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The difference Orlicz space of χ

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ABSTRACT: This paper is devoted to a study of the general properties of χ_M in respect of the difference sequence space $\chi_M(\Delta)$.

Key Words:: $\chi-$ sequence, analytic sequence, difference sequence, Orlicz space.

Contents

1. Introduction

A Complex sequence, whose k^{th} term is x_k is denoted by $\{x_k\}$ or simply x. Let ϕ be the set of all finite sequences. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence $x = \{x_k\}$ is said to be entire if $lim_{k\to\infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ. A sequence x is called gai sequence if $\lim_{k\to\infty} (k!\,|x_k|)^{1/k} = 0$. The vector space of all gai sequences will be denoted by χ. Kizmaz [19] defined the following difference sequence spaces

$$Z(\Delta) = \{x = (x_k) : \Delta x \in Z\}$$

for $Z = \ell_\infty, c, c_0$, where $\Delta x = (\Delta x)_{k=1}^{\infty} = (x_k - x_{k+1})_{k=1}^{\infty}$ and showed that these are Banach spaces with norm $\|x\| = |x_1| + \|\Delta x\|_\infty$. Later on Et and Colak [20] generalized the notion as follows:
Let $m \in \mathbb{N}$, $Z(\Delta^m) = \{x = (x_k) : \Delta^m x \in Z\}$ for $Z = \ell_\infty, c, c_0$ where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k)_{k=1}^{\infty} = \left(\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}\right)_{k=1}^{\infty}$.
The generalized difference has the following binomial representation:

$$\Delta^m x_k = \sum_{\gamma=0}^{m} (-1)^{\gamma} \binom{m}{\gamma} x_{k+\gamma},$$

They proved that these are Banach spaces with the norm

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty.$$

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Orlicz [1] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [2] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p(1 \leq p < \infty)$. Subsequently different classes of sequence spaces were defined by Parashar and Choudhary[3], Mursaleen et al.[4], Bektas and Altin[5], Tripathy et al.[6], Rao and subramanian[7],and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [8].

An Orlicz function is a function $M : [0, \infty) \to [o, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x)+M(y)$, then this function is called modulus function, introduced by Nakano[18] and further discussed by Ruckle[9] and Maddox[10], and many others.

An Orlicz function M is said to satisfy Δ_2- condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \leq KM(u)(u \geq 0)$. The Δ_2- condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of u and for $\ell > 1$. Lindenstrauss and Tzafriri[2] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}. \tag{1}$$

where $w = \{\text{all complex sequences}\}$. The space ℓ_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\} \tag{2}$$

becomes a Banach space which is called an Orlicz sequence space.For $M(t) = t^p, 1 \leq p < \infty$, the space ℓ_M coincide with the classical sequence space ℓ_p.

For $0 \leq r \leq 1$, a non-void subset U of a linear space is said to be absolutely $r-$ convex if $x, y \in U$ and $|\lambda|^r + |\mu|^r \leq 1$ together imply $\lambda x + \mu y \in U$, for $\lambda, \mu \in \mathbb{C}$. It is clear that if U is absolutely $r-$convex, then it is absolutely $t-$ convex for $t < r$. A linear topological space X is said to be $r-$convex if every neighbourhood of $0 \in X$ contains an absolutely $r-$ convex neighbourhood of $0 \in X$. The $r-$ convexity for $r > 1$ is of little interest, since X is $r-$ convex for $r > 1$ if and only if X is the only neighbourhood of $0 \in X$ (See Maddox and Roles [21]).
Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, ..., x_n, 0, 0, ...\}$. $s^{(k)} = (0, 0, ..., 1/k!, 0, 0, ...)$, $(1/k!)$ in the k^{th} place and zero's else where.

If X is a sequence space, we define:
(i)$X' =$ the continuous dual of X.
(ii)$X^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \, for each \, x \in X\}$;
(iii)$X^\beta = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \, is \, convergent, \, for each \, x \in X\}$;
(iv)$X^\gamma = \{a = (a_k) : \sup_n |\sum_{k=1}^{n} a_k x_k| < \infty, \, for each \, x \in X\}$;
(v)Let X be an FK-space$\supset \phi$. Then $X^f = \left\{ f(\delta^{(n)}) : f \in X' \right\}$.
$X^\alpha, X^\beta, X^\gamma$ are called the $\alpha-$(or Kö the-T öeplitz)dual of X, $\beta-$ (or generalized Kö the-T öeplitz)dual of X, $\gamma-$dual of X. Note that $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$

then $Y^{\mu} \subset X^{\mu}$, for $\mu = \alpha, \beta, or\, \gamma$.
An FK-space(Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k\,(k = 1, 2, ...)$ are continuous. We recall the following definitions[see [14]]. An FK-space is a locally convex Frechet space which is made up of sequences and has the property that coodinate projections are continuous. A metric space (X, d) is said to have AK (or sectional convergence) if and only if $d\left(x^{(n)}, x\right) \to 0$ as $n \to \infty$[see 14]. The space is said to have AD or be an AD space if ϕ is dense in X. We note that AK implies AD by [11].

2. Definitions and Preliminaries

Throughout the paper w, χ_M, Γ_M and Λ_M denote the spaces of all, Orlicz space of gai sequences, Orlicz space of entire sequences and Orlicz space of bounded sequences respectively. In this paper we define and study the Orlicz difference sequence spaces of gai sequences, entire sequences and analytic sequences. The idea of difference sequences was first introduced by Kizmaz[19]. Write $\Delta x_k = x_k - x_{k+1}$, for $k = 1, 2, 3, \cdots$.
Let w denote the set of all complex sequences $x = (x_k)_{k=1}^{\infty}, \Delta : w \to w$ be the difference operator defined by $\Delta x = (x_k - x_{k+1})_{k=1}^{\infty}$, and $M : [0, \infty) \to [0, \infty)$ be an Orlicz function. Define the sets

$$\chi_M = \left\{ x \in w : \left(M\left(\frac{(k!|x_k|)^{\frac{1}{k}}}{\rho} \right) \right) \to 0\, as\, k \to \infty\, for\ \ some\, \rho > 0 \right\};$$

$$\Gamma_M = \left\{ x \in w : \left(M\left(\frac{|x_k|^{\frac{1}{k}}}{\rho} \right) \right) \to 0\, as\, k \to \infty\, for\ \ some\, \rho > 0 \right\};$$

$$\Lambda_M = \left\{ x \in w : \sup_k \left(M\left(\frac{|x_k|^{\frac{1}{k}}}{\rho} \right) \right) < \infty\, for\, some\, \rho > 0 \right\}$$

Define the sets $\chi_M(\Delta) = \{x \in w : \Delta x \in \chi_M\}$; $\Gamma_M(\Delta) = \{x \in w : \Delta x \in \Gamma_M\}$; $\Lambda_M(\Delta) = \{x \in w : \Delta x \in \Lambda_M\}$ The space $\chi_M(\Delta)$ is a metric space with the metric

$$d(x, y) = \inf\left\{ \rho > 0 : \sup_k \left(M\left(\frac{(k!|\Delta x_k - \Delta y_k|)^{1/k}}{\rho} \right) \right) \le 1 \right\}$$

for all $x = \{x_k\}$ and $y = \{y_k\}$ in $\chi_M(\Delta)$. The space $\Lambda_M(\Delta), \Gamma_M(\Delta)$ is a metric space with the metric

$$d(x, y) = inf\left\{ \rho > 0 : \sup_k \left(M\left(\frac{|\Delta x_k - \Delta y_k|^{1/k}}{\rho} \right) \right) \le 1 \right\}$$

for all $x = \{x_k\}$ and $y = \{y_k\}$ in $\Gamma_M(\Delta), \Lambda_M(\Delta)$.

Definition 2.1 *A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$.*

Lemma 2.1 *(See [14, Theorem 7.2.7]). Let X be an FK-space$\supset \phi$. Then (i)$X^{\gamma} \subset X^f$ (ii)if X has AK, $X^{\beta} = X^f$; (iii)if X has AD, $X^{\beta} = X^{\gamma}$.*

3. Main Results

Proposition 3.1 $\chi_M(\Delta) \subset \Gamma_M(\Delta)$.

Proof: Proof is easy, so omitted.

Proposition 3.2 $\chi_M(\Delta)$ *has AK.*

Proof: Let $x \in \chi_M(\Delta)$ so that $\{\Delta x_k\} \in \chi_M$. Then $\lim_{k\to\infty}\left(M\left(\frac{(k!|\Delta x_k|)^{1/k}}{\rho}\right)\right) = 0$ and so that

$$\sup_{k\geq n+1}\left(M\left(\frac{(k!\,|\Delta x_k|)^{1/k}}{\rho}\right)\right) \to 0 \quad as \quad n\to\infty \tag{3}$$

$d\left(x, x^{[n]}\right) = \inf\left\{\rho > 0 : \sup_{k\geq n+1}\left(M\left(\frac{(k!|\Delta x_k|)^{1/k}}{\rho}\right)\right) \leq 1\right\} \to 0 \quad as\ n\to\infty$, by using (3).
$\Rightarrow x^{[n]} \to x\, as\, n \to \infty$, implying that $\chi_M(\Delta)$ has AK. This completes the proof.

Proposition 3.3 $\chi_M(\Delta)$ *is not solid.*

Example: Consider $(x_k) = (1) \in \chi_M(\Delta)$. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin \chi_M(\Delta)$. Hence $\chi_M(\Delta)$ is not solid.

Proposition 3.4 *Let M be an Orlicz function which satisfies Δ_2- condition. Then $\chi(\Delta) \subset \chi_M(\Delta)$.*

Proof: Let

$$x \in \chi(\Delta) \tag{4}$$

Then $(k!\,|\Delta x_k|)^{1/k} \leq \epsilon$ for sufficiently large k and every $\epsilon > 0$. But then, by taking $\rho \geq 1/2$.
$\left\{M\left(\frac{(k!|\Delta x_k|)^{1/k}}{\rho}\right)\right\} \leq \left(M\left(\frac{\epsilon}{\rho}\right)\right) \leq (M(2\epsilon))$ (because M is non-decreasing)
$\Rightarrow \left\{M\left(\frac{(k!|\Delta x_k|)^{1/k}}{\rho}\right)\right\} \leq K(M(\epsilon))$ by Δ_2- condition, for some $K > 0$.
$\leq \epsilon$
$\Rightarrow \left\{M\left(\frac{(k!|\Delta x_k|)^{1/k}}{\rho}\right)\right\} \to 0\, as\, k\to\infty$, by defining $M(\epsilon) < \frac{\epsilon}{K}$. Hence

$$x \in \chi_M(\Delta). \tag{5}$$

Hence (4) and (5) we get $\chi(\Delta) \subset \chi_M(\Delta)$. This completes the proof.

Proposition 3.5 *If M is a Orlicz function, then $\chi_M(\Delta)$ is a linear space over the set of complex numbers $\mathbb{C}$.*

Proof: Let $x, y \in \chi_M(\Delta)$ and $\alpha, \beta \in \mathbb{C}$. Let $x, y \in \chi_M(\Delta)$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\left(M\left(\frac{(k!\,|\Delta x_k|)^{1/k}}{\rho_1}\right)\right) \to 0\, as\, k\to\infty \tag{6}$$

$$\left(M\left(\frac{(k!\,|\Delta y_k|)^{1/k}}{\rho_2}\right)\right) \to 0 \ as \ k \to \infty \tag{7}$$

Let $\rho_3 = \max\{2A\rho_1, 2B\rho_2\}$, where $A = \sup_k |\alpha|^{1/k}$ and $B = \sup_k |\beta|^{1/k}$. Since M is a non decreasing modulus function, we have
$\left(M\left(\frac{(k!|\alpha\Delta x_k+\beta\Delta y_k|)^{1/k}}{\rho_3}\right)\right) \le \frac{1}{2}\left(M\left(\frac{(k!|\Delta x_k|)^{1/k}}{\rho_1}\right)\right) + \frac{1}{2}\left(M\left(\frac{(k!|\Delta y_k|)^{1/k}}{\rho_2}\right)\right)$
$\to 0$, as $k \to \infty$ by (6)and (7).
So $(\alpha x + \beta y) \in \chi_M(\Delta)$. Therefore $\chi_M(\Delta)$ is linear. This completes the proof.

Definition 3.1 *Let $p = (p_k)$ be any sequence of positive real numbers. Then we define*

$$\chi_M(\Delta, p) = \left\{x = \{x_k\} : \left(M\left(\frac{(k!|\Delta x_k|)^{1/k}}{\rho}\right)\right)^{p_k} \to 0ask \to \infty\right\}$$

when $p_k = p$, a constant for all $k \in \mathbb{N}$, then $\chi_M(\Delta, p) = \chi_M(\Delta)$.

The following result can be proved by using standard techniques, so we state the result with out proof.

Theorem 3.1 *(a) Let $0 < inf p_k \le p_k \le 1$. Then $\chi_M(\Delta, p) \subset \chi_M(\Delta)$*
(b) Let $1 \le p_k \le supp_k < \infty$. Then $\chi_M(\Delta) \subset \chi_M(\Delta, p)$
(c) Let $0 < p_k \le q_k$ and let $\left\{\frac{q_k}{p_k}\right\}$ be bounded. Then $\chi_M(\Delta, q) \subset \chi_M(\Delta, p)$.

Theorem 3.2 *$\chi_M(\Delta)$ is a r convex for all $r > 0$, where $0 \le r \le inf p_k$. Moreover if $p_k = p \le 1$ for all $k \in \mathbb{N}$, then $\chi_M(\Delta, p)$ is $p-$ convex.*

Proof: Let $x \in \chi_M(\Delta, p)$. But if $r \in (0, inf_k p_k)$ then clearly $r < p_k$ for all k. Let $g^*(x)$ define under the metric

$$g^*(x) = inf\left\{\rho > 0 : \sup_k \left(M\left(\frac{(k!|\Delta x_k|)^{1/k}}{\rho}\right)^{p_k}\right) \le 1\right\}$$

Since $r \le p_k \le 1$, for all $k > k_0$.
$g^*(x)$ is subadditive: Further, for $0 \le |\lambda| \le 1$. $|\lambda|^{p_k} \le |\lambda|^r$, for all $k > k_0$. Therefore, for each λ we have $g^*(\lambda x) \le |\lambda|^r . g^*(x)$. Now, for $0 < \delta < 1$,
$U = \{x : g^*(x) \le \delta\}$, which is an absolutely $r-$ convex set, for $|\lambda|^r + |\mu|^r \le 1$ and $x, y \in U$, Now

$$\begin{aligned} g^*(\lambda x + \mu y) &\le g^*(\lambda x) + g^*(\mu y) \\ &\le |\lambda|^r g^*(x) + |\mu|^r g^*(y) \\ &\le |\lambda|^r \delta + |\mu|^r \delta \\ &\le (|\lambda|^r + |\mu|^r)\delta \\ &\le \delta. \end{aligned}$$

If $p_k = p \le 1$ for all $k \in \mathbb{N}$, then $U = \{x : g^*(x) \le \delta\}$, is an absolutely $p-$ convex set. This can be obtained by a similar analysis and therefore we omit the details. This completes the proof.

Theorem 3.3 $(\chi_M(\Delta))^\beta = \Lambda$.

Proof: Step 1: $\chi_M(\Delta) \subset \Gamma_M(\Delta)$ by Proposition 4.1; then $(\Gamma_M(\Delta))^\beta \subset (\chi_M(\Delta))^\beta$. But we have $(\Gamma_M(\Delta))^\beta = \Lambda$.

$$\Lambda \subset (\chi_M(\Delta))^\beta \tag{8}$$

Step 2: Let $y \in (\chi_M(\Delta))^\beta$; $f(x) = \sum_{k=1}^{\infty} x_k y_k$ with $x \in \chi_M(\Delta)$. We recall that $s^{(k)}$ has $\frac{1}{k!}$ in the k^{th} place and zero's elsewhere, with $x = s^{(k)}, \left\{M\left(\frac{(k!|\Delta x_k|)^{1/k}}{\rho}\right)\right\} = \left\{0, 0, \cdots, M\left(\frac{(1)^{1/k}}{\rho}\right), 0, \cdots\right\}$ which converges to zero. Hence $s^{(k)} \in \chi_M(\Delta)$. Hence $d\left(s^{(k)}, 0\right) = 1$. But $|y_k| \leq \|f\| \, d\left(s^{(k)}, 0\right) < \infty$ for all k. Thus (y_k) is a bounded sequence and hence an analytic sequence. In other words $y \in \Lambda$.

$$(\chi_M(\Delta))^\beta \subset \Lambda \tag{9}$$

From (8) and (9) we obtain $(\chi_M(\Delta))^\beta = \Lambda$. This completes the proof.

Proposition 3.6 $(\chi_M(\Delta))^\mu = \Lambda$ *for* $\mu = \alpha, \beta, \gamma, f$

Step 1: $(\chi_M(\Delta))$ *has AK by Proposition 4.2. Hence by Lemma 3 (i) we get* $(\chi_M(\Delta))^\beta = (\chi_M(\Delta))^f$. *But* $(\chi_M(\Delta))^\beta = \Lambda$ *Hence*

$$(\chi_M(\Delta))^f = \Lambda. \tag{10}$$

Step 2: *Since AK implies AD. Hence by Lemma 3(iii) we get* $(\chi_M(\Delta))^\beta = (\chi_M(\Delta))^\gamma$. *Therefore*

$$(\chi_M(\Delta))^\gamma = \Lambda. \tag{11}$$

Step 3:$(\chi_M(\Delta))$ *is not normal by Proposition 4.3. Hence by Proposition 2.7 [13]. We get*

$$(\chi_M(\Delta))^\alpha \neq (\chi_M(\Delta))^\beta = \Lambda \tag{12}$$

From (10),(11) and (12) we have $(\chi_M(\Delta))^\alpha \neq (\chi_M(\Delta))^\beta = (\chi_M(\Delta))^\gamma = (\chi_M(\Delta))^f = \Lambda$

Proposition 3.7 *The continuous dual space of* $\chi_M(\Delta)$ *is* Λ. *In other words* $[\chi_M(\Delta)]^* = \Lambda$.

Proof: *We recall that* s^k *has* $(1/k!)$ *in the* k^{th} *place and zero's elsewhere with* $x = s^k$,

$$\left\{M\left(\frac{(k!\,|\Delta x_k|)^{1/k}}{\rho}\right)\right\} =$$
$$\left\{\frac{M\,(1!0)^{1/1}}{\rho}, \frac{M\,(2!0)^{1/2}}{\rho}, \cdots \frac{M\,((k-1)!0)^{1/k-1}}{\rho}, \frac{M\,(k!(1-0))^{1/k}}{\rho}, \frac{M\,((k+1)0)^{1/k+1}}{\rho}, \cdots\right\}$$
$$= \left\{0, 0, \cdots, \frac{M\,(1)^{1/k}}{\rho}, 0, \cdots\right\}$$

Hence $s^k \in \chi_M(\Delta) . f(x) = \sum_{k=1}^{\infty} \Delta x_k y_k$ *with* $\Delta x \in \chi_M(\Delta)$ *and* $f \in [\chi_M(\Delta)]^*$, *where* $[\chi_M(\Delta)]^*$ *is the continuous dual space of* $\chi_M(\Delta)$. *Take* $x = s^{(k)} \in \chi_M(\Delta)$. *Then*

$$|y_k| \le \|f\| \, d\left(s^k, 0\right) < \infty \text{ for all } k.$$

Thus (y_k) *is a bounded sequence and hence an analytic sequence. In other words,* $y \in \Lambda$. *Therefore* $[\chi_M(\Delta)]^* = \Lambda$. *This completes the proof.*

Proposition 3.8 $\chi_M(\Delta)$ *is a complete metric space under the metric*

$$d(x,y) = inf\left\{\rho > 0 : \sup_k \left(M\left(\frac{(k!|\Delta x_k - \Delta y_k|)^{1/k}}{\rho}\right)\right) \le 1\right\}$$

where $x = (x_k) \in \chi_M(\Delta)$ *and* $y = (y_k) \in \chi_M(\Delta)$.

Proof: *Let* $\left\{x^{(n)}\right\}$ *be a cauchy sequence in* $\chi_M(\Delta)$. *Then given any* $\epsilon > 0$ *there exists a positive integer* N *depending on* ϵ *such that* $d\left(x^{(n)}, x^{(m)}\right) < \epsilon$, *for all* $n \ge N$ *and for all* $m \ge N$. *Hence* $d(x,y) = inf\left\{\rho > 0 : \sup_k \left(M\left(\frac{(k!|\Delta x_k - \Delta y_k|)^{1/k}}{\rho}\right)\right) \le 1\right\} < \epsilon$ *for all* $n \ge N$ *and for all* $m \ge N$. *Consequently* $\left\{M\left(\frac{\left(k!\left|\Delta x_k^{(n)}\right|\right)}{\rho}\right)\right\}$ *is a Cauchy sequence in the metric* $\mathbb{C}$ *of complex numbers. But* $\mathbb{C}$ *is complete. So,* $\left\{M\left(\frac{\left(k!\left|\Delta x_k^{(n)}\right|\right)}{\rho}\right)\right\} \to \left\{M\left(\frac{(k!|\Delta x_k|)}{\rho}\right)\right\}$, *as* $n \to \infty$. *Hence there exists a positive integer* n_0 *such that* $inf\left\{\rho > 0 : \sup_k \left(M\left(\frac{(k!|\Delta x_k - \Delta y_k|)^{1/k}}{\rho}\right)\right) \le 1\right\} < \epsilon$ *for all* $n \ge n_0$. *In particular, we have* $\left\{M\left(\frac{\left(k!\left|\Delta x_k^{(n)} - \Delta x_k\right|\right)^{1/k}}{\rho}\right)\right\} < \epsilon$. *Now* $\left\{M\left(\frac{(k!|\Delta x_k|)^{1/k}}{\rho}\right)\right\} \le \left\{M\left(\frac{\left(k!\left|\Delta x_k - \Delta x_k^{(n_0)}\right|\right)^{1/k}}{\rho}\right)\right\} + \left\{M\left(\frac{\left(k!\left|\Delta x_k^{(n_0)}\right|\right)^{1/k}}{\rho}\right)\right\} < \epsilon + 0$ *as* $k \to \infty$. *Thus* $\left\{M\left(\frac{(k!|\Delta x_k|)^{1/k}}{\rho}\right)\right\} < \epsilon$ *as* $k \to \infty$. *That is* $(x_k) \in \chi_M(\Delta)$. *Therefore,* $\chi_M(\Delta)$ *is a complete metric space. This completes the proof.*

Lemma 3.1 (14, Theorem 8.6.1) $Y \supset X \Leftrightarrow Y^f \subset X^f$ *where* X *is an AD-space and* Y *be an FK-space.*

Proposition 3.9 *Let* Y *be any FK-space* $\supset \phi$. *Then* $Y \supset \chi_M(\Delta)$ *if and only if the sequence* $s^{(k)}$ *is weakly analytic.*

Proof: *The following implications establish the result.*
$Y \supset \chi_M(\Delta) \Leftrightarrow Y^f \subset (\chi_M(\Delta))^f$, *since* $\chi_M(\Delta)$ *has AD and by Lemma 4.13.*
$\Leftrightarrow Y^f \subset \Lambda$, *since* $(\chi_M(\Delta))^f = \Lambda$.
$\Leftrightarrow$ *for each* $f \in Y'$, *the topological dual of* $Y . f\left(s^{(k)}\right) \in \Lambda$.
$\Leftrightarrow f\left(s^{(k)}\right)$ *is analytic.*
$\Leftrightarrow s^{(k)}$ *is weakly analytic.*
This completes the proof.

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