exponential decay of serially connected elastic wave *

Luci Harue Fatori and Carolina Lupifierio Antonio

ABSTRACT: In this work we study a flexible structures which is formed by three serially connected elastic waves, more specifically on structure whose material consist of three different types of components where one is purely elastic component and two dissipative elastic. We show that for this types of materials the dissipation produced by the dissipative elastic part is strong enough to produce exponential decay of the solution, no matter how small is its size. We also show that the linear model is well posed.

Key Words: Transmission problem, mixed materials, exponential decay.

Contents

1. Introduction

Many flexible structures consist of a large number of components coupled end to end in the form of a chain. In this paper, we consider the simplest type of such structures which is formed by three serially connected wave propagation, more specifically we study the transversal vibrations for composite elastic strings of the material consisting of three different types of components. One component is a simple elastic part while the others are dissipative where dissipation of frictional type. In this case, the dissipation are effective only in a part of the domain. Model mathematical result is known as a transmission problem and is characterized by a system of partial differential equations with discontinuous coefficients. Several authors have studied problems transmission in materials made of two components (see, for example, References [1,3,14]). On the other hand, work with materials consisting of three or more components are not common in the literature. Among them we can cite the work of A. Marzocchi, J.E.M. Rivera and M.G. Naso [9], where authors showed stability results for a material consists of two components with thermoelastic properties, and one component at any temperature. In this sense, what we propose in this work is to study the wave propagation on a material consisting of three elastic components, which initially considered two of them with frictional dissipation. Then replaced by a dissipation of a thermal dissipation. More

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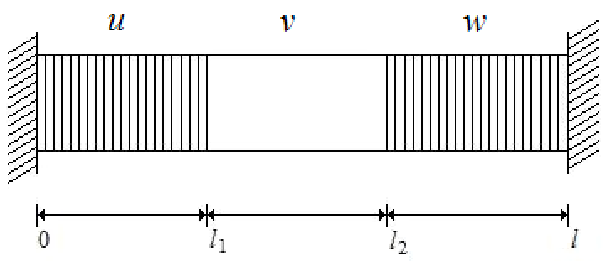
specifically, we consider a one-dimensional string defined on the interval $[0, l] \subset \mathbb{R}$, with the following composition:



where l_1, $l_2 \in (0, l)$, with $l_1 < l_2$. The system we will consider here is

$$u_{tt} - k_1 u_{xx} + a u_t = 0, \quad x \in (0, l_1),\ t > 0, \tag{1}$$
$$v_{tt} - k_2 v_{xx} = 0, \quad x \in (l_1, l_2),\ t > 0, \tag{2}$$
$$w_{tt} - k_3 w_{xx} + b w_t = 0, \quad x \in (l_2, l),\ t > 0, \tag{3}$$

where k_1, k_2, k_3, a and b are positive constants.

The functions $u = u(x,t)$, $v = v(x,t)$ and $w = w(x,t)$ satisfying the following boundary conditions

$$u(0,t) = w(l,t) = 0, \quad t > 0, \tag{4}$$

transmission condition

$$u(l_1,t) = v(l_1,t), \quad k_1 u_x(l_1,t) = k_2 v_x(l_1,t), \quad t > 0, \tag{5}$$
$$v(l_2,t) = w(l_2,t), \quad k_2 v_x(l_2,t) = k_3 w_x(l_2,t), \quad t > 0, \tag{6}$$

and initial conditions

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad x \in (0, l_1), \tag{7}$$
$$v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x), \quad x \in (l_1, l_2), \tag{8}$$
$$w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), \quad x \in (l_2, l). \tag{9}$$

Let us mention some papers related to problems we address. The asymptotic behavior as $t \to \infty$ of solution to the wave equation with different types of dissipative mechanism has been studied by many authors. For example, the frictional damping αu_t with dissipation works in the whole domain (see Reference [2]), or frictional boundary conditions as the work of [7,13] where the dissipation is working in a part of the boundary where the dissipation is working in a part of the boundary and also where the frictional damping is localized (see References [11,12,15]). In this sense, we can say that our contribution was to establish the exponential decay of the solution when time goes to infinity of a wave equation with discontinuous coefficients and frictional damping is localized because the system (1)-(9) is equivalent to the problem

$$z_{tt} - k(x) z_{xx} + c(x) z_t = 0 \quad \text{em } (0, l) \times (0, \infty)$$

with boundary condition

$$z(0,t) = z(l,t) = 0, \quad t > 0,$$

and initial condition

$$z(x,0) = z^0(x), \quad z_t(x,0) = z^1(x), \quad x \in (0,l)$$

where

$$z(x,t) = \begin{cases} u(x,t), & \text{if} \quad x \in (0,l_1) \\ v(x,t), & \text{if} \quad x \in (l_1,l_2) \\ w(x,t), & \text{if} \quad x \in (l_2,l) \end{cases}$$

$$k(x) = \begin{cases} k_1, & \text{if} \quad x \in (0,l_1) \\ k_2, & \text{if} \quad x \in (l_1,l_2) \\ k_3, & \text{if} \quad x \in (l_2,l) \end{cases} \qquad c(x) = \begin{cases} a, & \text{if} \quad x \in (0,l_1) \\ 0, & \text{if} \quad x \in (l_1,l_2) \\ b, & \text{if} \quad x \in (l_2,l). \end{cases}$$

We denote by Ω the set $(0,l_1) \cup (l_1,l_2) \cup (l_2,l)$ and $\mathcal{L}^2(\Omega)$, $\mathcal{H}^1(\Omega)$, $\mathcal{H}^2(\Omega)$ and $\mathcal{V}$ the spaces

$$\begin{aligned} \mathcal{L}^2(\Omega) &= L^2(0,l_1) \times L^2(l_1,l_2) \times L^2(l_2,l), \\ \mathcal{H}^1(\Omega) &= H^1(0,l_1) \times H^1(l_1,l_2) \times H^1(l_2,l), \\ \mathcal{H}^2(\Omega) &= H^2(0,l_1) \times H^2(l_1,l_2) \times H^2(l_2,l), \\ \mathcal{V} &= \{(u,v,w) \in \mathcal{H}^1(\Omega) : u(0) = w(l) = 0,\ u(l_1) = v(l_1),\ v(l_2) = w(l_2)\}. \end{aligned}$$

Observe that $\mathcal{V}$ is a Hilbert space with the norm

$$||(u,v,w)||^2_{\mathcal{V}} := \int_0^{l_1} |u_x|^2\,dx + \int_{l_1}^{l_2} (|v|^2 + |v_x|^2)\,dx + \int_{l_2}^{l} |w_x|^2\,dx.$$

The weak solutions of (1) − (9) are defined as follows

Definition 1.1 *The triple (u,v,w) is a weak solution of the system (1)−(9) when*

$$\begin{aligned} &(u,v,w) \in L^\infty(0,T;\mathcal{V}), \\ &(u_t,v_t,w_t) \in L^\infty(0,T;\mathcal{L}^2(\Omega)), \end{aligned}$$

and satisfies

$$\begin{aligned} &\frac{d}{dt}\int_0^{l_1} u_t\phi\,dx + k_1\int_0^{l_1} u_x\phi_x\,dx + a\int_0^{l_1} u_t\phi\,dx + \frac{d}{dt}\int_{l_1}^{l_2} v_t\psi\,dx \\ &+k_2\int_{l_1}^{l_2} v_x\psi_x\,dx + \frac{d}{dt}\int_{l_2}^{l} w_t\varphi\,dx + k_3\int_{l_2}^{l} w_x\varphi_x\,dx + b\int_{l_2}^{l} w_t\varphi\,dx = 0, \end{aligned}$$

in $\mathcal{D}'(0,T)$ for all $(\phi,\psi,\varphi) \in \mathcal{V}$.

For the existence result is

Theorem 1.1 *Suppose that the initial data* $(u^0, v^0, w^0) \in \mathcal{V}$, $(u^1, v^1, w^1) \in \mathcal{L}^2(\Omega)$ *and satisfy* (5) − (6)*. Then problem* (1) − (9) *has a unique weak solution* (u, v, w)*. Moreover, if* $(u^0, v^0, w^0) \in \mathcal{H}^2(\Omega) \cap \mathcal{V}$ *e* $(u^1, v^1, w^1) \in \mathcal{V}$*, then the solution satisfies*

$$(u, v, w) \in L^\infty(0, T; \mathcal{H}^2(\Omega) \cap \mathcal{V}), \qquad (u_t, v_t, w_t) \in L^\infty(0, T; \mathcal{V}),$$
$$(u_{tt}, v_{tt}, w_{tt}) \in L^\infty(0, T; \mathcal{L}^2(\Omega)).$$

In this case, we say that (u, v, w) *is a strong solution to the problem* (1) − (9)*.*

In the following we define the energy of the system (1) − (9)

$$E(t; u, v, w) = E_1(t; u) + E_2(t; v) + E_3(t; w) \tag{10}$$

where E_1, E_2 and E_3 we denote the first order energy associated to each equation,

$$E_1(t; u) = \frac{1}{2}\int_0^{l_1} |u_t|^2 + k_1|u_x|^2 dx,$$
$$E_2(t; v) = \frac{1}{2}\int_{l_1}^{l_2} |v_t|^2 + k_2|v_x|^2 dx,$$
$$E_3(t; w) = \frac{1}{2}\int_{l_2}^{l} |w_t|^2 + k_3|w_x|^2 dx.$$

Using the same procedure as in [4] we have our main result.

Theorem 1.2 *Let* (u, v, w) *be a strong solution of* (1) − (9) *given by Theorem 1.1. Then there exist positive constants* C_0 *and* γ *such that*

$$E(t; u, v, w) \le C_0 \mathcal{E}(0) e^{-2\gamma t},$$

where $\mathcal{E}(0)$ *will be defined later.*

2. Existence and Regularity

In this section we give the proof Theorem 1.1. We only show the main arguments of the proof which was based on the Faedo-Galerkin method.

Proof of Theorem 1.1: Let us denote by $\{(\phi^i, \psi^i, \varphi^i),\ i \in \mathbb{N}\}$ an orthonormal basis of $\mathcal{V}$, $V_m = \text{span}\{(\phi^1, \psi^1, \varphi^1), ..., (\phi^m, \psi^m, \varphi^m)\}$ and

$$(u^m(t), v^m(t), w^m(t)) = \sum_{j=1}^{m} h_{j,m}(t)(\phi^j, \psi^j, \varphi^j)$$

where the functions $(u^m(t), v^m(t), w^m(t))$ are given by the solution of the approximate system

$$\int_0^{l_1} u_{tt}^m \phi^j\, dx + k_1 \int_0^{l_1} u_x^m \phi_x^j\, dx + a\int_0^{l_1} u_t^m \phi^j\, dx + \int_{l_1}^{l_2} v_{tt}^m \psi^j\, dx + k_2 \int_{l_1}^{l_2} v_x^m \psi_x^j\, dx$$
$$+ \int_{l_2}^{l} w_{tt}^m \varphi^j\, dx + k_3 \int_{l_2}^{l} w_x^m \varphi_x^j\, dx + b\int_{l_2}^{l} w_t^m \varphi^j\, dx = 0, \tag{11}$$

$j=1,...,m$, with initial data

$$(u^m(0),v^m(0),w^m(0))=(u^0_m,v^0_m,w^0_m)\to(u^0,v^0,w^0)\quad in\quad \mathcal{V}, \tag{12}$$
$$(u^m_t(0),v^m_t(0),w^m_t(0))=(u^1_m,v^1_m,w^1_m)\to(u^1,v^1,w^1)\quad in\quad \mathcal{L}^2(\Omega). \tag{13}$$

Then from standard arguments on ODEs the system (11)−(13) has a local solution in t. To extend this solution to the whole interval $[0,\infty)$ it is enough to show that approximate solutions are bounded independently of m e t.

Let us define

$$E^m(t):=E(t,u^m,v^m,w^m).$$

Multiplying equation (11) by $h'_{j,m}(t)$, summing up on j and integrating from 0 to t, we get

$$E^m(t)=E^m(0)-a\int_0^t\int_0^{l_1}|u^m_t|^2\,dxdt-b\int_0^t\int_{l_2}^{l}|w^m_t|^2\,dxdt.$$

Therefore, there exists $M_1>0$ such that

$$E^m(t)\le M_1,\quad \forall\, m\in\mathbb{N},\ \forall\, t\in[0,T]. \tag{14}$$

Our next step is to estimate the second order energy. Differentiating relation (11) with respect to t and multiplying $h''_{j,m}(t)$, summing up on j we get

$$\frac{1}{2}\frac{d}{dt}\left\{\int_0^{l_1}|u^m_{tt}|^2+k_1|u^m_{xt}|^2\,dx+\int_{l_1}^{l_2}|v^m_{tt}|^2+k_2|v^m_{xt}|^2\,dx+\int_{l_2}^{l}|w^m_{tt}|^2+k_3|w^m_{xt}|^2\,dx\right\}$$
$$=-a\int_0^{l_1}|u^m_{tt}|^2\,dx-b\int_{l_2}^{l}|w^m_{tt}|^2\,dx.$$

Let us denote by $\mathcal{E}^m(t):=E(t,u^m_t,v^m_t,w^m_t)$, we obtain

$$\frac{d}{dt}\mathcal{E}^m(t)=-a\int_0^{l_1}|u^m_{tt}|^2\,dx-b\int_{l_2}^{l}|w^m_{tt}|^2\,dx$$

and integrating from 0 to t following that

$$\mathcal{E}^m(t)\le\mathcal{E}^m(0). \tag{15}$$

Now we must show that $\mathcal{E}^m(0)$ is bounded. In order to, multiplying (11) by $h''_{j,m}(t)$, summing up on j and letting $t\to0^+$ we get

$$\int_0^{l_1}|u^m_{tt}(0)|^2\,dx+\int_{l_1}^{l_2}|v^m_{tt}(0)|^2\,dx+\int_{l_2}^{l}|w^m_{tt}(0)|^2\,dx=-\int_{l_1}^{l_2}k_2v^m_x(0)v^m_{xtt}(0)\,dx$$
$$-\int_0^{l_1}(k_1u^m_x(0)u^m_{xtt}(0)+au^m_t(0)u^m_{tt}(0))\,dx-\int_{l_2}^{l}(k_3w^m_x(0)w^m_{xtt}(0)+bw^m_t(0)w^m_{tt}(0))\,dx.$$

After integrating, using the transmission condition and Youngťs Inequality we get there exist a positive constant $C > 0$ such that

$$\int_0^{l_1} |u_{tt}^m(0)|^2\,dx + \int_{l_1}^{l_2} |v_{tt}^m(0)|^2\,dx + \int_{l_2}^{l} |w_{tt}^m(0)|^2\,dx \leq$$
$$C\Big(\int_0^{l_1} |u_{xx}^m(0)|^2 + |u_t^m(0)|^2\,dx + \int_{l_1}^{l_2} |v_{xx}^m(0)|^2\,dx + \int_{l_2}^{l} |w_{xx}^m(0)|^2 + |w_t^m(0)|^2\,dx\Big).$$

This implies that the initial data satisfies

$$(u_{tt}^m(0), v_{tt}^m(0), w_{tt}^m(0)) \quad \text{is bounded in} \quad \mathcal{L}^2(\Omega),$$

and so is $\mathcal{E}^m(0)$. Whence that there exist $M_2 > 0$ such that

$$\mathcal{E}^m(t) \leq M_2, \quad \forall\, m \in \mathbb{N},\ \forall\, t \in [0,T]. \tag{16}$$

From (14) and (16) we see that there exists a subsequence of (u^m, v^m, w^m), still denoted by (u^m, v^m, w^m) such that

$$\begin{array}{lcl} (u^m, v^m, w^m) & \overset{*}{\rightharpoonup} & (u, v, w) \in L^\infty(0,T;\mathcal{V}), \\ (u_t^m, v_t^m, w_t^m) & \overset{*}{\rightharpoonup} & (u_t, v_t, w_t) \in L^\infty(0,T;\mathcal{V}). \\ (u_{tt}^m, v_{tt}^m, w_{tt}^m) & \overset{*}{\rightharpoonup} & (u_{tt}, v_{tt}, w_{tt}) \in L^\infty(0,T;\mathcal{L}^2(\Omega)). \end{array}$$

From this, letting $m \to \infty$ in (11) we conclude that

$$\int_0^T\!\!\int_0^{l_1} u_{tt}\vartheta_1\,dxdt + k_1\int_0^T\!\!\int_0^{l_1} u_x\vartheta_{1,x}\,dxdt + a\int_0^T\!\!\int_0^{l_1} u_t\vartheta_1\,dxdt$$
$$+\int_0^T\!\!\int_{l_1}^{l_2} v_{tt}\vartheta_2\,dxdt + k_2\int_0^T\!\!\int_{l_1}^{l_2} v_x\vartheta_{2,x}\,dxdt$$
$$+\int_0^T\!\!\int_{l_2}^{l} w_{tt}\vartheta_3\,dxdt + k_3\int_0^T\!\!\int_{l_2}^{l} w_x\vartheta_{3,x}\,dxdt + b\int_0^T\!\!\int_{l_2}^{l} w_t\vartheta_3\,dxdt = 0,$$

for all $(\vartheta_1, \vartheta_2, \vartheta_3) \in \mathcal{D}(0,T;\mathcal{D}(\Omega))$. Therefore we have that

$$\begin{array}{l} (u, v, w) \in L^\infty(0,T;\mathcal{H}^2(\Omega) \cap \mathcal{V}), \\ (u_t, v_t, w_t) \in L^\infty(0,T;\mathcal{V}), \\ (u_{tt}, v_{tt}, w_{tt}) \in L^\infty(0,T;\mathcal{L}^2(\Omega)). \end{array}$$

Verification of the initial and transmission conditions are a matter of routine. The uniqueness to weak solution we follows by Visik-Ladyzhenskaya methods and to strong solution follows by standard methods for hyperbolic equations. This ends the proof of Theorem 1.1. □

3. Exponential decay

In this section we prove by using multipliers techniques the solution (1) – (9) decays exponentially to zero as time goes to infinity. To do this, let us denote by $U(x,t) = u(x,t)e^{\gamma t}$, $V(x,t) = v(x,t)e^{\gamma t}$ and $W(x,t) = w(x,t)e^{\gamma t}$. Then (U,V,W) satisfies

$$U_{tt} - k_1 U_{xx} + aU_t = Q, \quad x \in (0,l_1),\ t > 0, \tag{17}$$
$$V_{tt} - k_2 V_{xx} = R, \quad x \in (l_1,l_2),\ t > 0, \tag{18}$$
$$W_{tt} - k_3 W_{xx} + bW_t = S, \quad x \in (l_2,l),\ t > 0, \tag{19}$$

where

$$Q := 2\gamma U_t + (a-\gamma)\gamma U, \tag{20}$$
$$R := 2\gamma V_t - \gamma^2 V, \tag{21}$$
$$S := 2\gamma W_t + (b-\gamma)\gamma W. \tag{22}$$

The functions U, V and W satisfying the following boundary condition

$$U(0,t) = W(l,t) = 0, \quad t > 0 \tag{23}$$

transmission condition

$$U(l_1,t) = V(l_1,t), \quad k_1 U_x(l_1,t) = k_2 V_x(l_1,t), \quad t > 0, \tag{24}$$
$$V(l_2,t) = W(l_2,t), \quad k_2 V_x(l_2,t) = k_3 W_x(l_2,t), \quad t > 0, \tag{25}$$

and initial condition

$$U(x,0) = u^0(x), \quad U_t(x,0) = u^1(x) + \gamma u^0(x), \quad x \in (0,l_1), \tag{26}$$
$$V(x,0) = v^0(x), \quad V_t(x,0) = v^1(x) + \gamma v^0(x) \quad x \in (l_1,l_2), \tag{27}$$
$$W(x,0) = w^0(x), \quad W_t(x,0) = w^1(x) + \gamma w^0(x), \quad x \in (l_2,l). \tag{28}$$

Let us consider

$$\mathcal{E}(t) := E(t;U,V,W) = E_1(t;U) + E_2(t;V) + E_3(t;W)$$

where $E(t;U,V,W)$ is given by (10). In order to show the exponential decay of (u,v,w) is enough to show that $\mathcal{E}(t)$ is limited. To this end, prove a series of results. Now we consider (u,v,w) strong solution of (1) - (9). In our arguments (Lemma 3.11) we make use of a convergence result due to Kim [5] and result related to the wave equation (vide [10]), which is recalled below.

Lemma 3.1 *Let us denote by $\{w^k\}$ a sequence of functions satisfying*

$$w^k \overset{*}{\rightharpoonup} w \quad in \quad L^\infty(0,T;H^\beta(\Omega)),$$
$$w_t^k \rightharpoonup w_t \quad in \quad L^2(0,T;H^\theta(\Omega)),$$

as $k \to \infty$, where $\theta < \beta$. Then we have that

$$w^k \to w \ in \ C([0,T];H^r(\Omega)),$$

for any $r < \beta$.

Lemma 3.2 *Suppose that the initial data* $z^0 \in H_0^1(0,l)$, $z^1 \in L^2(0,l)$ *and* $z : (0,l) \times (0,T) \to \mathbb{R}$ *is the solution of the problem*

$$\begin{cases} z_{tt} - kz_{xx} = 0, \\ z(0,t) = z(l,t) = 0, \\ z_x(0,t) = z_x(l,t) = 0, \\ z(x,0) = z^0(x),\ z_t(x,0) = z^1(x). \end{cases} \tag{29}$$

Then, $z = 0$ *a. e. in* $(0,l) \times (0,T)$.

Lemma 3.3 *Let* (U, V, W) *be a solution of (17)-(28) then there is exist positive constant* C *such that*

$$\frac{d}{dt}\mathcal{E}(t) \le -a\int_0^{l_1} |U_t|^2 dx - b\int_{l_2}^{l} |W_t|^2 dx + C\gamma\mathcal{E}(t).$$

Proof: Multiplying equation (17), (18) and (19) by U_t, V_t and W_t, respectively, and integrating by parts from 0 to l_1 , from l_1 to l_2 and l_2 to l, we conclude using the boundary and transmission conditions that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= -a\int_0^{l_1} |U_t|^2\, dx - b\int_{l_2}^{l} |W_t|^2\, dx \\ &\quad + \int_0^{l_1} QU_t\, dx \int_{l_1}^{l_2} RV_t\, dx + \int_{l_2}^{l} SW_t\, dx. \end{aligned} \tag{30}$$

From (20), (21), (22) and using Hölder's, Young's and Poincare's inequalities, we find

$$\int_0^{l_1} QU_t\, dx + \int_{l_1}^{l_2} RV_t\, dx + \int_{l_2}^{l} SW_t\, dx \quad \le \quad C\gamma\mathcal{E}(t), \tag{31}$$

where C is a positive constant. Therefore combining (30) and (31) is follows that

$$\frac{d}{dt}\mathcal{E}(t) \le -a\int_0^{l_1} |U_t|^2 dx - b\int_{l_2}^{l} |W_t|^2 dx + C\gamma\mathcal{E}(t).$$

□

Lemma 3.4 *Let* (U, V, W) *be a solution of (17)-(28) and consider the functionals*

$$F_1(t) = \int_0^{l_1} UU_t\, dx \qquad F_3(t) = \int_{l_2}^{l} WW_t\, dx.$$

Then there exists a positive constant C_1 *and* C_2 *such that*

$$\begin{aligned} \frac{d}{dt}F_1(t) &\le C_1\gamma\mathcal{E}(t) + C_1\int_0^{l_1} |U_t|^2\, dx - \frac{7k_1}{8}\int_0^{l_1} |U_x|^2\, dx + k_1U_x(l_1,t)U(l_1,t), \\ \frac{d}{dt}F_3(t) &\le C_2\gamma\mathcal{E}(t) + C_2\int_{l_2}^{l} |W_t|^2\, dx - \frac{7k_3}{8}\int_{l_2}^{l} |W_x|^2\, dx - k_3W_x(l_2,t)W(l_2,t). \end{aligned}$$

Proof: From (17) we get

$$\frac{d}{dt}F_1(t) = \int_0^{l_1} |U_t|^2\,dx + k_1\int_0^{l_1} U_{xx}U\,dx - a\int_0^{l_1} U_tU\,dx + \int_0^{l_1} QU\,dx.$$

After integrating by parts, using boundary condition and Young's inequality we get

$$\begin{aligned}\frac{d}{dt}F_1(t) \quad &\leq \quad \left(1+\frac{a}{2\epsilon_1}\right)\int_0^{l_1} |U_t|^2\,dx + k_1U_x(l_1,t)U(l_1,t) - k_1\int_0^{l_1} |U_x|^2\,dx \\ &\quad + \frac{a\epsilon_1 c_p}{2}\int_0^{l_1} |U_x|^2\,dx + \int_0^{l_1} QU\,dx. \end{aligned} \tag{32}$$

where ϵ_1 is a positive constant satisfying $\epsilon_1 < \dfrac{k_1}{4ac_p}$ and c_p is Poincaret́s constant. On the other side, from (20) is easy to see that

$$\int_0^{l_1} QU\,dx \quad \leq \quad C\gamma E_1(t;U), \tag{33}$$

where C is a positive constant. Combining (32) and (33) our first conclusion follows. Similarly, from (19) we get

$$\frac{d}{dt}F_3(t) = \int_{l_2}^{l} |W_t|^2\,dx + k_3\int_{l_2}^{l} W_{xx}W\,dx - b\int_{l_2}^{l} W_tW\,dx + \int_{l_2}^{l} SW\,dx.$$

After integrating by parts, using boundary condition and Young's inequality we get

$$\begin{aligned}\frac{d}{dt}F_3(t) \quad &\leq \quad \left(1+\frac{b}{2\epsilon_3}\right)\int_{l_2}^{l} |W_t|^2\,dx - k_3W_x(l_2,t)W(l_2,t) - k_3\int_{l_2}^{l} |W_x|^2\,dx \\ &\quad + \frac{b\epsilon_3 c_p}{2}\int_{l_2}^{l} |W_x|^2\,dx + \int_{l_2}^{l} SW\,dx, \end{aligned} \tag{34}$$

where ϵ_3 is a positive constant satisfying $\epsilon_3 < \dfrac{k_3}{4bc_p}$. On the other side, from (22) we have

$$\int_{l_2}^{l} SW\,dx \leq C\gamma E_3(t;W).$$

Replacing this inequality in (34) our second conclusion follows. □

Lemma 3.5 *Let (U,V,W) be a solution of (17)-(28) and let functional $J_1(t)$ given by*

$$J_1(t) = -\int_0^{l_1} x\,U_xU_t\,dx.$$

Then there exist a positive constant $C_3 > 0$ such that

$$\begin{aligned}\frac{d}{dt}J_1(t) &\leq C_3\gamma\mathcal{E}(t) + C_3\int_0^{l_1}|U_t|^2\,dx + \frac{5k_1}{8}\int_0^{l_1}|U_x|^2\,dx\\ &\quad -\frac{l_1}{2}|U_t(l_1,t)|^2 - \frac{k_1 l_1}{2}|U_x(l_1,t)|^2.\end{aligned}$$

Proof: Multiplying the equation (17) by $\sigma_1(x)U_x$, $\sigma_1 \in C^1(0,l_1)$, and integrating from 0 to l_1, we have

$$\begin{aligned}\frac{d}{dt}\left\{\int_0^{l_1}\sigma_1(x)U_xU_t\,dx\right\} &= \frac{1}{2}\int_0^{l_1}\sigma_1(x)\frac{d}{dx}|U_t|^2\,dx + \frac{k_1}{2}\int_0^{l_1}\sigma_1(x)\frac{d}{dx}|U_x|^2\,dx\\ &\quad -a\int_0^{l_1}\sigma_1(x)U_xU_t\,dx + \int_0^{l_1}\sigma_1(x)U_xQ\,dx\\ &= \frac{1}{2}\left[\sigma_1(x)|U_t|^2\Big|_0^{l_1} - \int_0^{l_1}\sigma_1'(x)|U_t|^2\,dx\right] + \frac{k_1}{2}\sigma_1(x)|U_x|^2\Big|_0^{l_1}\\ &\quad -\frac{k_1}{2}\int_0^{l_1}\sigma_1'(x)|U_x|^2\,dx - a\int_0^{l_1}\sigma_1(x)U_xU_t\,dx + \int_0^{l_1}\sigma_1(x)U_xQ\,dx.\end{aligned}$$

Taking $\sigma_1(x) = -x$ and using Young's inequality we obtain

$$\begin{aligned}\frac{d}{dt}J_1(t) \leq \left(\frac{1}{2}+\frac{al_1}{2\eta}\right)\int_0^{l_1}|U_t|^2\,dx + \left(\frac{k_1}{2}+\frac{al_1\eta}{2}\right)\int_0^{l_1}|U_x|^2\,dx - \frac{l_1}{2}|U_t(l_1,t)|^2\\ -\frac{k_1l_1}{2}|U_x(l_1,t)|^2 + l_1\int_0^{l_1}|U_xQ|\,dx, \qquad (35)\end{aligned}$$

where η is a positive constant satisfying $\eta < \dfrac{k_1}{4al_1}$. On the other side, from (20) we have that there exist a positive constant C such that

$$\int_0^{l_1}|QU_x|\,dx \leq C\gamma E_1(t;U).$$

Therefore, using the last estimate in (35) our conclusion follows. □

Lemma 3.6 *Let (U,V,W) be a solution of (17)-(28) and consider the functional $J_2(t)$*

$$J_2(t) = \int_{l_1}^{l_2}\frac{(l_2+l_1)x - 2l_1l_2}{(l_2-l_1)}\,V_xV_t\,dx.$$

Then there exist a positive constant $C_4 > 0$ such that

$$\begin{aligned}\frac{d}{dt}J_2(t) &\leq -\frac{(l_2+l_1)}{l_2-l_1}E_2(t;V) + C_4\gamma\mathcal{E}(t) + \frac{l_2}{2}|V_t(l_2,t)|^2 + \frac{l_1}{2}|V_t(l_1,t)|^2\\ &\quad +\frac{k_2l_2}{2}|V_x(l_2,t)|^2 + \frac{k_2l_1}{2}|V_x(l_1,t)|^2.\end{aligned}$$

Proof: Multiplying the equation (18) by $\sigma_2(x)V_x$, $\sigma_2 \in C^1(l_1, l_2)$, and integrating from l_1 to l_2, we have

$$\frac{d}{dt}\left\{\int_{l_1}^{l_2} \sigma_2(x)V_xV_t\,dx\right\} = \frac{1}{2}\left[\sigma_2(x)|V_t|^2\Big|_{l_1}^{l_2} - \int_{l_1}^{l_2} \sigma_2'(x)|V_t|^2\,dx\right] + \frac{k_2}{2}\sigma_2(x)|V_x|^2\Big|_{l_1}^{l_2} - \frac{k_2}{2}\int_{l_1}^{l_2} \sigma_2'(x)|V_x|^2\,dx + \int_{l_1}^{l_2} \sigma_2(x)V_xR\,dx.$$

Taking $\sigma_2(x) = \dfrac{(l_2+l_1)x - 2l_1l_2}{(l_2-l_1)}$ we get

$$\frac{d}{dt}J_2(t) \leq \frac{1}{2}\left[l_2|V_t(l_2,t)|^2 + l_1|V_t(l_1,t)|^2 - \frac{(l_2+l_1)}{(l_2-l_1)}\int_{l_1}^{l_2}|V_t|^2\,dx\right] + \frac{k_2l_2}{2}|V_x(l_2,t)|^2 + \frac{k_2l_1}{2}|V_x(l_1,t)|^2 - \frac{k_2}{2}\frac{(l_2+l_1)}{(l_2-l_1)}\int_{l_1}^{l_2}|V_x|^2\,dx + l_2\int_{l_1}^{l_2}|V_xR|\,dx. \tag{36}$$

On the other side, we see that there exist positive constant C such that

$$\int_{l_1}^{l_2} |V_xR|\,dx \leq C\gamma\mathcal{E}(t).$$

Therefore, using the above inequality in (36) our conclusion follows. □

Lemma 3.7 *Consider the functional* $J_3(t)$

$$J_3(t) = \int_{l_2}^{l} \frac{l_2(l-x)}{l-l_2}\,W_xW_t\,dx$$

where (U,V,W) *be a solution of (17)-(28). Then there exist a positive constant* $C_5 > 0$ *such that*

$$\frac{d}{dt}J_3(t) \leq C_5\int_{l_2}^{l}|W_t|^2\,dx + \frac{5k_3l_2}{8(l-l_2)}\int_{l_2}^{l}|W_x|^2\,dx - \frac{l_2}{2}|W_t(l_2,t)|^2 - \frac{k_3l_2}{2}|W_x(l_2,t)|^2 + C_5\gamma\mathcal{E}(t).$$

Proof: Multiplying the equation (19) by $\sigma_3(x)W_x$, $\sigma_3 \in C^1(l_2, l)$, and integrating from l_2 to l, we have

$$\frac{d}{dt}\left\{\int_{l_2}^{l} \sigma_3(x)W_xW_t\,dx\right\} = \frac{1}{2}\left[\sigma_3(x)|W_t|^2\Big|_{l_2}^{l} - \int_{l_2}^{l} \sigma_3'(x)|W_t|^2\,dx\right] + \frac{k_3}{2}\sigma_3(x)|W_x|^2\Big|_{l_2}^{l} - \frac{k_3}{2}\int_{l_2}^{l}\sigma_3'(x)|W_x|^2\,dx - b\int_{l_2}^{l}\sigma_3(x)W_xW_t\,dx + \int_{l_2}^{l}\sigma_3(x)W_xS\,dx.$$

Taking $\sigma_3(x)=\dfrac{l_2(l-x)}{l-l_2}$ and using Young's inequality we get that there exist a positive constant C_5 such that

$$\begin{aligned}\frac{d}{dt}J_3(t) &\leq -\frac{l_2}{2}|W_t(l_2,t)|^2+C_5\int_{l_2}^{l}|W_t|^2\,dx-\frac{k_3l_2}{2}|W_x(l_2,t)|^2\\ &\quad+\frac{k_3l_2}{2(l-l_2)}\int_{l_2}^{l}|W_x|^2\,dx+\frac{bl_2\eta}{2}\int_{l_2}^{l}|W_x|^2\,dx+l_2\int_{l_2}^{l}|W_xS|\,dx,\end{aligned}\tag{37}$$

where η is a positive constant satisfying $\eta<\dfrac{k_3}{4b(l-l_2)}$. On the other side, we get there exist a positive constant C such that

$$\int_{l_2}^{l}|W_xS|\,dx \quad\leq\quad C\gamma E_3(t;W),$$

Therefore, using the last estimate in (37) our conclusion follows. □

Lemma 3.8 *Consider the functional $H_1(t)$ given by*

$$H_1(t)=F_1(t)+J_1(t)$$

where F_1 and J_1 are defined in Lemma 3.4 and 3.5. Then there exist a positive constant C_6 such that

$$\frac{d}{dt}H_1(t)\leq C_6\gamma\mathcal{E}(t)+C_6\int_0^{l_1}|U_t|^2\,dx-\frac{k_1}{4}\int_0^{l_1}|U_x|^2\,dx+C_6|U(l_1,t)|^2,$$

Proof: Combining the first estimate of the lemma 3.4 and from lemma 3.5. □

Lemma 3.9 *Let $H_2(t)$ the functional*

$$H_2(t)=J_2(t)+C_0J_1(t)+K_0J_3(t)$$

where J_1, J_2 and J_3 are functionals defined in Lemmas 3.5, 3.6 and 3.7 and the constants satisfies $C_0=\max\left\{1,\dfrac{k_1}{k_2}\right\}$ and $K_0=\max\left\{1,\dfrac{k_3}{k_2}\right\}$. Then, there exist positive constant $C_7>0$ such that

$$\begin{aligned}\frac{d}{dt}H_2(t) &\leq C_7\gamma\mathcal{E}(t)+C_7\int_0^{l_1}|U_t|^2+|U_x|^2\,dx\\ &+ C_7\int_{l_2}^{l}|W_t|^2+|W_x|^2\,dx-\frac{l_2+l_1}{l_2-l_1}E_2(t;V).\end{aligned}$$

Proof: From lemmas 3.5, 3.6, 3.7 and using the transmission conditions we have that there exist a positive constant C such that

$$\begin{aligned}\frac{d}{dt}H_2(t) &\leq C\gamma\mathcal{E}(t) + C_0C_3\int_0^{l_1}|U_t|^2\,dx + \frac{5k_1C_0}{8}\int_0^{l_1}|U_x|^2\,dx \\ &+ K_0C_5\int_{l_2}^{l}|W_t|^2\,dx + \frac{5k_3l_2K_0}{8(l-l_2)}\int_{l_2}^{l}|W_x|^2\,dx - \frac{(l_2+l_1)}{l_2-l_1}E_2(t;V) \\ &+ \frac{l_2}{2}\Big(1-K_0\Big)|V_t(l_2,t)|^2 + \frac{l_1}{2}\Big(1-C_0\Big)|V_t(l_1,t)|^2 \\ &+ \frac{k_2l_2}{2}\left(1-\frac{K_0k_2}{k_3}\right)|V_x(l_2,t)|^2 + \frac{k_2l_1}{2}\left(1-\frac{C_0k_2}{k_1}\right)|V_x(l_1,t)|^2.\end{aligned}$$

Therefore, the choice of the constants C_0 and K_0 our conclusion follows. □

Lemma 3.10 *Let the functional* $H_3(t)$

$$H_3(t) = \frac{l_2}{l-l_2}F_3(t) + J_3(t).$$

Then there exist a positive constant C_8 *such that*

$$\frac{d}{dt}H_3(t) \leq C_8\int_{l_2}^{l}|W_t|^2\,dx - \frac{k_3l_2}{4(l-l_2)}\int_{l_2}^{l}|W_x|^2\,dx + C_8|W(l_2,t)|^2 + C_8\gamma\mathcal{E}(t).$$

Proof: Using the lemmas 3.4 and 3.7 we have that there exist a positive constant C such that

$$\begin{aligned}\frac{d}{dt}H_3(t) \leq &\left(\frac{l_2}{l-l_2}C_2 + C_5\right)\int_{l_2}^{l}|W_t|^2\,dx - \frac{l_2k_3}{4(l-l_2)}\int_{l_2}^{l}|W_x|^2\,dx \\ &-\frac{k_3l_2}{l-l_2}W_x(l_2,t)W(l_2,t) - \frac{k_3l_2}{2}|W_x(l_2,t)|^2 \\ &-\frac{l_2}{2}|W_t(l_2,t)|^2 + C\gamma\mathcal{E}(t).\end{aligned}$$

Therefore, applying the Young's inequality in the last term we get to $\epsilon > 0$

$$\begin{aligned}\frac{d}{dt}H_3(t) \leq \left(\frac{l_2}{l-l_2}C_2 + C_5\right)\int_{l_2}^{l}|W_t|^2\,dx - \frac{k_3l_2}{4(l-l_2)}\int_{l_2}^{l}|W_x|^2\,dx \\ +\frac{k_3}{2\epsilon}|W(l_2,t)|^2 + C\gamma\mathcal{E}(t)\end{aligned}$$

and hence our conclusion follows. □

Lemma 3.11 *For any $\delta > 0$ there exist C_δ, independently of the initial data, such that*

$$\int_0^T |U(l_1,t)|^2\,dt + \int_0^T |W(l_2,t)|^2\,dt \le \delta \int_0^T \mathcal{E}(t)\,dt + + C_\delta \left\{ \int_0^T \int_0^{l_1} |U_t|^2\,dxdt + \int_0^T \int_{l_2}^{l} |W_t|^2\,dxdt \right\},$$

for any solution (U,V,W) of system (17) − (28), provided that T is large enough.

Proof: We argue by contradiction. Let us suppose that there exists a sequence of initial data $(U^{0,\nu}, V^{0,\nu}, W^{0,\nu}) \in \mathcal{H}^2(\Omega) \cap \mathcal{V}$ and $(U^{1,\nu}, V^{1,\nu}, W^{1,\nu}) \in \mathcal{V}$, and a positive constant δ_0 such that the corresponding solutions (U^ν, V^ν, W^ν) of system

$$U^\nu_{tt} - k_1 U^\nu_{xx} + aU^\nu_t = 2\gamma U^\nu_t + (a-\gamma)\gamma U^\nu, \qquad x \in (0,l_1), \quad t>0, \tag{38}$$
$$V^\nu_{tt} - k_2 V^\nu_{xx} = 2\gamma V^\nu_t - \gamma^2 V^\nu, \qquad x \in (l_1,l_2), \qquad t>0, \tag{39}$$
$$W^\nu_{tt} - k_3 W^\nu_{xx} + bW^\nu_t = 2\gamma W^\nu_t + (b-\gamma)\gamma W^\nu, \qquad x \in (l_2,l), \quad t>0, \tag{40}$$

$$U^\nu(0,t) = U^\nu(l,t) = 0,$$
$$U^\nu(l_1,t) = V^\nu(l_1,t); \quad k_1 U^\nu_x(l_1,t) = k_2 V^\nu_x(l_1,t), \quad t>0,$$
$$V^\nu(l_2,t) = W^\nu(l_2,t); \quad k_2 V^\nu_x(l_2,t) = k_3 W^\nu_x(l_2,t), \quad t>0,$$

$$U^\nu(x,0) = U^{0,\nu}(x), \quad U^\nu_t(x,0) = U^{1,\nu}(x), \qquad x \in (0,l_1),$$
$$V^\nu(x,0) = V^{0,\nu}(x), \quad V^\nu_t(x,0) = V^{1,\nu}(x), \qquad x \in (l_1,l_2),$$
$$W^\nu(x,0) = W^{0,\nu}(x), \quad W^\nu_t(x,0) = W^{1,\nu}(x), \quad x \in (l_2,l),$$

satisfying

$$\int_0^T |U^\nu(l_1,t)|^2\,dt + \int_0^T |W^\nu(l_2,t)|^2\,dt = 1 \tag{41}$$

and verifying the inequality

$$1 > \delta_0 \int_0^T \mathcal{E}^\nu(t)\,dt + \nu \left\{ \int_0^T \int_0^{l_1} |U^\nu_t|^2\,dxdt + \int_0^T \int_{l_2}^{l} |W^\nu_t|^2\,dxdt \right\}$$

for any ν, where $\mathcal{E}^\nu(t) = E(t; U^\nu, V^\nu, W^\nu)$. This implies that

$$\int_0^T \mathcal{E}^\nu(t)\,dt \quad \text{is bounded for any} \quad \nu,$$

and also that

$$\int_0^T \int_0^{l_1} |U^\nu_t|^2\,dxdt \to 0 \quad \text{and} \quad \int_0^T \int_{l_2}^{l} |W^\nu_t|^2\,dxdt \to 0 \quad \text{when} \quad \nu \to \infty. \tag{42}$$

Then

$$\begin{array}{lll}(U^\nu,\, V^\nu,\, W^\nu) & \text{is bounded in} & L^\infty(0,T;\mathcal{H}^1(\Omega)),\\ (U^\nu_t,\, V^\nu_t,\, W^\nu_t) & \text{is bounded in} & L^\infty(0,T;\mathcal{L}^2(\Omega)).\end{array}$$

Hence there exists a subsequence of (U^ν, V^ν, W^ν), which we denote in the same way, such that

$$(U^\nu, V^\nu, W^\nu) \overset{*}{\rightharpoonup} (U, V, W) \quad \text{in} \quad L^\infty(0,T;\mathcal{H}^1(\Omega)),$$

$$(U^\nu_t, V^\nu_t, W^\nu_t) \rightharpoonup (U_t, V_t, W_t) \quad \text{in} \quad L^2(0,T;\mathcal{L}^2(\Omega)).$$

Then applying the Lemma 3.1 of Kim, with $a = 0$ and $b = 1$, we get

$$(U^\nu, V^\nu, W^\nu) \to (U, V, W) \quad \text{in} \quad C(0,T;\mathcal{H}^r(\Omega)),$$

for $r < 1$. Using (41) we have

$$\int_0^T |U(l_1,t)|^2\, dt + \int_0^T |W(l_2,t)|^2\, dt = 1. \tag{43}$$

On the other hand, from the converge (42) we conclude that

$$U_t = 0 \quad \text{q.s. in} \quad (0,l_1)\times(0,T), \tag{44}$$
$$W_t = 0 \quad \text{q.s. in} \quad (l_2,l)\times(0,T). \tag{45}$$

Hence (U, V, W) satisfy

$$-k_1 U_{xx} = (a-\gamma)\gamma U, \tag{46}$$
$$V_{tt} - k_2 V_{xx} = 2\gamma V_t - \gamma^2 V, \tag{47}$$
$$-k_3 W_{xx} = (b-\gamma)\gamma W. \tag{48}$$

Multiplying (46) by U and integrating by parts from 0 to l_1 we obtain

$$-\, k_1 U_x(l_1,t)U(l_1,t) + k_1 \int_0^{l_1} |U_x|^2\, dx \le (a-\gamma)\gamma c_p \int_0^{l_1} |U_x|^2 dx. \tag{49}$$

Multiplying (48) by W and integrating by parts from l_2 to l we get

$$k_3 W_x(l_2,t)W(l_2,t) + k_3 \int_{l_2}^{l} |W_x|^2\, dx \le (b-\gamma)\gamma c_p \int_{l_2}^{l} |W_x|^2 dx. \tag{50}$$

On the other hand, differentiating equation (47) with respect to t and taking $\varphi = V_t$ we get

$$\left\{\begin{array}{l}\varphi_{tt} - k_2\varphi_{xx} = 2\gamma\varphi_t - \gamma^2\varphi,\\ \varphi(l_1,t) = \varphi(l_2,t) = 0,\\ \varphi_x(l_1,t) = \varphi_x(l_2,t) = 0,\\ \varphi(x,0) = \varphi^0,\\ \varphi_t(x,0) = \varphi^1,\end{array}\right.$$

with $\varphi^0 \in \mathcal{V}$ and $\varphi^1 \in L^2(l_1, l_2)$. Let us denote $\widetilde{v} = e^{-\gamma t}\varphi$. Then $\widetilde{v}$ satisfy

$$\begin{cases} \widetilde{v}_{tt} - k_2 \widetilde{v}_{xx} = 0, \\ \widetilde{v}(l_1, t) = \widetilde{v}(l_2, t) = 0, \\ \widetilde{v}_x(l_1, t) = \widetilde{v}_x(l_2, t) = 0, \\ \widetilde{v}(x, 0) = \widetilde{v}^0, \\ \widetilde{v}_t(x, 0) = \widetilde{v}^1, \end{cases}$$

with $\widetilde{v}^0 \in \mathcal{V}$ e $\widetilde{v}^1 \in L^2(l_1, l_2)$. Then using Lemma 3.2 we get $\widetilde{v} \equiv 0$ and consequently $\varphi \equiv 0$. Hence $V_t \equiv 0$ and from (47) we conclude that

$$-k_2 V_{xx} = -\gamma^2 V \quad \text{in} \quad (l_1, l_2) \times (0, T).$$

From this and integrating by parts we get

$$\begin{aligned} -\gamma^2 \int_{l_1}^{l_2} |V|^2 dx &= -k_2 \int_{l_1}^{l_2} V_{xx} V \, dx \\ &= -k_2 \left[V_x V \Big|_{l_1}^{l_2} - \int_{l_1}^{l_2} |V_x|^2 \, dx \right] \\ &= -k_2 V_x(l_2, t) V(l_2, t) + k_2 V_x(l_1, t) V(l_1, t) + k_2 \int_{l_1}^{l_2} |V_x|^2 \, dx. \end{aligned}$$

Using the transmission conditions we get

$$-k_3 W_x(l_2, t) W(l_2, t) + k_1 U_x(l_1, t) U(l_1, t) + k_2 \int_{l_1}^{l_2} |V_x|^2 \, dx = -\gamma^2 \int_{l_1}^{l_2} |V|^2 dx. \quad (51)$$

Summing (49), (50) and (51) we obtain

$$c_1 \int_0^{l_1} |U_x|^2 \, dx + k_2 \int_{l_1}^{l_2} |V_x|^2 \, dx + c_3 \int_{l_2}^{l} |W_x|^2 \, dx \le -\gamma^2 \int_{l_1}^{l_2} |V|^2 dx \le 0,$$

where c_1 and c_3 are positive constants. Therefore

$$\int_0^{l_1} |U_x|^2 \, dx + \int_{l_2}^{l} |W_x|^2 \, dx \le 0. \quad (52)$$

On the other side, using Gagliardo-Nirenberg's inequality and Young's we get

$$\begin{aligned} |U(l_1, t)| &\le ||U(t)||_{L^\infty(0,l_1)} \le ||U(t)||_{L^2(0,l_1)}^{\frac{1}{2}} ||U(t)||_{H^1(0,l_1)}^{\frac{1}{2}} \\ &\le \frac{1}{2} ||U(t)||_{L^2(0,l_1)} + \frac{1}{2} ||U(t)||_{H^1(0,l_1)} \\ &\le C ||U_x(t)||_{L^2(0,l_1)}, \end{aligned}$$

C is a positive constant. Using the last estimate in (52) we get

$$|U(l_1,t)|^2 \leq C\int_0^{l_1} |U_x|^2\,dx,$$

and same way

$$|W(l_2,t)|^2 \leq C\int_{l_2}^{l} |W_x|^2\,dx,$$

Hence, we arrive at

$$\int_0^T |U(l_1,t)|^2\,dt + \int_0^T |W(l_2,t)|^2\,dt \leq C\int_0^T\int_0^{l_1} |U_x|^2\,dxdt + C\int_0^T\int_{l_2}^{l} |W_x|^2\,dxdt \leq 0,$$

but this contradicts (43) and therefore our conclusion follows. □

Now we use the above auxiliary lemmas to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2 Let us introduce the following

$$\mathcal{L}(t) = N\mathcal{E}(t) + M_0(H_1(t) + H_3(t)) + H_2(t).$$

Then we see that for N_1 and N_2 large we have

$$N_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq N_2\mathcal{E}(t). \tag{53}$$

Now, combining the conclusions of Lemmas 3.3, 3.8, 3.9 and 3.10 we have that there exist a positive constant $C > 0$ such that

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \quad \leq \quad & -\left(aN - C_6M_0 - C_7\right)\int_0^{l_1} |U_t|^2\,dx + M_0C_6|U(l_1,t)|^2 \\
& -\left(bN - C_8M_0 - C_7\right)\int_{l_2}^{l} |W_t|^2\,dx + M_0C_8|W(l_2,t)|^2 \\
& -\left(\frac{k_1}{4}M_0 - C_7\right)\int_0^{l_1} |U_x|^2\,dx - \frac{l_2+l_1}{l_2-l_1}\,E_2(t;V) + C\gamma\mathcal{E}(t) \\
& -\left(\frac{k_3 l_2}{4(l-l_2)}M_0 - C_7\right)\int_{l_2}^{l} |W_x|^2\,dx.
\end{aligned}$$

Integrating the above identities from 0 to t, $t \gg T > 0$ and using the Lemma 3.11

we obtain

$$\begin{aligned}\mathcal{L}(t)-\mathcal{L}(0) \quad \leq \quad & -\left(aN-C_6M_0-C_7\right)\int_0^t\int_0^{l_1}|U_t|^2\,dxds \\ & -\left(\frac{k_1}{4}M_0-C_7\right)\int_0^t\int_0^{l_1}|U_x|^2\,dxds \\ & -\left(bN-C_8M_0-C_7\right)\int_0^t\int_{l_2}^{l}|W_t|^2\,dxds \\ & -\left(\frac{k_3l_2}{4(l-l_2)}M_0-C_7\right)\int_0^t\int_{l_2}^{l}|W_x|^2\,dxds \\ & +k\delta\int_0^t E(s)\,ds+kC_\delta\int_0^t\int_0^{l_1}|U_t|^2\,dxds+kC_\delta\int_0^t\int_{l_2}^{l}|W_t|^2\,dxds \\ & -\frac{l_2+l_1}{l_2-l_1}\int_0^t E_2(s;V)\,ds+C\gamma\int_0^t\mathcal{E}(s)\,ds\end{aligned}$$

where $k=\max\left\{M_0C_6, M_0C_8\right\}$. Fixing δ and γ small, we can take N and M_0 sufficiently large, with $N>>M_0$ so there is a positive constant N_0 such that

$$\mathcal{L}(t)<\mathcal{L}(0).$$

Therefore, from (53) we arrive at

$$\mathcal{E}(t)\leq K_0\mathcal{E}(0), \tag{54}$$

where $K_0=N_2/N_1$. Now we see that there exist a positive constant $K_1>0$ satisfying

$$E(t;u,v,w)e^{2\gamma t}\leq K_1\mathcal{E}(t).$$

Using the above inequality and (54) we conclude that

$$E(t;u,v,w)e^{2\gamma t}\leq C_0\mathcal{E}(0),$$

where $C_0=K_1K_0$. Therefore

$$E(t;u,v,w)\leq C_0\mathcal{E}(0)e^{-2\gamma t}.$$

This ends the proof of Theorem 1.2. □

Remark. We can extend the previous theorem to the weak solutions by using simple density argument and the laws of semi-continuity for the energy functional.

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Luci Harue Fatori and Carolina Lupifierio Antonio
Department of Mathematics,
Universidade Estadual de Londrina,
Londrina - Paraná-Brasil
lucifatori@uel.br and carollupi.cla@hotmail.com