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Odd-order quasilinear evolution equations posed on a bounded interval

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Contents

1. Introduction

We study in a rectangle $Q_T = (0,T)\times(0,1)$ global well-posedness of nonhomogeneous initial-boundary value problems for general odd-order quasilinear partial differential equations

$$u_t + (-1)^{l+1}\partial_x^{2l+1}u + \sum_{j=0}^{2l} a_j\partial_x^j u + uu_x = f(t,x) \tag{1.1}$$

with initial data

$$u(0,x) = u_0(x), \quad x\in(0,1) \tag{1.2}$$

and boundary data

$$\partial_x^j u(t,0) = \mu_j(t), \quad j=0,\dots,l-1, \tag{1.3}$$

$$\partial_x^j u(t,1) = \nu_j(t), \quad j=0,\dots,l, \quad t\in(0,T), \tag{1.4}$$

where $l\in\mathbb{N}$, a_j are real constants. This class of equations includes well-known Korteweg–de Vries and Kawahara equations which model the dynamics of long small-amplitude waves in various media.

Our study is motivated by physics and numerics and our main goal is to formulate a correct nonhomogeneous initial-boundary value problem for (1.1) in a bounded interval and to prove the existence and uniqueness of global in time weak and regular solutions in a large scale of Sobolev spaces as well as to study decay of solutions while $t\to\infty$.

For reasonable initial and boundary conditions we prove existence and uniqueness of global weak and regular solutions as well as the exponential decay while $t\to\infty$ of the obtained solution with small boundary conditions, the right-hand side and initial data.

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Dispersive equations such as KdV and Kawahara equations have been developed for unbounded regions of wave propagations, however, if one is interested in implementing numerical schemes to calculate solutions in these regions, there arises the issue of cutting off a spatial domain approximating unbounded domains by bounded ones. In this occasion some boundary conditions are needed to specify the solution. Therefore, precise mathematical analysis of boundary value problems in bounded domains for general dispersive equations is welcome and attracts attention of specialists in the area of dispersive equations, especially KdV and BBM equations, [3,5,6,7,8,9,12,15,16,17,18,21,25,26,27,28,31,37,38]. Cauchy problem for dispersive equations of high orders was successfully explored by various authors, [2,10,11,15,23,33,36]. On the other hand, we know few published results on initial-boundary value problems posed on a finite interval for general nonlinear odd-order dispersive equations, such as the Kawahara equation, see [13,14,29,20].

Well-posedness of such a problem for a linearized version of (1.1) with homogeneous initial and boundary data (1.2)–(1.4) was established in [32]. It should be noted that imposed boundary conditions are reasonable at least from mathematical point of view, see comments in [13].

The theory of global solvability of dispersive equations is based on conservation laws, the first one — in L^2. Let $u(t,x)$ be a sufficiently smooth and decaying while $|x| \to \infty$ solution of an initial value problem for (1.1) (where $a_{2j} = 0, j = 1, \ldots, l,\ f \equiv 0$), then

$$\int_{\mathbb{R}} u^2\, dx = \text{const}.$$

The analogous equality can be written for problem (1.1)–(1.4) in the case of zero boundary data. In the general case one has to make this data zero with the help of a certain auxiliary function. In our paper [20] we constructed a solution of an initial-boundary value problem for the linear homogeneous equation

$$u_t + (-1)^{l+1}\partial_x^{2l+1} u = 0 \tag{1.5}$$

with the same initial and boundary data (1.2)–(1.4) and used it as such an auxiliary function. This idea gives us an opportunity to establish our existence results for (1.1) under natural assumptions on boundary data (see Remark 2.2 below).

Another important fact is extra smoothing of solutions in comparison with initial data. In a finite domain it was first established for the KdV equation in [25,7] based on multiplication of the equation by $(1+x)u$ and consequent integration. In our case, we also have an extra smoothing effect. Roughly speaking, if $u_0 \in H^{(2l+1)k}(0,1)$, then $u \in L^2(0,T; H^{(2l+1)k+l}(0,1))$.

It has been shown in [27,28] that the KdV equation is implicitly dissipative. This means that for small initial data the energy decays exponentially as $t \to +\infty$ without any additional damping terms in the equation. Moreover, the energy decays even for the modified KdV equation with a linear source term, [28]. In [20] we proved that this phenomenon takes place for general dispersive equations of odd-orders for homogeneous boundary data and the right-hand side. Here we generalize this result proving the exponential stability for small nonhomogeneous boundary data and the right-hand side.

2. Notations. Statement of main results

For any space of functions, defined on the interval $(0,1)$, we omit the symbol $(0,1)$, for example, $L^p = L^p(0,1)$, $H^k = H^k(0,1)$, $C_0^\infty = C_0^\infty(0,1)$ etc.

Define linear differential operators in L^2 with constant coefficients

$$P_0 \equiv \sum_{j=0}^{2l} a_j \partial_x^j, \quad P \equiv (-1)^{l+1}\partial_x^{2l+1} + P_0.$$

The main assumption on P_0 is the following.

Definition 2.1. We say that the operator P_0 satisfies Assumption A if either

$$(-1)^j a_{2j} \geq 0, \quad j = 1, \dots, l,$$

or there is a natural number $m \leq l$ such, that

$$(-1)^m a_{2m} > 0 \qquad \text{and} \qquad a_{2j} = 0, \quad j = m+1, \dots, l.$$

Lemma 2.1. *Assumption A is equivalent to the following property: there exists a constant $c_0 \geq 0$ such that for any function $\varphi \in H^{2l+1}$, $\varphi(0) = \dots = \varphi^{(l-1)}(0) = 0$, $\varphi(1) = \dots = \varphi^{(l-1)}(1) = 0$,*

$$(P_0\varphi, \varphi) \geq -c_0 \|\varphi\|_{L^2}^2 \tag{2.1}$$

(here and further $(\cdot,\cdot)$ denotes the scalar product in L^2).

Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be respectively the direct and inverse Fourier transforms of a function f. For $s \in \mathbb{R}$ define the fractional order Sobolev space

$$H^s(\mathbb{R}) = \left\{ f : \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi)] \in L_2(\mathbb{R}) \right\}$$

and for a certain interval $I \subset \mathbb{R}$ let $H^s(I)$ be a space of restrictions on I of functions from $H^s(\mathbb{R})$. Define also

$$H_0^s(I) = \left\{ f \in H^s(\mathbb{R}) : \operatorname{supp} f \subset \overline{I} \right\}.$$

If ∂I is a finite part of the boundary of the interval I, then for $s \in (k+1/2, k+3/2)$, where $k \geq 0$ – integer,

$$H_0^s(I) = \left\{ f \in H^s(I) : f^{(j)}\big|_{\partial I} = 0, \quad j = 0, \dots, k \right\}.$$

Note, that $H_0^s(I) = H^s(I)$ for $s \in [0, 1/2)$.

If $\mathcal{X}$ is a certain Banach (or full countable–normed) space, define by $C_b(\overline{I}; \mathcal{X})$ a space of continuous bounded mappings from $\overline{I}$ to $\mathcal{X}$. Let

$$C_b^k(\overline{I}; \mathcal{X}) = \left\{ f(t) : \partial_t^j f \in C_b(\overline{I}; \mathcal{X}), \quad j = 0, \dots, k \right\},$$
$$C_b^\infty(\overline{I}; \mathcal{X}) = \left\{ f(t) : \partial_t^j f \in C_b(\overline{I}; \mathcal{X}), \quad \forall j \geq 0 \right\}.$$

If I is a bounded interval, the index b is omitted.

The symbol $L^p(I;\mathcal{X})$ is used in the usual sense for the space of Bochner measurable mappings from I to $\mathcal{X}$, summable with order p (essentially bounded if $p=+\infty$).

Next we introduce some special functional spaces.

Definition 2.2. For integer $k\ge 0$, $T>0$ and an interval (bounded or unbounded) $I\subset\mathbb{R}$ define

$$X_k((0,T)\times I)=\{u(t,x):\\ \partial_t^n u\in C([0,T];H^{(2l+1)(k-n)}(I))\cap L^2(0,T;H^{(2l+1)(k-n)+l}(I)),\quad n=0,\dots,k\},$$

$$M_k((0,T)\times I)=\{f(t,x):\partial_t^k f\in L^2(0,T;H^{-l}(I)),\\ \partial_t^n f\in C([0,T];H^{(2l+1)(k-n-1)}(I))\cap L^2(0,T;H^{(2l+1)(k-n)-l-1}(I)),\\ n=0,\dots,k-1\}.$$

Obviously,

$$\|P_0u\|_{M_k((0,T)\times I))}\le c\|u\|_{X_k((0,T)\times I))}. \tag{2.2}$$

In fact, we construct solutions to problem (1.1)–(1.4) in the spaces $X_k(Q_T)$ for the right parts of equation (1.1) in the spaces $M_k(Q_T)$.

To describe properties of boundary functions μ_j, ν_j we use the following functional spaces.

Definition 2.3. Let $s\ge 0$, $m=l-1$ or $m=l$, define

$$\mathcal{B}_s^m(0,T)=\prod_{j=0}^{m}H^{s+(l-j)/(2l+1)}(0,T).$$

We also use auxiliary subsets of $\mathcal{B}_s^m(0,T)$:

$$\mathcal{B}_{s0}^m(0,T)=\prod_{j=0}^{m}H_0^{s+(l-j)/(2l+1)}(\mathbb{R}_+)\big|_{(0,T)},\quad \mathbb{R}_+=(0,+\infty).$$

For the study of properties of equation (1.5) we need more sophisticated spaces than X_k.

Definition 2.4. For $s\ge 0$, $I\subset\mathbb{R}$ define

$$Y_s((0,T)\times I)=\{u(t,x):\partial_t^n u\in C([0,T];H^{(2l+1)(s-n)}(I)),\quad n=0,\dots,[s],\\ \partial_x^j u\in C_b(\overline{I};H^{s+(l-j)/(2l+1)}(0,T)),\quad j=0,\dots,[(2l+1)s]+l\}.$$

Obviously, $Y_k(Q_T) \subset X_k(Q_T)$.

The spaces Y_s originate from internal properties of the linear operator $\partial_t + (-1)^{l+1}\partial_x^{2l+1}$. In fact, consider an initial value problem in a strip $\Pi_T = (0,T)\times\mathbb{R}$ for (1.5) with the initial data (1.2). This problem was studied in [23]. In particular, if $u_0 \in H^{(2l+1)s}(\mathbb{R})$, then for any $T > 0$ there exists a solution of (1.5), (1.2), $S(t,x;u_0)$, given by the formula

$$S(t,x;u_0) = \mathcal{F}_x^{-1}\left[e^{i\xi^{2l+1}t}\widehat{u}_0(\xi)\right](x). \tag{2.3}$$

For this solution for any $t\in\mathbb{R}$ and integer $0 \le n \le s$, $0 \le j \le (2l+1)(s-n)$

$$\|\partial_t^n\partial_x^j S(t,\cdot;u_0)\|_{L^2(\mathbb{R})} = \|u_0^{((2l+1)n+j)}\|_{L^2(\mathbb{R})}, \tag{2.4}$$

and for any $x\in\mathbb{R}$ and integer $0\le j\le (2l+1)s+l$

$$\|D_t^{s+(l-j)/(2l+1)}\partial_x^j S(\cdot,x;u_0)\|_{L^2(\mathbb{R})} = c(l)\|D_x^{(2l+1)s}u_0\|_{L^2(\mathbb{R})}, \tag{2.5}$$

where the symbol D^s denotes the Riesz potential of the order $-s$. In particular, the traces of $\partial_x^j S$ for $x=0$, $j=0,\dots,m=l-1$, and $x=1$, $j=0,\dots,m=l$ lie in $\mathcal{B}_s^m(0,T)$.

In order to formulate compatibility conditions for the original problem we now introduce certain special functions.

Definition 2.5. Let $\Phi_0(x)\equiv u_0(x)$ and for natural n

$$\Phi_n(x) \equiv \partial_t^{n-1}f(0,x) - P\Phi_{n-1}(x) - \sum_{m=0}^{n-1}\binom{n-1}{m}\Phi_m(x)\Phi'_{n-m-1}(x).$$

The following theorems have been proved in [20].

Theorem 2.1 (local well-posedness)**.** *Let the operator P_0 satisfy Assumption A. Let $u_0 \in H^{(2l+1)k}(0,1)$, $(\mu_0,\dots,\mu_{l-1}) \in \mathcal{B}_k^{l-1}(0,T)$, $(\nu_0,\dots,\nu_l)\in\mathcal{B}_k^l(0,T)$, $f\in M_k(Q_T)$ for some $T>0$ and integer $k\ge 0$. Assume also that $\mu_j^{(n)}(0) = \Phi_n^{(j)}(0)$, $j=0,\dots,l-1$, $\nu_j^{(n)}(0)=\Phi_n^{(j)}(1)$, $j=0,\dots,l$, for $0\le n\le k-1$. Then there exists $t_0\in(0,T]$ such that problem (1.1)–(1.4) is well–posed in $X_k(Q_{t_0})$.*

Theorem 2.2 (global well-posedness)**.** *Let the hypothesis of Theorem 2.1 be satisfied and, in addition, if $k=0$, then $f\in L^1(0,T;L^2)$, and if $l=1$, $k=0$, then $\mu_0,\nu_0 \in H^{1/3+\varepsilon}(0,T)$ for certain $\varepsilon>0$. Then problem (1.1)–(1.4) is well–posed in $X_k(Q_T)$.*

Remark 2.1. We mean that the problem is well-posed in the space X_k, if there exists a unique solution $u(t,x)$ in this space and the map $(u_0,(\mu_0,\dots,\mu_{l-1}),(\nu_0,\dots,\nu_l),f)\mapsto u$ is Lipschitz continuous on any ball in the corresponding norms.

Remark 2.2. The properties (2.5) of the solution S to the initial-value problem (1.5), (1.2) show that the smoothness conditions on the boundary data in our results are natural (with the only exception in the case $l=1$, $k=0$ for global results) because they originate from the properties of the operator $\partial_t + (-1)^{l+1}\partial_x^{2l+1}$.

Remark 2.3. All these well-posedness results can be easily generalized for an equation of (1.1) type with a nonlinear term $g(u)u_x$, where a sufficiently smooth function g has not more than linear rate of growth.

3. Decay of small solutions

Consider in $Q_\infty = (0,+\infty)\times(0,1)$ the equation

$$u_t + P(\partial_x)u + uu_x = f(x,t), \tag{3.1}$$

where

$$P(\partial_x) = (-1)^{l+1}\partial_x^{2l+1} + \sum_{j=0}^{2l} a_j\partial_x^j,$$

with initial and boundary data:

$$u(0,x) = u_0(x),\ x\in(0,1), \tag{3.2}$$

$$\partial_x^j u(t,0) = \mu_j(t),\ j=0,\dots,l-1, \tag{3.3}$$

$$\partial_x^j u(t,1) = \nu_j(t),\ j=0,\dots,l,\ t>0. \tag{3.4}$$

Let for $j=0,\dots,l-1$

$$\psi_j(x) = \frac{x^j\eta(1-x)}{j!},$$

where η is a certain smooth "cut-off" function, namely, $\eta\ge 0,\ \eta'\ge 0,\ \eta(x)=0$ for $x\le 1/4,\ \eta(x)=1$ for $x\ge 3/4,\ \eta(x)+\eta(1-x)\equiv 1$. One can see that uniformly on j for a certain positive constant c^* and $\forall x\in(0,1)$

$$|\ \psi_j\ |, |\ \psi_j'\ | \le c^*. \tag{3.5}$$

Let

$$\psi(t,x) = \sum_{j=0}^{l-1}(\mu_j(t)\psi_j(x) + \nu_j(t)\psi_j(1-x)).$$

Then for a function

$$U(t,x) \equiv u(t,x) - \psi(t,x)$$

problem (3.1)-(3.4) becomes

$$U_t + P(\partial_x)U + UU_x + (\psi U)_x = f - \psi_t - P(\partial_x)\psi - \psi\psi_x \equiv F(t,x), \tag{3.6}$$

$$U(0,x) = u_0(x) - \psi(0,x) \equiv U_0(x), \tag{3.7}$$

$$\partial_x^j U(t,0) = \partial_x^j U(t,1) = 0,\ j=0,\dots,l-1, \tag{3.8}$$

$$\partial_x^l U(t,1) = \nu_l(t),\ t>0. \tag{3.9}$$

Since

$$\|\psi(0,\cdot)\|_{L^2(0,1)} \le c^* l\sum_{j=0}^{l-1}(|\ \mu_j(0)\ | + |\ \nu_j(0)\ |)$$

and

$$\|\psi_t + P(\partial_x)\psi + \psi\psi_x\|_{L^2(0,1)}$$

$$\leq c\sum_{j=0}^{l-1}(\mid \mu_j(t) \mid + \mid \mu_j'(t) \mid + \mu_j^2(t) + \mid \nu_j(t) \mid + \mid \nu_j'(t) \mid + \nu_j^2(t)),$$

where c depends on l, the values of a_j and properties of the function η, it easy to see that U_0 and F are small if u_0, f, μ_j, ν_j, $j = 0, \ldots, l-1$, are small.

Define

$$A_j = (-1)^{j+1}(2j+1)a_{2j+1} + (-1)^j \sigma a_{2j}, \quad j = 0, \ldots, l,$$

where $\sigma = 2$ if $(-1)^j a_{2j} \geq 0$, $\sigma = 4$ if $(-1)^j a_{2j} < 0$; $(-1)^{l+1} a_{2l+1} = 1$.

Theorem 3.1. *Let Assumption A is satisfied and*

$$A_l + \sum_{j:A_j<0} 2^{3(j-l)} A_j = 2K > 0. \tag{3.10}$$

Let$u_0 \in L^2, f \in L^2(0,+\infty; L^2(0,1))$, $\mu_j, \nu_j \in H^1(0,+\infty)$, $j = 0, \ldots, l-1$; $\nu_l \in L^2(0,+\infty)$. *Assume also that for a certain* $\delta \in (0,1]$

$$3c^* 2^{-3l} \sum_{j=0}^{l-1}(\|\mu_j\|_{L^\infty(0+\infty)} + \|\nu_j\|_{L^\infty(0+\infty)}) \leq \frac{(1-\delta)K}{2}, \tag{3.11}$$

$$\{\|(1+x)^{1/2}U_0\|_{L^2}^2 + \frac{2^{3(1-l)}}{\delta K}\int_0^{+\infty} \|F(t,\cdot)\|_{L^2}^2 dt + 2\int_0^{+\infty} \nu_l^2(t) dt\}^{1/2}$$
$$< 3K2^{3(l-1)}, \tag{3.12}$$

and for all $t > 0$

$$\int_0^t e^{\kappa\tau}\{\frac{1}{\delta K} 2^{3(1-l)}\|F(\tau,\cdot)\|_{L^2}^2 + 2\nu_l^2(\tau)\}dt \leq Me^{\gamma t}, \tag{3.13}$$

where M *is a positive constant,* $\gamma \in (0,\kappa)$ *and*

$$2\kappa = 2^{3l}K + \sum_{j<l:A_j\geq 0} 2^{3j} A_j.$$

Then a unique solution $u(t,x)$ *to problem (3.1)-(3.4), such that* $u \in X_0(Q_T)$ *for all* $T > 0$, *satisfies for all* $t > 0$ *the inequality*

$$\|U(t,\cdot)\|_{L^2}^2 \leq 2e^{-\kappa t}\|U_0\|_{L^2}^2 + Me^{(\gamma-\kappa)t}. \tag{3.14}$$

Proof. First of all note that the hypothesis of Theorem 2.2 are satisfied, hence such a unique solution exists. By Assumption A, $(-1)^l a_{2l} \geq 0$, hence $A_l \geq 2l+1 > 0$. Multiplying (3.6) by $2(1+x)U(t,x)$ and integrating, we find

$$\frac{d}{dt}\int_0^1 (1+x)U^2(t,x)\,dx+\sum_{j=0}^{l}\int_0^1 \big[(-1)^{j+1}(2j+1)a_{2j+1}+(-1)^j 2a_{2j}(1+x)\big](\partial_x^j U)^2\,dx$$

$$-\frac{2}{3}\int_0^1 U^3\,dx+\int_0^1\big[(1+x)\psi_x-\psi\big]U^2dx$$

$$\leq 4\|U\|_{L^2}\|F\|_{L^2}+2\nu_l^2(t). \tag{3.15}$$

(In fact, such a calculation must be first performed for smooth solutions and the general case can be obtained via closure). We use the Friedrichs inequality as follows: for any $\varphi\in H_0^l$

$$\|\varphi\|_{L^\infty}\leq 2^{1-3l/2}\|\varphi^{(l)}\|_{L^2},\qquad \|\varphi\|_{L^2}\leq 2^{-3l/2}\|\varphi^{(l)}\|_{L^2}.$$

Then

$$\Big|\int_0^1 U^3\,dx\Big|\leq \|U\|_{L^\infty}\|U\|_{L^2}^2\leq 2^{1-3l}\|U(t,\cdot)\|_{L^2}\|\partial_x^l U\|_{L^2}^2,$$

and

$$|\int_0^1\big[(1+x)\psi_x-\psi\big]U^2dx\,|$$

$$\leq 3c^*2^{-3l}\sum_{j=0}^{l-1}(\|\mu_j\|_{L^\infty(0,+\infty)}+\|\nu_j\|_{L^\infty(0,+\infty)})\|\partial_x^l U\|_{L^2}^2\leq\frac{(1-\delta)K}{2}\|\partial_x^l U\|_{L^2}^2.$$

Taking this into account, we rewrite (3.15) as follows:

$$\frac{d}{dt}\int_0^1(1+x)U^2(t,x)dx+\frac{3K}{2}\int_0^1(\partial_x^l U)^2dx$$

$$+\int_0^1\big[\frac{K}{2}-\frac{1}{3}2^{(2-3l)}\|(1+x)^{1/2}U(t,\cdot)\|_{L^2}\big](\partial_x^l U)^2dx$$

$$\leq\frac{K}{2}\int_0^1(\partial_x^l U)^2dx+\frac{1}{\delta K}2^{3(1-l)}\|F\|_{L^2}^2+2\nu_l^2(t).$$

Since

$$\frac{1}{3}2^{2-3l}\|(1+x)^{1/2}U_0\|_{L^2}<\frac{K}{2},$$

exploiting standard arguments, one can prove that

$$\frac{1}{3}2^{2-3l}\|(1+x)^{1/2}U(t,\cdot)\|_{L^2}<\frac{K}{2},\ \forall t\geq 0.$$

Returning to (3.15), we get

$$\frac{d}{dt}\int_0^1 (1+x)U^2(t,x)dx + \int_0^1 \big(2^{3l}K + \sum_{j<l:A_j\geq 0} 2^{3j}A_j\big)U^2 dx$$

$$\leq \frac{1}{\delta K}2^{3(1-l)}\|F\|^2_{L^2} + 2\nu_l^2(t),$$

whence

$$\frac{d}{dt}\int_0^1 (1+x)U^2(t,x)dx + \kappa\int_0^1 (1+x)U^2(t,x)dx \leq \frac{1}{\delta K}2^{3(1-l)}\|F\|^2_{L^2} + 2\nu_l^2(t).$$

From here follows (3.14). □

Remark 3.1. Inequality (3.13) is valid if $\|f\|$ and functions $\nu_j(t), \mu_j(t)$, $j = 0,\ldots,l-1$; $\nu_l(t)$ and their first derivatives are exponentially decreasing and small.

Remark 3.2. In [21] a non-trivial stationary solution to the initial-boundary value problem for the homogeneous KdV equation under zero boundary data was constructed. Therefore certain assumptions on the initial data u_0 are necessary for the decay of the corresponding solution as $t \to +\infty$.

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