Bol. Soc. Paran. Mat. (3s.) **v. 28** 1 (2010): 51–65.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/spm doi:10.5269/bspm.v28i1.10815

Eigenvalues of an Operator Homogeneous at the Infinity

Aomar Anane, Omar Chakrone, Mohammed Filali And Belhadj Karim

Abstract: In this paper, we show the existence of a sequences of eigenvalues for an operator homogenous at the infinity, we give his variational formulation and we establish the simplicity of all eigenvalues in the case $N = 1$. Finally we study the solvability of the problem

$$\begin{cases} \mathcal{A}(u) := -div(A(x, \nabla u)) & = & f(x, u) + h & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{cases}$$

as well as the spectrum of

$$\begin{cases} G_0'(u) & = & \lambda m|u|^{p-2}u & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{cases}$$

Key Words:: Operator homogeneous at infinity; Eigenvalues; Boundary Value problem.

Contents

1. Introduction

Consider the quasilinear problem

$$\begin{cases} \mathcal{A}(u) := -div(A(x, \nabla u)) & = & f(x, u) + h & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $h \in W^{-1,p'}(\Omega)$ an arbitrary function, p' is the Hölder conjugate exponent of p, $(1 < p < \infty)$ and $A(x, \xi) = (A_i(x, \xi))_{1 \leq i \leq N}$ such that $A_i(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}$

2000 *Mathematics Subject Classification:* 35P30, 34L30 are functions satisfying the usual growth conditions. We require some conditions on the functional A_i such that the operator $\mathcal{A}(u)$ will be homogenous at the infinity and derive from a potential $G(u)$ (i.e.,$G' = \mathcal{A}$). For example, for $\varepsilon > 0$, $\mathcal{A}(u) = -\triangle_p^{\varepsilon} u = -div((\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u)$ is an homogenous operator at the infinity and $G_0'(u) = -\triangle_p u = -div(|\nabla u|^{p-2}\nabla u)$ is an associated homogenous operator. The Problem (1) has been studied by Anane in [2], he showed the existence of the weak solutions of the problem (1) with conditions of nonresonance under (the first eigenvalue of the operator $\mathcal{A}$). This paper is organized as follows. In section 2, we recall some results about our operators. In section 3, we show (see Theorem 3.1) the existence of sequences of eigenvalues $\lambda_n(m, \Omega)$ for the following problem

$$\begin{cases} G_0'(u) &= \lambda m |u|^{p-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where G_0' (not necessarily equal to $-\triangle_p$) is an associated homogenous operator of $\mathcal{A}$, G_0 is a potential associated to G_0', $p > 1$ and $m \in M^+(\Omega) = \{m \in L^\infty(\Omega);\ \text{meas}\{x \in \Omega; m(x) > 0\} \neq 0\}$ is the weight. In section 4, we give (see Proposition 4.1) the variational formulation of $\lambda_n(m, \Omega)$ and some properties. In section 5 we show a Theorem of nonresonance (see Theorem (5.1)). In section 6 we study (see Theorem 6.3) the Fredholm Alternative for the operators $\mathcal{A}$ and G_0'(i.e., if λ does not belong to the spectrum of G_0'), then the problem (1) (with $f(x, u) = \lambda m|u|^{p-2}u$), and the following problem

$$\begin{cases} G_0'(u) &= \lambda m |u|^{p-2} u + h \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

admit a solution for all $h \in W^{-1,p'}(\Omega)$. Finally in section 7, in the case $N = 1$, we establish the simplicity of all eigenvalues (the simplicity of the first eigenvalue remains open in the general case) and we study the problem (1), when $\frac{f(x,s)}{|s|^{p-2}s}$ and $\frac{pF(x,s)}{|s|^p}$ are situated between two consecutively eigenvalues, where $F(x, s) = \int_0^s f(x, t)dt$ (see Theorem 7.3).

2. Preliminaries

Consider the problem (1) with $A(x, \xi) = ((A_i(x, \xi))_{1 \le i \le N}$, satisfies the hypotheses:

(H_1) $A_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and there exist $c \geq 0$, $k \in L^{p'}(\Omega)$ such that

$$|A_i(x, \xi)| \le c|\xi|^{p-1} + k(x), \forall \xi \in \mathbb{R}^N, a.e. x \in \Omega. \tag{4}$$

(H_2) There exists a function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies:

i) $a(x, .) : \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable a.e. $x \in \Omega$ and $\frac{\partial a(x,\xi)}{\partial \xi_i} = A_i(x, \xi)$.

ii) $a(x, .) : \mathbb{R}^N \to \mathbb{R}$ is convex and there exists $\delta > 0$ such that

$$a(x, \xi) \ge \delta |\xi|^p, \quad \forall \xi \in \mathbb{R}^N, \ a.e.\ x \in \Omega. \tag{5}$$

(H_3) There exists a Carathéodory function $a_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}$, where $a_0(x,.)$ is even and strictly convex such that

$$|a(x,t\xi) - t^p a_0(x,\xi)| \leq t^p C(t)(|\xi|^p + k_1(x)), \ \forall \xi \in \mathbb{R}^N, t > 0, \quad a.e. \ x \in \Omega,$$

for a certain function C of t such that $\lim\limits_{t\to+\infty} C(t) = 0$ and $k_1 \in L^1(\Omega)$.

(H_4) $a_0(x,.) : \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable and

1. There exist $c' \geq 0$, $k' \in L^{p'}(\Omega)$ such that $|\frac{\partial a_0(x,\xi)}{\partial \xi_i}| \leq c'|\xi|^{p-1} + k'(x), \forall \xi \in \mathbb{R}^N, a.e. x \in \Omega$.

2. $\sum_{i=1}^{i=N} \frac{\partial a_0(x,\xi)}{\partial \xi_i}\xi_i \geq C_0|\xi|^p - K_0(x)$, for all $x \in \Omega$, $\xi \in \mathbb{R}^N$ with $C_0 > 0$ some constant and $K_0 \in L^1(\Omega)$.

Remarks 2.1 *1. From (H_1) the operator $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$: $\mathcal{A}(u) = -div(A(x,\nabla u))$, with $\langle \mathcal{A}(u), v\rangle = \int_\Omega A(x,\nabla u)\nabla v = \sum_{i=1}^{i=N} \int_\Omega A_i(x,\nabla u)\frac{\partial v}{\partial x_i}$, is well defined, continuous on $W_0^{1,p}(\Omega)$.*

2. Let the functional $G : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by $G(u) = \int_\Omega a(x,\nabla u)dx$. Under the hypotheses (H_1), (H_2) and (H_3), G is well defined, weakly lower semi-continuous, continuously differentiable and $G'(u) = \mathcal{A}(u)$.

3. we consider the functional $G_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$: $G_0(u) = \int_\Omega a_0(x,\nabla u)dx$. By the hypotheses (H_1), (H_2) and (H_3), the operator G_0 is well defined continuous and weakly lower semicontinuous.

Proposition 2.1 *Assume that (H_1), (H_2) and (H_3) hold. Then a_0 is unique and verifies the following conditions*

1. $a_0(x,r\xi) = |r|^p a_0(x,\xi)$, for all $\xi \in \mathbb{R}^N$ and $r \in \mathbb{R}$.

2. We have $\lim\limits_{||u||_{1,p}\to+\infty} \frac{G(u)-G_0(u)}{||u||_{1,p}^p} = 0$ and $G_0(ru) = |r|^p G_0(u)$, for all $r \in \mathbb{R}$.

3. $G_0(u) \geq \delta||u||_{1,p}^p$, for all $u \in W_0^{1,p}(\Omega)$, where $||u||_{1,p} = (\int_\Omega |\nabla u(x)|^p dx)^{\frac{1}{p}}$ the norm of $W_0^{1,p}(\Omega)$ and δ is defined in (5).

4. If (H_4) holds, then G_0 is continuously differentiable and G_0' satisfies the (S^+) property, i.e., if $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $\limsup\limits_{n\to+\infty}\langle G_0'(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$).

Denoted

$$\frac{\partial a_0(x,\xi)}{\partial \xi_i} = A_i^0(x,\xi), \ A_0(x,\xi) = (A_i^0(x,\xi))_{1\leq i\leq N}. \tag{6}$$

such that $G_0' : W_0^{1,p} \to W_0^{-1,p'}(\Omega) : G_0'(u) = -div(A_0(x,\nabla u)$, is the unique homogenous operator associated to the operator $\mathcal{A} = G'$.

Proof:

1. By (H_3), it is clear that $a_0(x,\xi) = \lim\limits_{t\to+\infty} \frac{a(x,t\xi)}{t^p}$ e.a $.x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, this proves that a_0 is unique. For $r > 0$, $a_0(x, r\xi) = r^p \lim\limits_{t\to+\infty} \frac{a(x,rt\xi)}{(rt)^p}$, so $a_0(x,r\xi) = r^p a_0(x,\xi)$. For $r < 0$, we have $a_0(x,-r\xi) = (-r)^p a_0(x,\xi)$, since $a_0(x,.)$ is even, thus $a_0(x,r\xi) = |r|^p a_0(x,\xi)$.

2. Results by 1.

3. From (H_3), we obtain $a(x,t\nabla u) - t^p C(t)(|\nabla u|^p + k_1(x)) \leq t^p a_0(x,\nabla u)$ and by (5), we conclude that $(\delta - C(t))|\nabla u|^p \leq a_0(x,\nabla u)$, thus $\delta|\nabla u|^p \leq a_0(x,\nabla u)$, consequently $G_0(u) \geq \delta ||u||_{1,p}^p$ for all $u \in W_0^{1,p}(\Omega)$.

4. From 1) of (H_4), G_0 is continuously differentiable and we have $\langle G_0'(u), v\rangle = \sum_{i=1}^{i=N} \int_\Omega A_i^0(x,\nabla u)\frac{\partial v}{\partial x_i}$. Since G_0 is convex strictly in ξ, then $\langle G_0'(u) - G_0'(v), u-v\rangle > 0$ for all $u,v \in W_0^{1,p}(\Omega)$ with $u \neq v$. The conditions 1), 2) of (H_4) and the fact that $\langle G_0'(u) - G_0'(v), u-v\rangle > 0$ for all $u,v \in W_0^{1,p}(\Omega)$, $(u \neq v)$ imply that G_0' satisfies the (S^+) property (see [7] pp,25).

□

In the continuation we consider that the hypotheses (H_1), (H_2), (H_3) and (H_4) are verified.

3. Eigenvalues Problem

Consider the eigenvalues problem, find $(u,\lambda) \in W_0^{1,p}(\Omega) \setminus \{0\} \times \mathbb{R}_+$ such that

$$\int_\Omega A_0(x,\nabla u)\nabla v dx = \lambda \int_\Omega m|u|^{p-2}uv dx \tag{7}$$

for all $v \in W_0^{1,p}(\Omega)$, where $A_0(x,\nabla u) = (A_i^0(x,\nabla u))_{1\leq i\leq N}$, is defined in (6). Consider $B : W_0^{1,p}(\Omega) \to \mathbb{R}$: $B(u) = \frac{1}{p}\int_\Omega m|u|^p dx$.

Lemma 3.1 *If (u,λ) is a solution of (7), then $v = [\frac{1}{2\lambda G_0(u)}]^{\frac{1}{p}} u$ is a critical point of $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$, with $\Phi(v) = G_0^2(v) - B(v)$, corresponding to the critical value $c = -\frac{1}{4\lambda^2}$. Reciprocally if $(u \neq 0)$ is a critical point of Φ corresponding to the critical value c, then (u,λ) is a solution of (7), where $\lambda = \frac{1}{2\sqrt{-c}}$.*

Proof: Let (u,λ) be a solution of (7), from Proposition (2.1) we conclude that for all $\beta \in \mathbb{R}^*$, βu is also eigenvalue corresponding to λ. For $\beta = [\frac{1}{2\lambda G_0(u)}]^{\frac{1}{p}}$, $v = \beta u$ verifies $G_0(v) = \frac{1}{2\lambda}$, thus $\lambda = \frac{1}{2G_0(v)}$ and $B(v) = \frac{1}{2\lambda^2}$. Consequently $\Phi'(v) = 0$ and $\Phi(v) = -\frac{1}{4\lambda^2}$. On the other hand if $u \neq 0$ is eigenvalue of Φ corresponding to the critical value c, then $\Phi(u) = -G_0^2(u) = c$, thus $G_0(u) = \sqrt{-c}$ and $\langle G_0'(u), v\rangle = \frac{1}{2G_0(u)}\langle B'(u), v\rangle$, for all $v \in W_0^{1,p}(\Omega)$. □

Theorem 3.1 *The problem (7) admits an increasing positive sequences of the eigenvalue* $(\lambda_n)_{n\in\mathbb{N}^*}$*, with* $\lim\limits_{n\to+\infty}\lambda_n = +\infty$.

Proof: Throughout this paper we put

$$C_n = \inf_{K\in A_n(\gamma)} \sup_{v\in K} \Phi(v), \tag{8}$$

where

$$A_n(\gamma) = \{K \subset W_0^{1,p}(\Omega)\setminus\{0\}; K \text{ compact, symmetric, and } \gamma(K)\geq n\}, \tag{9}$$

with $\gamma(K)$ indicates the genus of K (see [9]). As Φ is even and of C^1, to prove the existence of the sequences $(\lambda_n)_{n\geq1}$, it is sufficient to applied the fundamental multiplicity theorem (see [8]), i.e,.(to show that: **(i)** Φ is bounded below, **(ii)** Φ satisfies the Palais–Smale condition, **(iii)** for all $n\in\mathbb{N}^*$, there exists $K\in A_n(\gamma)$ such that $\sup_{v\in K}\Phi(v) < 0$. In fact (**i**), for all $v\in W_0^{1,p}(\Omega)$, we have $\Phi(v)\geq \delta^2||v||_{1,p}^{2p} - \frac{1}{p}||m||_\infty||v||_p^p$, thus $\Phi(v)\geq ||v||_{1,p}^p(\delta^2||v||_{1,p}^p - C^p\frac{1}{p}||m||_\infty)$, where C is the Sobolev constant. Hence Φ is bounded from below and coercive. (**ii**) Φ satisfies the Palais–Smale condition; indeed, let (u_n) be a sequences of $W_0^{1,p}(\Omega)$ such that $(\Phi(u_n))$ is bounded and $\Phi'(u_n)\to 0$ in $W_0^{1,p}(\Omega)$. Since Φ is coercive, (u_n) is bounded. It follows that there exists a subsequences, still denoted by (u_n), such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, and $u_n\to u$ in $L^p(\Omega)$, on the other hand $||u_n||_{1,p}$ is bounded in $\mathbb{R}$, hence $||u_n||_{1,p}\to a\in\mathbb{R}$, with $a\geq 0$. If $a=0$, we conclude that $u_n\to 0$ in $W_0^{1,p}(\Omega)$. If $a>0$, there exists $n_0\in\mathbb{N}$ such that $||u_n||_{1,p} > \frac{a}{2}$ for all $n\geq n_0$, thus $G_0(u_n) > \delta(\frac{a}{2})^p$, for all $n\geq n_0$. Now, for all $n\geq n_0$

$$\frac{\Phi'(u_n)}{2G_0(u_n)} = G_0'(u_n) - \frac{B'(u_n}{2G_0(u_n)}. \tag{10}$$

Since $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, by (10), we have

$$\frac{1}{2G_0(u_n)}\langle\Phi'(u_n), u_n - u\rangle = \langle G_0'(u_n), u_n-u\rangle - \frac{1}{2G_0(u_n)}\langle B'(u_n), u_n - u\rangle$$

for all $n\geq n_0$. on account of the fact that $B'(u_n)$ is bounded in $L^{p'}(\Omega)$, then we obtain $\lim\limits_{n\to+\infty}\langle G_0'(u_n), u_n-u\rangle = 0$, hence $\limsup\limits_{n\to+\infty}\langle G_0'(u_n), u_n - u\rangle \leq 0$, and since G_0' posses the (S^+) property, then $u_n\to u$. (**iii**) Since $meas(\Omega)^+ = meas\{x\in\Omega; m(x)>0\} > 0$, then for all $n\in\mathbb{N}^*$, there exist $u_1, u_2, \ldots u_n \in W_0^{1,p}(\Omega)$, such that $suppu_i \cap suppu_j = \emptyset$ if $i\neq j$, and $B(u_i)=1$. Let $F_n = span\{u_1, u_2, \ldots u_n\}$ be the subspace of $W_0^{1,p}(\Omega)$, spanned by $\{u_1, u_2, \ldots u_n\}$. For all $v = \sum_{i=1}^{i=n}\alpha_i u_i \in F_n$, we have $B(v) = \sum_{i=1}^{i=n}|\alpha_i|^p B(u_i) = \sum_{i=1}^{i=n}|\alpha_i|^p$, hence the function: $v\to B(v)^{\frac{1}{p}}$ is a norm on F_n, therefore there exist $\alpha_1, \beta_1 > 0$ such that $\alpha_1 A_1(v)\leq B(v)\leq \beta_1 A_1(v)$, where $A_1(v) = \frac{1}{p}||v||_{1,p}^p$.

Let $\mathbf{A} = \{v\in W_0^{1,p}(\Omega); G_0(v)\leq \frac{R}{p}||v||_{1,p}^p, R\gg\delta\}$. For all $v\in \mathbf{A}\cap F_n$ we have

$\frac{\alpha_1}{R}G_0(v) \le B(v) \le \frac{\beta_1}{p\delta}G_0(v)$. Now let $K = \{v \in F_n \cap \mathbf{A}; \frac{\alpha_1^2}{4R^2} \le B(v) \le \frac{\alpha_1^2}{3R^2}\}$. For all $v \in K$, we have

$$\begin{cases} \Phi(v) = G_0^2(v) - B(v), \\ \qquad \le \frac{R^2}{\alpha_1^2}B^2(v) - B(v), \\ \qquad \le \frac{\alpha_1^2}{9R^2} - \frac{\alpha_1^2}{4R^2}. \end{cases}$$

Hence for all $v \in K, \Phi(v) < 0$ and $\gamma(K) \ge n$, consequently C_n is a critical value and $\lambda_n = \frac{1}{2\sqrt{-C_n}}$ is an eigenvalue. Now we prove that $\lim_{n\to+\infty} C_n = 0$ (see also [1]). □

It suffices to show that, for all $\varepsilon > 0$, there exists $n_\varepsilon \ge 1$ such that $\sup_{v\in K}\Phi(v) \ge -\varepsilon$, for all $K \in A_{n_\varepsilon}(\gamma)$, with $K \subset E$ where $E = \{v \in W_0^{1,p}(\Omega); \Phi(v) \le 0\}$. Since Φ is coercive then E is bounded in $W_0^{1,p}(\Omega)$. It results from it, by using the fact that $I : W_0^{1,p}(\Omega) \to L^p(\Omega)$ is compact that for all $n > 0$, there exist a subspace $F_n \subset L^p(\Omega)$ and $I_n : E \to F_n$ continuous such that $\sup_{v\in E}||v - I_n(v)||_p \le n$. Putting: $J_n(v) = \frac{1}{2}(I_n(v) - I_n(-v))$, for all $v \in E$. It is clear that J_n is well defined, odd, continuous and satisfies: $\sup_{v\in E}||v - J_n(v)||_p \le n$. Lets $\varepsilon > 0$, since $\overline{E}$ is compact in $L^p(\Omega)$ then there exists $n_\varepsilon > 0$ such that $|B(v) - B(J_{n_\varepsilon}(v))| \le \frac{\varepsilon}{2}$ for all $v \in E$. Let $\delta_\varepsilon > 0$ such that $B(v) \le \frac{\varepsilon}{2}$ for $||v||_p \le \delta_\varepsilon$. Thus for all $v \in E$, with $||J_{n_\varepsilon}(v)||_p \le \delta_\varepsilon$, we have $B(v) \le |B(v) - B(J_{n_\varepsilon}(v))| + |B(J_{n_\varepsilon}(v))| \le \varepsilon$. This last inequality implies that for each compact K symmetric, with $K \subset E \cap \{v \in W_0^{1,p}(\Omega); B(v) \ge \varepsilon\}$, we have $J_{n_\varepsilon}(K) \subset \{v \in F_{n_\varepsilon}; ||v||_p \ge \delta_\varepsilon\}$. Since $J_{n_\varepsilon}(K)$ is symmetric and compact in $L^p(\Omega)$, then $\overline{\gamma}(J_{n_\varepsilon}(K)) \le dim(F_{n_\varepsilon})$, where $\overline{\gamma}(K')$ indicates the genus in $L^p(\Omega)$ of K'. Finally since J_{n_ε} is continuous and odd then $\gamma(K) \le \overline{\gamma}(J_{n_\varepsilon}(K)) \le dim(F_{n_\varepsilon})$. Consequently for all compact symmetric $K \subset E$ such that $\gamma(K) \ge dim(F_{n_\varepsilon}) + 1$, there exists $v_0 \in K$ such that $\inf_{v\in K} B(v) \le B(v_0) < \varepsilon$ and since $\Phi(v) \ge -B(v)$, then we have $\sup_{v\in K}\Phi(v) \ge -\inf_{v\in K} B(v) \ge -\varepsilon$, the proof is complete.

4. Variational Formulation

Lemma 4.1 *Let $S_p = \{v \in W_0^{1,p}(\Omega); pG_0(v) = 1\}$, and $S = \{v \in W_0^{1,p}(\Omega); ||v||_{1,p}^p = 1\}$, then S_p and S are homeomorphic by an odd homomorphism, more precisely $\Psi : S_p \to S \ : \ \Psi(v) = \frac{v}{||v||_{1,p}}$.*

Proof: Consider $\Psi : S_p \to S, v \mapsto \frac{v}{||v||_{1,p}}$. Ψ is an odd and continuous function. Suppose that $\Psi(v) = \Psi(v')$ i.e., $\frac{v}{||v||_{1,p}} = \frac{v'}{||v'||_{1,p}}$, thus $\frac{pG_0(v)}{||v||_{1,p}^p} = \frac{pG_0(v')}{||v'||_{1,p}^p}$, therefore $\frac{1}{||v||_{1,p}^p} = \frac{1}{||v'||_{1,p}^p}$ hence $v = v'$, then Ψ is an injection. Let $u \in S$ and putting $v = \frac{u}{(pG_0(u))^{\frac{1}{p}}} \in S_p$, $\Psi^{-1} : S \to S_p : u \to \frac{u}{(pG_0(u))^{\frac{1}{p}}}$, this proves that Ψ is a surjection and Ψ^{-1} is continuous. □

Lemma 4.2 *There exist α, $\beta > 0$ such that for all $v \in S_p$, we have $\alpha \leq ||v||_{1,p}^p \leq \beta$.*

Proof: For all $v \in W_0^{1,p}(\Omega)$, we have $G_0(v) \geq \delta ||v||_{1,p}^p$ in particular $||v||_{1,p}^p \leq \frac{1}{\delta p}$, for all $v \in S_p$. There exists $\alpha > 0$, such that $\alpha \leq ||v||_{1,p}^p$, for all $v \in S_p$, otherwise, for all $n > 0$, there exists $v_n \in S_p$, such that $\frac{1}{n} > ||v_n||_{1,p}^p$ thus $\lim\limits_{n\to+\infty} v_n = 0$, but $pG_0(v_n) = 1$, this contradicts the continuity of G_0, finally there exist α, $\beta > 0$, such that for all $v \in S_p$, $\alpha \leq ||v||_{1,p}^p \leq \beta$. □

Putting

$$\Gamma_n(\gamma) = \{K \subset W_0^{1,p}(\Omega) \setminus \{0\}; K \text{ compact, symmetric, of } S_p \text{ and } \gamma(K) \geq n\}. \quad (11)$$

Proposition 4.1 *For all $n \geq 1$*

$$\frac{1}{\lambda_n(\gamma)} = \sup_{K \in \Gamma_n(\gamma)} \inf_{u \in K} \int_\Omega m|u|^p dx, \quad (12)$$

where $\Gamma_n(\gamma)$ is defined in (11).

Proof: Putting $d_n = \sup\limits_{\widetilde{K} \in \Gamma_n(\gamma)} \inf\limits_{v \in \widetilde{K}} \int_\Omega m|v|^p dx$, Previously we show that d_n is well defined and strictly positive. Let F_n the subspace (defined in (**iii**) proof of theorem (3.1)), $K = \{u \in F_n, ||u||_{1,p} = 1\}$ and $v \in \widetilde{K} = \Psi^{-1}(K)$, $\Psi(v) = u$, (Lemma (4.1)) so $\frac{v}{||v||_{1,p}} = u$, $\int_\Omega m|u|^p dx = \frac{1}{||v||_{1,p}^p} \int_\Omega m|v|^p dx$, where $v \in \widetilde{K}$ and $u \in K = \Psi(\widetilde{K})$. Since $u \in K \subset F_n$,(B and A_1 are equivalent), then there exists $c > 0$ such that $c\frac{1}{p}||u||_{1,p} \leq \frac{1}{p}\int_\Omega m|u|^p dx \leq \frac{1}{cp}||u||_{1,p}$ and $v \in \widetilde{K} \subset S_p$, hence $\alpha \leq ||v||_{1,p}^p$ (Lemma (4.2)). Consequently $0 < \alpha c \leq ||v||_{1,p}^p \int_\Omega m|u|^p dx = \int_\Omega m|v|^p dx$, this result shows that $\inf\limits_{V \in \widetilde{K}} \int_\Omega m|v|^p dx \geq \alpha c$, finally $d_n > 0$. On one hand, let $\widetilde{K} \in \Gamma_n(\gamma)$, and $i : \widetilde{K} \to K_1 = \{tv/v \in \widetilde{K}, t > 0\} : i(v) = tv$, i is an odd continuous homomorphism. By definition of C_n, the number defined in (8), for all $t > 0$, we have $\frac{1}{\lambda_n^2} \geq 4 \inf\limits_{u \in \widetilde{K}} (\frac{t^p}{p}\int_\Omega m|u|^p dx - \frac{t^{2p}}{p^2})$. For $t = (\frac{pd_n}{2})^{\frac{1}{p}}$, we obtain $(\frac{1}{\lambda_n^2} + d_n^2)\frac{1}{2d_n} \geq \inf\limits_{u \in \widetilde{K}} \int_\Omega m|u|^p dx$, hence $\lambda_n \leq d_n^{-1}$. On the other hand $\frac{1}{4\lambda_n^2} = \sup\limits_{K \in A_n(\gamma)} \min\limits_{v \in K}(B(v) - G_0^2(v))$, where $A_n(\gamma)$ is defined in (9). For $0 < \varepsilon < \frac{1}{\lambda_n^2}$, there exists a compact $K_\varepsilon \in A_n(\gamma)$, such that $B(v) > 0$, for all $v \in K_\varepsilon$. Thus from (5), we have $G_0(v) > 0$, for all $v \in K_\varepsilon$. Consequently $2\left(\frac{1}{4\lambda_n^2} - \varepsilon\right)^{\frac{1}{2}} \leq \inf\limits_{v \in K_\varepsilon}\left(\frac{B(v)}{G_0(v)}\right)$. Now let $h : W_0^{1,p}(\Omega) \setminus \{0\} \to S_p : h(v) = \frac{v}{[pG_0(v)]^{\frac{1}{p}}}$, h is an odd continuous function and $h(K_\varepsilon) \in \Gamma_n(\gamma)$, hence $2\left(\frac{1}{4\lambda_n^2} - \varepsilon\right)^{\frac{1}{2}} \leq \inf\limits_{u \in h(K_\varepsilon)} \int_\Omega m|u|^p dx \leq d_n$, therefore $\lambda_n \geq d_n^{-1}$, finally $\lambda_n^{-1} = d_n$. □

From this proposition we can easily obtain the following result

Corollary 4.0A *1.* $\lambda_n(\Omega, \alpha m) = \frac{\lambda_n(\Omega,m)}{\alpha}$, *for all* $\alpha > 0$.

2. $\lambda_n(\Omega, \lambda_n(\Omega, 1)) = 1$, *for all* $n \geq 1$.

3. $\lambda_1(\Omega, m) = \inf_{v \in W_0^{1,p}(\Omega)} \left(\frac{pG_0(v)}{\int_\Omega m|v|^p dx} \right)$, *with* $\int_\Omega m|v|^p dx > 0$.

4. $\frac{1}{\lambda_1(\Omega,m)} = \sup_{v \in S_p} \int_\Omega m|v|^p dx$.

5. If $m_1, m_2 \in M^+(\Omega)$, *and* $m_1 < m_2$ *a.e, then* $\lambda_1(m_1, \Omega) > \lambda_1(m_2, \Omega)$.

6. $m \in L^\infty(\Omega) \to \lambda_n(m)$ *is continuous (see [6]).*

5. Quasilinear problem

Consider the problem (1), where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $h \in W^{-1,p'}(\Omega)$. Lets the energy functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated with this problem, $\Phi(u) = G(u) - \int_\Omega F(x, u)dx - \langle h, u\rangle$, where $F(x, s) = \int_0^s f(x, t)dt$. Now suppose the following conditions on f and F.
(f) : There exist $a \geq 0, b \in L^{p'}(\Omega)$ such that $|f(x, s)| \leq a|s|^{p-1} + b(x)$ a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$.
(F) : $\beta(x) \equiv \limsup_{|s|\to+\infty} \frac{pF(x,s)}{|s|^p} < \lambda_1(\Omega, 1)$ a.e uniformly in x, i.e., there exist $\gamma \in L^1(\Omega)$ such that $F(x, s) \leq \frac{\beta(x)}{p}|s|^p + \gamma(x)$, $\beta \in L^\infty(\Omega)$ and $\beta(x) < \lambda_1(\Omega, 1)$ a.e. $x \in \Omega$.

Theorem 5.1 *Assume that the hypotheses* (H_1), (H_2), (H_3) *and* (H_4) *hold. If the conditions* (f) *and* (F) *are verified, then for all* $h \in W^{-1,p'}(\Omega)$ *the problems (1) admits a solution that minimizes* $\Phi(u) = G(u) - \int_\Omega F(x, u)dx - \langle h, u\rangle$.

Proof: In our conditions Φ is continuously differentiable, weakly lower semicontinuous and to finish the proof, it suffices to show that Φ is coercive. Let $\Phi(u) = G(u) - \int_\Omega F(x, u)dx - \langle h, u\rangle$. Suppose by contradiction that there exist a sequences (u_n) and a real c such that $||u_n||_{1,p} \to +\infty$ and $\Phi(u_n) \leq c$. we know that, $\lim_{||u_n||_{1,p}\to+\infty} \frac{G(u_n)-G_0(u_n)}{||u_n||^p_{1,p}} = 0$, thus from Proposition (2.1), for all $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$, $(1 - \varepsilon)G_0(u_n) \leq G(u_n) \leq (1 + \varepsilon)G_0(u_n)$, for all $n \geq n_0$. Therefore we have $(1 - \varepsilon)G_0(u_n) \leq \frac{1}{p} \int_\Omega \beta(x)|u_n|^p dx + \int_\Omega \gamma(x)dx + \langle h, u_n\rangle + c$. Putting $v_n = \frac{u_n}{||u_n||_{1,p}}$, since v_n is bounded in $W_0^{1,p}(\Omega)$ then there exists a subsequences still denoted by (v_n) such that $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$ and $v_n \to v$ strongly in $L^p(\Omega)$. Consequently from Proposition (2.1), we have $\delta(1 - \varepsilon) \leq (1 - \varepsilon)G_0(v_n) \leq \frac{1}{p} \int_\Omega \beta(x)|v_n|^p dx + \frac{1}{||u_n||^p_{1,p}} \int_\Omega \gamma(x)dx + \frac{c}{||u_n||^p_{1,p}} + \frac{1}{||u_n||^p_{1,p}} \langle h, u_n\rangle$, we passe to limit and by Remarks (2.1), we obtain $\delta(1-\varepsilon) \leq (1-\varepsilon)G_0(v) \leq \frac{1}{p} \int_\Omega \beta(x)|v|^p dx$, for all $\varepsilon > 0$, so $v \neq 0$. On the other hand $p(1 - \varepsilon)G_0(v) \leq \int_\Omega \beta(x)|v|^p dx \leq \lambda_1(\Omega, 1) \int_\Omega |v|^p dx$, for all $\varepsilon > 0$, this proves that $pG_0(v) \leq \int_\Omega \beta(x)|v|^p dx \leq \lambda_1(\Omega,1)\int_\Omega |v|^p dx$, therefore v is a solution of equation $G_0'(u) = \beta(x)|u|^{p-2}u$ and 1 is an eigenvalue. But $\beta(x) < \lambda_1(\Omega,1)$ and by Corollary (4.0A), we conclude that $\lambda_1(\beta(x)) > \lambda_1(\lambda_1) = 1$, this contradicts that $\lambda_1(\beta(x))$ is the first positive eigenvalue. Finally Φ is coercive. □

It is easily to show that the problem

$$\begin{cases} G_0'(u) &= f(x,u) + h \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{13}$$

admits a solution that minimizes $\Phi_0(u) = G_0(u) - \int_\Omega F(x,u)dx - \langle h, u\rangle$, in the conditions of Theorem (5.1).

Remark 5.2 *The condition* (f), *can be replaced by the condition* $\max_{|s|\leq R}|f(x,s)| \in L^1_{loc}(\Omega)$, *for all* $R > 0$, *in this case* Φ *is not of class* C^1 *on* $W_0^{1,p}(\Omega)$. *In [4], the authors showed that the problem* (1), *with* $G' = -\triangle_p$ *admits a solution.*

6. Fredholm Alternative

In the following section we show the Fredholm Alternative, this is the reason we will announce a definition, lemmas and a corollary, whose be frequently used later. Let X be a Banach space and Sym(X) the class of all closed and symmetric parties (in comparison with origin) of $X \setminus \{0\}$. Let $S^{K-1} = \{x \in \mathbb{R}^k; ||x||_{\mathbb{R}^k} = 1\}$.

Definition 6.1 *(cf [3]) The function* $\theta : Sym(X) \to \mathbb{N} \cup +\infty$ *is defined by*

1. $\theta(\emptyset) = 0$
2. *If* $F \neq \emptyset$, *then* $\theta(F) = \sup\{k \in \mathbb{N}; \textit{there exist an odd } f \in C(S^{K-1}, F)\}$.

Let us recall that the numbers $C_n(\gamma) = \inf\limits_{K\in A_n(\gamma)} \sup\limits_{v\in K} \Phi(v)$ defined in (8), where $A_n(\gamma) = \{K \in W_0^{1,p}(\Omega)\setminus\{0\}/K$ compact, symmetric and $\gamma(K) \geq n\}$ are critical points, corresponding to the eigenvalues $\lambda_n(\gamma)$ defined in (12), we define $C_n(\theta)$ and $\lambda_n(\theta)$ in substitute in (8) γ by θ, we obtain

Lemma 6.1 *(cf [3])*

1. *For all* $n \geq 1$, $C_n(\theta)$ *is a critical point of* Φ.
2. $-\infty < \inf_{W_0^{1,p}(\Omega)} \Phi = C_1(\theta) \leq C_2(\theta) \leq \ldots \leq C_n(\theta) < 0 = \Phi(0)$.
3. $\lim\limits_{n\to+\infty} C_n(\theta) = 0$.

Lemma 6.2 *(cf [3]) For all* $n \geq 1$, *we have* $C_n(\theta) = -\frac{1}{4(\lambda_n(\theta))^2}$, *where* $C_n(\theta)$ *and* $\lambda_n(\theta)$ *are defined respectively by* (8) *and* (12) *in substitute* γ *by* θ.

Corollary 6.1A *(cf [3]) Let $\Phi \in C^1(X,\mathbb{R})$ be a functional satisfied the Palais–Smale condition (PS) on X, $K_0 \in Sym(X)$ a compact and $A_1 \subset X$ a no empty symmetrical set. If the following conditions are verified*
(P_1) If $K \in Sym(X)$ compact with $\gamma(K) \geq \theta(K_0)+1$, then $K \cap A_1 \neq \emptyset$.
(P_2) $\alpha := \max_{K_0} \Phi < \inf_{A_1} \Phi := \beta$. Then the value

$$C = \inf_{h\in\Gamma} \max_{u\in h(\bar{D})} \Phi(u)$$

where $D = co(K_0) := \{tx+(1-t)x';\ x,x' \in K_0,\quad 0 \leq t \leq 1\}$ and $\Gamma = \{h \in C(\bar{D}, X\setminus\{0\})/h = id \text{ on } K_0\}$ is a critical point of the functional Φ. Moreover $C \geq \beta$.

Now we consider the hypothesis
(H_5) There exists a Carathéodory function $a_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}$ such that $a_0(x,.)$ is even, strictly convex and continuously differentiable such that

$$|A_i(x,t\xi) - t^{p-1}A_i^0(x,\xi)| \leq t^{p-1}C(t)(|\xi|^{p-1} + K_2(x)),\ a.e. x \in \Omega,\ \forall \xi \in \mathbb{R}^N,\ t > 0,$$

where $K_2 \in L^{p'}(\Omega)$, $A_i(x,\xi) = \frac{\partial a(x,\xi)}{\partial \xi_i}$, $A_i^0(x,\xi_i) = \frac{\partial a_0(x,\xi)}{\partial \xi_i}$ and $C(t)$ a certain function of t such that $\lim_{t\to+\infty} C(t) = 0$ and $a_0(x,0) = 0,\ \forall x \in \Omega$.

Remark 6.2 *The hypotheses (H_1), (H_2) and (H_5) imply that $\lim_{||v||_{1,p}\to+\infty} \frac{G(v)-G_0(v)}{||v||_{1,p}^p} = 0$. For all $v \in W_0^{1,p}(\Omega)$, $r \in \mathbb{R}$, we have $G_0(rv) = |r|^p G_0(v)$ and $G_0(v) \geq \delta||v||_{1,p}^p$, where δ is defined in (5).*

Consider the problem

$$\begin{cases} -div(A(x,\nabla u)) = \lambda m|u|^{p-2}u + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $m \in M^+(\Omega)$ and $h \in W^{-1,p'}(\Omega)$.

Theorem 6.3 *Assume that the hypotheses (H_1), (H_2) and (H_5) hold. Then for all λ positive that does not belong to the spectrum of G_0', the problem (14) admits a solution.*

Example 6.4 *$\mathcal{A}(u) = -div((\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u)$, with $\varepsilon > 0$, $G(u) = \frac{1}{p}\int_\Omega (\varepsilon + |\nabla u|^2)^{\frac{p}{2}} dx$ and $G_0(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx$.*

Proof: [Proof of Theorem (6.3).] Consider the energy functional $\Phi : W_0^{1,p}(\Omega) \to \overline{\mathbb{R}}$ associated to the problem (14)

$$\Phi(u) = G(u) - \frac{\lambda}{p}\int_\Omega m|u|^p dx - \langle h,u\rangle, \text{ and } \Phi'(u) = G'(u) - \lambda m|u|^{p-2}u - h, \tag{15}$$

where $G'(u) = -div(A(x,\nabla u))$. If $0 \leq \lambda < \lambda_1(\Omega, m)$, then Φ is coercive, and from our hypotheses the problem admits a solution. If $\lambda_1(\Omega, m) < \lambda$, applying the Corollary 6.1A. Previously we show that the functional Φ satisfies the Palais–Smale condition, otherwise suppose that there exists a sequences (u_n) in $W_0^{1,p}(\Omega)$ such that $(\Phi(u_n))$ is bounded and $\Phi^{'}(u_n) \to 0$ in $W_0^{1,p}(\Omega)$, and $||u_n||_{1,p} \to +\infty$. Put $v_n = \frac{u_n}{||u_n||_{1,p}}$ and $t_n = ||u_n||_{1,p}$, (v_n) is bounded in $W_0^{1,p}(\Omega)$, so there exists a subsequences still denoted by (v_n) such that $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$, and $v_n \to v$ strongly in $L^P(\Omega)$. Let

$$\Phi_0(u) = G_0(u) - \frac{\lambda}{p}\int_\Omega m|u|^p dx - \langle h, u\rangle, \Phi_0^{'}(u) = G_0^{'}(u) - \lambda m|u|^{p-2}u - h. \quad (16)$$

From (15) and (16), we obtain

$$\frac{\Phi^{'}(u_n)}{||u_n||_{1,p}^{p-1}} - \frac{\Phi_0^{'}(u_n)}{||u_n||_{1,p}^{p-1}} = \frac{G'(u_n)}{||u_n||_{1,p}^{p-1}} - \frac{G_0^{'}(u_n)}{||u_n||_{1,p}^{p-1}}. \quad (17)$$

For all $\varphi \in W_0^{1,p}(\Omega) \setminus \{0\}$, we have

$$|\langle \frac{G'(u_n)}{||u_n||_{1,p}^{p-1}} - \frac{G_0^{'}(u_n)}{||u_n||_{1,p}^{p-1}}, \varphi\rangle| \leq C(t_n)(||v_n||_{1,p}^{p-1} + ||K_2||_{L^{P'}(\Omega)})\sum_{i=1}^{i=N}(\int_\Omega |\frac{\partial\varphi}{\partial x_i}|^p dx)^{\frac{1}{p}}. \quad (18)$$

Consequently from the hypotheses (H_5), we conclude that

$$\lim_{n\to+\infty} \frac{G'(u_n)}{||u_n||_{1,p}^{p-1}} - \frac{G_0^{'}(u_n)}{||u_n||_{1,p}^{p-1}} = 0. \quad (19)$$

(17), (19) and $\Phi'(u_n) \to 0$ in $W_0^{1,p}(\Omega)$, show that

$$\lim_{n\to+\infty} \frac{\Phi_0'(u_n)}{||u_n||_{1,p}^{p-1}} = 0. \quad (20)$$

From (16), we have

$$\frac{\Phi_0'(u_n)}{||u_n||_{1,p}^{p-1}} = G_0'(v_n) - \lambda m|v_n|^{p-2}v_n - \frac{h}{||u_n||_{1,p}^{p-1}}, \quad (21)$$

therefore $\langle \frac{\Phi_0'(u_n)}{||u_n||_{1,p}^{p-1}}, v_n - v\rangle = \langle G_0'(v_n) - \lambda m|v_n|^{p-2}v_n - \frac{h}{||u_n||_{1,p}^{p-1}}, v_n - v\rangle$. By (20) and (21), we have $\lim_{n\to+\infty}\langle G_0^{'}(v_n), v_n - v\rangle = 0$, since $G_0^{'}$ posses the (S^+) property, we conclude that $v_n \to v$. From (21), we have $G_0'(v) = \lambda m|v|^{p-2}v$, this contradicts our assumption, finally Φ satisfies the Palais–Smale condition. According to the hypothesis of our Theorem there exists $n \in \mathbb{N}^*$ such that $\lambda_n(\theta, m) < \lambda < \lambda_{n+1}(\theta, m)$. Now we must verify the conditions (P_1) and (P_2) of Corollary (6.1A). Consider the set

$$A_1 = \{v \in W_0^{1,p}(\Omega) \setminus \{0\}; \lambda_{n+1}(\theta, m)\int_\Omega m|v|^p dx \leq pG_0(v)\}, \quad (22)$$

we have $\Phi(u) = G(u) - \frac{\lambda}{p}\int_\Omega m|u|^p dx - \langle h, u\rangle$, from the Remark (6.2) we conclude that for $\varepsilon > 0$, there exists $R > 0$ such that $G(u) \geq (1-\varepsilon)G_0(u)$ for all $||u||_{1,p} > R$, therefore $\Phi(u) \geq G_0(u)(1 - \varepsilon - \frac{\lambda}{\lambda_{n+1}(\theta,m)}) - \langle h, u\rangle$, for $||u||_{1,p} > R$ and $u \in A_1$. Hence for ε rather small and $p > 1$, Φ is coercive on A_1 and the value $\beta := \inf_{u\in A_1} \Phi(u)$ is well defined. On the other hand let $\varepsilon > 0$, from (12), there exists $K' \in \Gamma_n(\theta)$ such that for all $u \in K'$

$$\frac{1}{\lambda_n(\theta, m)} - \varepsilon \leq \min_{u\in K'} \int_\Omega m|u|^p dx \leq \int_\Omega m|u|^p dx,$$

hence for all $v \in \mathbb{R}K'$, $pG_0(v)\left(\frac{1}{\lambda_n(\theta,m)} - \varepsilon\right) \leq \int_\Omega m|v|^p dx$, we have $\Phi(v) \leq G(v) - \frac{\lambda}{\lambda_n(\theta,m)}G_0(v) + \varepsilon\lambda G_0(v) - \langle h, v\rangle$ and from the Remark (5.2) there exists $R > 0$ such that for all $v \in \mathbb{R}K'$ and $||v||_{1,p} > R$.

$$\Phi(v) \leq G_0(v)(1 + \varepsilon - \frac{\lambda}{\lambda_n(\theta, m)} + \varepsilon\lambda) - \langle h, v\rangle.$$

Consequently for ε rather small $\Phi(v) \to -\infty$ when $||v||_{1,p} \to +\infty$. Since K' is a compact there exists t_0 rather big such that $\alpha := \max_{v\in t_0K'} \Phi(v) < \beta$. Next putting $K_0 = t_0K'$, we have $K_0 \in Sym(W_0^{1,P}(\Omega))$, K_0 is a compact and $\theta(K_0) \geq n$, therefore (P_2) is verified. There remains to verify (P_1), let K a compact, symmetric and $\gamma(K) \geq n + 1$, we put $\widetilde{K} = \{\frac{u}{(pG_0(u))^{\frac{1}{p}}}; u \in K\}$, we have $\widetilde{K} \in \Gamma_{n+1}(\theta)$ and $\min_{u\in\widetilde{K}} \int_\Omega m|u|^p dx \leq \frac{1}{\lambda_{n+1}(\theta,m)}$, finally there exists $u_0 \in K$ such that $\lambda_{n+1}(\theta, m)\int_\Omega m|u_0|^p dx \leq pG_0(u_0)$ i.e., $K \cap A_1 \neq$. □

7. The eigenvalue in the case N=1

In this section we consider that $N = 1$.

Proposition 7.1 *Assume that the hypotheses (H_1), (H_2) and (H_5) hold. Then there exists $\delta' > 0$ such that $A^0(x, 1) > \delta'$, a.e.$x \in \Omega$, and $\langle G_0'(u), u\rangle = \int_\Omega A^0(x, 1)|u'|^p dx = pG_0(u)$, for all $u \in W_0^{1,p}(\Omega)$, where $A^0(x, \xi) = \frac{\partial a_0(x,\xi)}{\partial\xi}$, is defined in (6).*

Proof: From (6) and Proposition (2.1) , we have $A^0(x, r) = r^{p-1}A^0(x, 1)$, for all $r > 0$, hence there exits $c > 0$ such that $a_0(x, 1) = c^{p-1}A^0(x, 1)$, consequently from (5) there exists $\delta' > 0$ such that $A^0(x, 1) > \delta'$, a.e.$x \in \Omega$. On the other hand consider the function $f(t) = G_0(tu)$, $t \in \mathbb{R}$, from Proposition (2.1), we have $\langle G_0'(u), u\rangle = \int_\Omega A^0(x, 1)|u'|^p dx = pG_0(u)$. □

Remark 7.1 *From (12) and Proposition (7.1), we conclude that for all $n \geq 1$,*

$$\frac{1}{\lambda_n(\gamma)} = \sup_{K\in\Gamma_n(\gamma)} \inf_{u\in K} \int_\Omega m|u|^p dx, \tag{23}$$

where $\Gamma_n(\gamma)$ *is defined in* (11) *and*

$$S_p = \{u \in W_0^{1,p}(\Omega); \int_\Omega A^0(x,1)|u'|^p dx = 1\}. \tag{24}$$

Let $\rho(x) = a_0(x,1)$ and $\Omega = I = (a,b)$ such that $a < b$, if $\rho \in C^1(I) \cap C^0(\overline{I})$, then we have

Theorem 7.2 *([5]) For all* $p > 1$, $m \in M^+(\Omega)$ *the problem* (2), *has a non trivial solution if and only if* λ *belongs to an increasing sequence* $(\lambda_n)_{n\geq 1}$. *Moreover*

1. *Each* λ_n *is simple and any corresponding eigenfunction takes the forme* $\alpha v_n(x)$ *with* $\alpha \in \mathbb{R}$, *namely the multiplicity of each eigenfunction is* 1. *Moreover* $v_n(x)$ *has exactly* $n-1$ *simple zeros.*
2. *Each* λ_n *verifies the strict monotonicity with respect to the weight and the domain* Ω.
3. $\sigma^+(G_0) = \{\lambda_n, n = 1,2\dots\}$. *The eigenvalues are ordered as* $0 < \lambda_1(m) < \lambda_2(m) < \lambda_3(m) < \dots \lambda_n(m) \to +\infty$ *as* $n \to +\infty$.

7.1. APPLICATION. Consider the Dirichlet problem

$$\begin{cases} -(A(x,u'))' &= f(x,u) + h \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{25}$$

where $A : \Omega \times \mathbb{R} \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, satisfies the Carathéodory conditions and $h \in W^{-1,p'}(\Omega)$. Now supposing that f satisfies the hypotheses $(H_{\alpha,\beta})$: for $\alpha, \beta \in \mathbb{R}$, with $\alpha < \beta$, we have

1. for all $R > 0$, there exists $\phi_R \in L^{p'}(\Omega)$ such that
$$\max_{|s|\leq R} |f(x,s)| \leq \phi_R(x) \text{ a.e. } x \in \Omega. \tag{26}$$
2. $(f_{\alpha,\beta})$ for all $\varepsilon > 0$ there exists $b_\varepsilon \in L^{p'}(\Omega)$ such that a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, we have
$$-b_\varepsilon(x) + (\alpha - \varepsilon)|s|^p \leq sf(x,s) \leq (\beta + \varepsilon)|s|^p + b_\varepsilon(x). \tag{27}$$
3. $(F_{\alpha,\beta})$ $\alpha \leq\neq l(x) := \liminf_{|s|\to+\infty} \frac{pF(x,s)}{|s|^p}, \limsup_{|s|\to+\infty} \frac{pF(x,s)}{|s|^p} := k(x) \leq\neq \beta$ *a.e.*$x \in \Omega$

   and for all $\varepsilon > 0$, there exists $d_\varepsilon \in L^1(\Omega)$, such that *a.e.*$x \in \Omega$, for all $s \in \mathbb{R}$, we have
$$-d_\varepsilon(x) + (l(x) - \varepsilon)\frac{|s|^p}{p} \leq F(x,s) \leq (k(x) + \varepsilon)\frac{|s|^p}{p} + d_\varepsilon(x), \tag{28}$$
   where $F(x,s) = \int_0^s f(x,t)dt$, $m_1(x) \leq\neq m_2(x)$, "i.e.," $m_1(x) \leq m_2(x)$ a.e.$x \in \Omega$ and $m_1(x) < m_2(x)$, in some subset of Ω of nonzero measure, for all $m_1, m_2 \in M^+(\Omega)$. Let the energy functional Φ corresponding to the problem (25), we have $\Phi(u) = G(u) - \int_\Omega F(x,u)dx - \langle h, u\rangle$, where G is defined in Remarks (2.1).

Proposition 7.2 *Assume that the hypotheses* (H_1), (H_2), (H_5) *hold and* f *satisfies the hypotheses* $(H_{\alpha,\beta})$. *If* Φ *does not satisfied the Palais–Smale condition* (PS), *then there exist* $m(x) \in L^\infty(\Omega)$, $v \in W_0^{1,p}(\Omega) \setminus \{0\}$, *and* $(u_n) \subset W_0^{1,p}(\Omega)$ *such that* v *is nontrivial solution of the problem*

$$(P_m) \begin{cases} G_0^{'}(u) = m|u|^{p-2}u & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \end{cases}$$

and

$$\begin{cases} \alpha \leq\neq m(x) \leq\neq \beta, \\ ||u_n||_{1,p} \to +\infty, \frac{u_n}{||u_n||_{1,p}} \to v \ in \ W_0^{1,p}(\Omega), \\ (\Phi(u_n)) \ is \ a \ bounded \ sequences. \end{cases}$$

Proof: The proof is an adaptation of the Theorem ((4.1) see [3]) and the Theorem (6.3). □

Theorem 7.3 *Assume that the hypotheses* (H_1), (H_2) *and* (H_5) *hold. If* f *satisfies* $(H_{\lambda_n(1),\lambda_{n+1}(1)})$, *for* $n \geq 1$, *then* Φ *will satisfy the Palais–Smale condition* (PS) *and the problem* (25) *admits a solution.*

Proof: If Φ does not satisfied (PS), then from Proposition (7.2), there exists $m(x) \in L^\infty(\Omega)$ such that $\lambda_n(1) \leq\neq m(x) \leq\neq \lambda_{n+1}(1)$, this contradicts with Theorem (7.2), the rest of the proof is an adaptation of the Theorem (6.3). □

References

1. Anane A.: Simplicité et isolation de la première valeur propre du p–Laplacien avec poids. C.R.Acad.Sci.Paris t.305, pp. 725-728, 1987.

2. Anane A.: Etude des valeurs propres et de la résonance pour l'opérateur p–laplacien, These de Doctorat, Université Libre de Bruxelles, 1987.

3. Anane A., Chakrone O.: Sur un théorème de point critique et application à un problème de non-résonance entre deux valeurs propre du p–laplacien. Annales de la faculté des sciences de Toulouse Sèr. 6,9 no. 1(2000), p. 5-30.

4. Anane A., Chakrone O.: Strongly nonlinear elliptic problem without growth condition, 2002–Fez Conference on Partial Differential Equations, Electronic Journal of Differential Equations. Conference 09, 2002, pp. 41-47.

5. Anane A., Chakrone O. and Moussa M.: Spectrum of one dimensional p–laplacian operator with indefinite weight. E.J. Qualitative Theory of Diff.Equ.,no. 17.(2002), pp. 1-11.

6. Anane A., Tsouli N.: On a resonance condition between the first and the second eigenvalues for the p–laplacien.IJMMS 26:10(2001)625-634.

7. Berkovits J and Vesa Mustonen.: V(1988) Nonlinear Mappings of Monotone Type, Classification and Degree Theory. Number 2/88 in Mathematics. Faculty of Science, University of Oulu.

8. Costa D G.: Tôpicos en anàlise nào-lineare e aplicaçoes às equaçoes diferenciais, Proc. E.L.A.M., Rio de Janeiro(1986), Sprenger Lect. Notes.

9. Zeidler E.: Nonlinear functional analysis and its applications, Tome III, variational Methods and Optimization, Springer Verlag(1984).

Aomar Anane – ananeomar@yahoo.fr
Omar Chakrone – chakrone@yahoo.fr
Mohammed Filali– filali@sciences.univ-oujda.ac.ma
Belhadj Karim – karembelf@hotmail.com
Université Mohamed I, Faculté des Sciences,
Département de mathématiques et Informatique,
Oujda, Maroc.