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On the stabilization of the Korteweg–de Vries equation *

Vilmos Komornik

Dedicated to D. L. Russell on the occasion of his 70th birthday.

ABSTRACT: We consider the Korteweg–de Vries equation on a bounded interval with periodic boundary conditions. We prove that a natural mass conserving global feedback exponentially stabilizes the system in all Sobolev norms and we obtain explicit decay rates. The proofs are based on the family of conservation laws for the Korteweg–de Vries equation.

Key Words: KdV equation. conservation laws, stabilization, decay rate.

Contents

1. Introduction

Let $\Omega = (0,1)$, $k > 0$, $\mathbb{R}_+ = [0,\infty)$ and consider the problem

$$\begin{cases} w' + ww_x + w_{xxx} = -k(w - [w]) & \text{in } \Omega \times \mathbb{R}_+, \\ w(0,t) = w(1,t) & \text{for } t \in \mathbb{R}_+, \\ w_x(0,t) = w_x(1,t) & \text{for } t \in \mathbb{R}_+, \\ w_{xx}(0,t) = w_{xx}(1,t) & \text{for } t \in \mathbb{R}_+, \\ w(0) = w^0 & \text{on } \Omega, \end{cases} \tag{1.1}$$

where $'$ stands for the time derivative, the subscript x for the spatial derivative, $[w]$ denotes the mean-value of w defined by

$$[w] := \int_\Omega w \; dx.$$

For $k = 0$ the equation (1.1) is a good model of shallow water: $w(x,t)$ denotes the depth of water at a point x at time t; see [6], [9]. The periodic boundary conditions correspond to a circular movement. In this model $[w]$ denotes the total volume of water.

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For $k > 0$ the action of the "feedback" $-k(w - [w])$ consists in balancing the level of water, conserving at the same time its total volume. Indeed, the latter property follows, at least formally, from (1.1):

$$\begin{aligned}
[w]' &= \int_\Omega w' \, dx \\
&= -\int_\Omega ww_x + w_{xxx} + k(w - [w]) \, dx \\
&= -\int_\Omega \left(\frac{1}{2}w^2 + w_{xx}\right)_x \, dx + k[w] - k\int_\Omega w \, dx \\
&= -\int_\Omega \left(\frac{1}{2}w^2 + w_{xx}\right)_x \, dx \\
&= \left[\frac{1}{2}w^2 + w_{xx}\right]_0^1 \\
&= 0,
\end{aligned}$$

whence

$$[w(t)] = [w^0], \quad \text{for all} \quad t \in \mathbb{R}_+. \tag{1.2}$$

The following formal computation shows that $w(t)$ converges exponentially to the constant $M := [w^0] = [w]$ in $L^2(\Omega)$ as $t \to \infty$:

$$\begin{aligned}
\left(\int (w - [w])^2 \, dx\right)' &= \int 2(w - M)w' \, dx \\
&= \int -2(w - M)(ww_x + w_{xxx} + k(w - M)) \, dx \\
&= \int -2w^2 w_x + 2Mww_x - 2ww_{xxx} - 2k(w - M)^2 \, dx \\
&= \left[-\frac{2}{3}w^3 + Mw^2 - 2ww_{xx} + w_x^2\right]_0^1 - 2k\int (w - M)^2 \, dx \\
&= -2k\int (w - M)^2 \, dx,
\end{aligned}$$

whence

$$\|w(t) - [w^0]\|_{L^2(\Omega)} = \|w^0 - [w^0]\|_{L^2(\Omega)} e^{-kt} \quad \text{for all} \quad t \geq 0. \tag{1.3}$$

Let us introduce the Hilbert spaces

$$\begin{aligned}
&H_p^0 := L^2(\Omega), \\
&H_p^m := \left\{w \in H^m(\Omega) : w^j(0) = w^j(1), \ j = 0, \dots, m-1\right\}, \ m = 1, 2, \dots,
\end{aligned}$$

and the dual space

$$H_p^{-1} := (H_p^1)'.$$

Identifying $L^2(\Omega)$ with its dual $(L^2(\Omega))'$ we obtain the algebraical and topological inclusions

$$\cdots \subset H_p^2 \subset H_p^1 \subset H_p^0 \subset H_p^{-1}.$$

We recall from [8] that the problem (1.1) is well-posed in the following sense:

Theorem 1.1 *Let $m \geq 2$ and $w^0 \in H_p^m$. Then the problem*

$$\begin{cases} w' + ww_x + w_{xxx} = -k(w - [w]) & \text{in } \Omega \times \mathbb{R}_+, \\ \frac{\partial^j}{\partial x^j} w(0,t) = \frac{\partial^j}{\partial x^j} w(1,t) & \text{for } t \in \mathbb{R}_+, \quad j = 0, \dots, m-1, \\ w(0) = w^0 & \text{on } \Omega \end{cases} \tag{1.4}$$

has a unique solution

$$w \in C(\mathbb{R}_+; H_p^m) \cap C^1(\mathbb{R}_+; H_p^{m-3}). \tag{1.5}$$

Furthermore, the mapping $w^0 \mapsto (w, w')$ is continuous from H_p^m into $H^m \times H^{m-3}$.

The purpose of this paper is to extend the estimate (1.3) on the asymptotic behaviour of the solutions of (1.5):

Theorem 1.2 *Let $m \geq 2$ and $w^0 \in H_p^m$. Then for every fixed $0 < k' < k$ there exists a constant $C = C(w^0, k')$ such that the solution of (1.4) satisfies the estimate*

$$\|(w(t) - [w^0], w'(t))\|_{H_p^m \times H_p^{m-3}} \leq Ce^{-k't}, \quad t \in \mathbb{R}_+. \tag{1.6}$$

In order to convince ourselves about the validity of these estimates let us consider for a moment the linearized problem

$$\begin{cases} w' + w_{xxx} = -k(w - [w]) & \text{in } \Omega \times \mathbb{R}_+, \\ \frac{\partial^j}{\partial x^j} w(0,t) = \frac{\partial^j}{\partial x^j} w(1,t) & \text{for } t \in \mathbb{R}_+, \quad j = 0, \dots, m-1, \\ w(0) = w^0 & \text{on } \Omega. \end{cases}$$

Assuming that the solutions satisfy the regularity properties (1.6) (see [7]), the desired estimates follow by applying the multiplier method. Indeed, $[w]$ is constant again because

$$\begin{aligned} [w]' &= \int_\Omega w' \, dx \\ &= -\int_\Omega w_{xxx} + k(w - [w]) \, dx \\ &= -[w_{xx}]_0^1 + k[w] - k \int_\Omega w \, dx \\ &= 0. \end{aligned}$$

Denoting this constant by M, the function $v := w - M$ has the same regularity properties as w and it solves the following problem:

$$\begin{cases} v' + v_{xxx} = -kv & \text{in } \Omega \times \mathbb{R}_+, \\ \frac{\partial^j}{\partial x^j} v(0,t) = \frac{\partial^j}{\partial x^j} v(1,t) & \text{for } t \in \mathbb{R}_+, \quad j = 0, \dots, m-1, \\ v(0) = w^0 - M & \text{on } \Omega. \end{cases} \tag{1.7}$$

By approximating the initial value by smoother functions it is sufficient to prove the estimates for periodic solutions belonging to H_p^{2m}. Multiplying the differential equation in (1.7) by $\frac{\partial^{2j} v}{\partial x^{2j}}$ for $j = 0, \dots, m$, integrating by parts in Ω and using the periodic boundary conditions we obtain that

$$\begin{aligned} 0 &= \int_\Omega (v' + v_{xxx} + kv) \frac{\partial^{2j} v}{\partial x^{2j}} \ dx \\ &= (-1)^j \int_\Omega \frac{\partial^j v'}{\partial x^j} \cdot \frac{\partial^j v}{\partial x^j} - \frac{\partial^{j+2} v}{\partial x^{j+2}} \cdot \frac{\partial^{j+1} v}{\partial x^{j+1}} + k \left(\frac{\partial^j v}{\partial x^j} \right)^2 \ dx \\ &= (-1)^j \frac{\partial}{\partial t} \left(\frac{1}{2} \int_\Omega \left(\frac{\partial^j v}{\partial x^j} \right)^2 \ dx \right) + (-1)^{j+1} \left(\frac{1}{2} \int_\Omega \frac{\partial}{\partial x} \left(\frac{\partial^{j+1} v}{\partial x^{j+1}} \right)^2 \ dx \right) \\ &\quad + (-1)^j \int_\Omega k \left(\frac{\partial^j v}{\partial x^j} \right)^2 \ dx. \end{aligned}$$

Since

$$\frac{1}{2} \int_\Omega \frac{\partial}{\partial x} \left(\frac{\partial^{j+1} v}{\partial x^{j+1}} \right)^2 \ dx = \frac{1}{2} \left[\left(\frac{\partial^{j+1} v}{\partial x^{j+1}} \right)^2 \right]_0^1 = 0$$

by the boundary conditions, we conclude that

$$\frac{\partial}{\partial t} \int_\Omega \left(\frac{\partial^j v}{\partial x^j} \right)^2 \ dx = -2k \int_\Omega \left(\frac{\partial^j v}{\partial x^j} \right)^2 \ dx$$

and hence

$$\int_\Omega \left(\frac{\partial^j}{\partial x^j} v(t) \right)^2 \ dx = e^{-2kt} \int_\Omega \left(\frac{\partial^j}{\partial x^j} v(0) \right)^2 \ dx$$

for all $t \geq 0$ and $j = 0, \dots, m$. Therefore

$$\|v(t)\|_{H_p^m} = e^{-kt} \|v(0)\|_{H_p^m}, \quad t \geq 0,$$

and then, using the equation

$$v' = -v_{xxx} - kv$$

we obtain that

$$\|v'(t)\|_{H_p^{m-3}} \leq (1+k) e^{-kt} \|v(0)\|_{H_p^m}, \quad t \geq 0.$$

Taking the definition of v into account we obtain finally that

$$\|(w(t)-[w^0],w'(t))\|_{H_p^m\times H_p^{m-3}}\le Ce^{-kt},\quad t\in\mathbb{R}_+. \tag{1.8}$$

The presence of the nonlinear term creates serious difficulties with respect to the linearized problems but as we will see, the final estimates are only slightly weaker than (1.8): we have a decay rate $k-\varepsilon$ with arbitrarily small $\varepsilon>0$ instead of k.

Theorem 1.2 has been planned to be a part of a joint work with D. L. Russell and B.-Y. Zhang (see [2], [3]) but, due to some mismanagement, it has never been published before. The author is indebted to D. L. Russell and B.-Y. Zhang for many helpful conversations on this subject.

2. Proof of Theorem 1.2 for $m=2$

We shall often use the equality (1.2) and therefore we shall write $[w]$ instead of $[w^0]$. For brevity we shall write $\int$ instead of $\int_\Omega$.

Applying a usual density argument it is sufficient to prove the estimates (1.6) for $w^0\in H_p^5$. According to Theorem 1.1 thus we may assume that

$$w\in C(\mathbb{R}_+;H_p^5)\cap C^1(\mathbb{R}_+;H_p^2). \tag{2.1}$$

This regularity property is sufficient to justify all computations which follow.

It is convenient to introduce the notations

$$M:=[w^0],\quad v:=w-M,\quad v^0:=w^0-M; \tag{2.2}$$

then we deduce from (1.1), (1.2) and (2.1) that

$$v\in C(\mathbb{R}_+;H_p^5), \tag{2.3}$$
$$v\in C^1(\mathbb{R}_+;H_p^2), \tag{2.4}$$
$$[v(t)]=0\quad\text{for all}\quad t\in\mathbb{R}_+, \tag{2.5}$$
$$v'+vv_x+Mv_x+v_{xxx}+kv=0\quad\text{in}\quad\Omega\times\mathbb{R}_+, \tag{2.6}$$
$$v(0)=v^0\quad\text{on}\quad\Omega, \tag{2.7}$$

and the estimates (1.6) take the following form:

$$\|(v(t),v'(t))\|_{H_p^2\times H_p^{-1}}\le Ce^{-k't},\quad t\in\mathbb{R}_+. \tag{2.8}$$

Lemma 2.1 *The function*

$$t\mapsto\int v(t)^2\ dx,\quad t\in\mathbb{R}_+ \tag{2.9}$$

is continuously differentiable, and

$$\Big(\int v^2\ dx\Big)'=-2k\int v^2\ dx. \tag{2.10}$$

Proof: Since H_p^2 is a Banach algebra, it follows from (2.4) that the function v^2 is continuously differentiable. Hence the function (2.9), being the composition of two function of class C^2, is also continuously differentiable.

Using (2.6) and the periodicity of v (see (2.3)) we easily obtain the identity (2.10):

$$\begin{aligned}\Big(\int v^2\ dx\Big)' &= \int 2vv'\ dx\\ &= \int -2v(vv_x + Mv_x + v_{xxx} + kv)\ dx\\ &= \int -2v^2v_x - 2Mvv_x - 2vv_{xxx} - 2kv^2\ dx\\ &= \Big[-\frac{2}{3}v^3 - Mv^2 - 2vv_{xx} + v_x^2\Big]_0^1 - 2k\int v^2\ dx\\ &= -2k\int v^2\ dx.\end{aligned}$$

□

Lemma 2.2 *The function*

$$t \mapsto \int v_x(t)^2 - \frac{1}{3}v^3(t)^2\ dx, \quad t\in\mathbb{R}_+ \tag{2.11}$$

is continuously differentiable and

$$\Big(\int v_x(t)^2 - \frac{1}{3}v^3(t)^2\ dx\Big)' = -2k\int v_x(t)^2 - \frac{1}{2}v^3(t)^2\ dx. \tag{2.12}$$

It follows easily from (2.4) that the function (2.11) is continuously differentiable. Using (2.6) and the periodicity of v hence the identity (2.12) follows:

$$\begin{aligned}&\Big(\int v_x^2 - \frac{1}{3}v^3\ dx\Big)'\\ &= \int 2v_xv_x' - v^2v'\ dx\\ &= \big[2v_xv'\big]_0^1 + \int -v'(2v_{xx} + v^2)\ dx\\ &= \int (vv_x + Mv_x + v_{xxx} + kv)(2v_{xx} + v^2)\ dx\\ &= \int (v^2)_xv_{xx} + \Big(Mv_x^2 + v_{xx}^2 + \frac{1}{4}v^4 + \frac{M}{3}v^3\Big)_x + v^2v_{xxx} - 2kv_x^2 + kv^3\ dx\\ &= \Big[v^2v_{xx} + Mv_x^2 + v_{xx}^2 + \frac{1}{4}v^4 + \frac{M}{3}v^3\Big]_0^1 + \int -2kv_x^2 + kv^3\ dx\\ &= -2k\int -\frac{1}{2}v^3\ dx.\end{aligned}$$

Lemma 2.3 *The function*

$$t \mapsto \int v_{xx}(t)^2 - \frac{5}{3}v_x^2 v(t)^2 + \frac{5}{36}v(t)^4 \ dx, \quad t \in \mathbb{R}_+ \tag{2.13}$$

is continuously differentiable and

$$\Big(\int v_{xx}^2 - \frac{5}{3}v_x^2 v + \frac{5}{36}v^4 \ dx\Big)' = -2k \int v_{xx}^2 - \frac{5}{2}v_x^2 v + \frac{5}{18}v^4 \ dx. \tag{2.14}$$

By (2.4) the function (2.13) is continuously differentiable. To show the identity (2.14) first we deduce from (2.6), using the periodicity of v, the following identity:

$$\begin{aligned}
&\Big(\int v_{xx}^2 - \frac{5}{3}v_x^2 v + \frac{5}{36}v^4 \ dx\Big)' \\
&= \int 2v_{xx}v_{xx}' - \frac{10}{3}v_x v_x' v - \frac{5}{3}v_x^2 v' + \frac{5}{9}v^3 v' \ dx \\
&= \Big[2v_{xx}v_x' - 2v_{xxx}v' - \frac{10}{3}v v_x v'\Big]_0^1 \\
&\qquad + \int v'\Big(2v_{xxxx} + \frac{5}{3}v_x^2 + \frac{10}{3}v v_{xx} + \frac{5}{9}v^3\Big) \ dx \\
&= \int v'\Big(2v_{xxxx} + \frac{5}{3}v_x^2 + \frac{10}{3}v v_{xx} + \frac{5}{9}v^3\Big) \ dx \\
&= -\int (v v_x + M v_x + v_{xxx} + kv)\Big(2v_{xxxx} + \frac{5}{3}v_x^2 + \frac{10}{3}v v_{xx} + \frac{5}{9}v^3\Big) \ dx \\
&= -k\int 2v v_{xxxx} + \frac{5}{3}v v_x^2 + \frac{10}{3}v^2 v_{xx} + \frac{5}{9}v^4 \ dx \\
&\qquad - M\int 2v_x v_{xxxx} + \frac{5}{3}v_x^3 + \frac{10}{3}v v_x v_{xx} + \frac{5}{9}v^3 v_x \ dx \\
&\qquad - \int 2v v_x v_{xxxx} + \frac{5}{3}v v_x^3 + \frac{10}{3}v^2 v_x v_{xx} + \frac{5}{9}v^4 v_x + 2v_{xxx}v_{xxxx} \\
&\qquad + \frac{5}{3}v_x^2 v_{xxx} + \frac{10}{3}v v_{xx} v_{xxx} + \frac{5}{9}v^3 v_{xxx} \ dx \\
&=: -kI_1 - MI_2 - I_3.
\end{aligned}$$

It suffices to show that

$$I_1 = \int 2v_{xx}^2 - 5v v_x^2 + \frac{5}{9}v^4 \ dx \quad \text{and} \quad I_2 = I_3 = 0.$$

We have

$$\begin{aligned}
I_1 &= \Big[2v v_{xxx} - 2v_x v_{xx} + \frac{10}{3}v^2 v_x\Big]_0^1 \\
&\qquad + \int 2v_{xx}^2 + \frac{5}{3}v v_x^2 - \frac{20}{3}v v_x^2 + \frac{5}{9}v^4 \ dx \\
&= \int 2v_{xx}^2 - 5v v_x^2 + \frac{5}{9}v^4 \ dx
\end{aligned}$$

and

$$I_2 = \Big[2v_x v_{xxx} - v_{xx}^2 + \frac{5}{3}vv_x^2 + \frac{5}{36}v^4\Big]_0^1 + \int \frac{5}{3}v_x^3 - \frac{5}{3}v_x^3 \ dx$$
$$= 0.$$

Finally, we have

$$\begin{aligned}
I_3 &= \int 2vv_xv_{xxxx} + \frac{5}{3}vv_x^3 + \frac{5}{3}v_x^2v_{xxx} + \frac{10}{3}v^2v_xv_{xx} + \frac{10}{3}vv_{xx}v_{xxx} \\
&\quad + \frac{5}{9}v^4v_x + \frac{5}{9}v^3v_{xxx} + 2v_{xxx}v_{xxxx} \ dx \\
&= \Big[2vv_xv_{xxx} + \frac{1}{9}v^5]_0^1 + \int -2v_x^2v_{xxx} - 2vv_{xx}v_{xxx} + \frac{5}{3}vv_x^3 \\
&\quad + \frac{5}{3}v_x^2v_{xxx} + \frac{10}{3}v^2v_xv_{xx} + \frac{10}{3}vv_{xx}v_{xxx} + \frac{5}{9}v^3v_{xxx} \ dx \\
&= \int -2v_x^2v_{xxx} - 2vv_{xx}v_{xxx} + \frac{5}{3}vv_x^3 + \frac{5}{3}v_x^2v_{xxx} + \frac{10}{3}v^2v_xv_{xx} \\
&\quad + \frac{10}{3}vv_{xx}v_{xxx} + \frac{5}{9}v^3v_{xxx} + \frac{5}{9}v^3v_{xx} \ dx \\
&= \Big[-2v_x^2v_{xx} - vv_{xx}^2 + \frac{5}{3}vv_x^2 + \frac{5}{3}vv_{xx}^2]_0^1 \\
&\quad + \int 4v_xv_{xx}^2 + v_xv_{xx}^2 \\
&\quad + \frac{5}{3}vv_x^3 - \frac{10}{3}v_xv_{xx}^2 \\
&\quad + \frac{10}{3}v^2v_xv_{xx} - \frac{5}{3}v_xv_{xx}^2 - \frac{5}{3}v^2v_xv_{xx} \ dx \\
&= \int \frac{5}{3}vv_x^3 + \frac{5}{3}v^2v_xv_{xx} \ dx = 0.
\end{aligned}$$

In order to simplify the notation we shall write $\|\cdot\|_p$ for the norm of $L^p(\Omega)$, $1 \le p \le \infty$. Since Ω is the unit interval, the Hölder inequality is particularly simple:

$$\|v\|_p \le \|v\|_q \quad \text{for all} \quad v \in L^q(\Omega), \quad 1 \le p \le q \le \infty. \tag{2.15}$$

We shall also use the Poincaré–Wirtinger inequality:

$$\|v\|_\infty \le \|v_x\|_1 \quad \text{for all} \quad v \in H^1(\Omega) \quad \text{satisfying} \quad [v] = 0. \tag{2.16}$$

The proof is simple: since v is continuous, there exists $a \in \Omega$ such that $v(a) = 0$. Then for any $y \in \Omega$ we have

$$|v(y)| = |v(y) - v(a)| = \Big|\int_a^y v_x \ dx\Big| \le \int_\Omega |v_x| \ dx = \|v_x\|_1.$$

Noe that Lemma 2.1 implies that

$$\|v(t)\|_2 = \|v^0\|_2 e^{-kt} \quad \text{for all} \quad t \in \mathbb{R}_+. \tag{2.17}$$

Now let us show that for each fixed $k' \in (0,k)$ there exists a positive constant C' such that

$$\|v_x(t)\|_2 = C' e^{-k't} \quad \text{for all} \quad t \geq 0. \tag{2.18}$$

Using (2.15)–(2.17) we have

$$\begin{aligned} \|v^3(t)\|_1 &\leq \|v(t)\|_\infty^2 \|v(t)\|_1 \\ &\leq \|v_x(t)\|_1^2 \|v(t)\|_2 \\ &\leq \|v_x(t)\|_2^2 \|v^0\|_2 e^{-kt}; \end{aligned}$$

consequently, for any fixed $\varepsilon > 0$ (to be chosen later) there exists $T' > 0$ such that

$$\int v^3(t)\ dx \leq \varepsilon \int v_x^2\ dx \quad \text{for all} \quad t > T'. \tag{2.19}$$

If $\varepsilon \leq 2$, then we deduce from (2.19) the inequalities

$$\int \Big(v_x^2 - \frac{1}{3}v^3\Big)(t)\ dx \geq \frac{1}{3}\int v_x^2(t)\ dx \geq 0 \quad \text{for all} \quad t > T'. \tag{2.20}$$

If ε is sufficiently small, then we also deduce from (2.19) that

$$-2k \int \Big(v_x^2 - \frac{1}{2}v^3\Big)(t)\ dx \leq -2k' \int \Big(v_x^2 - \frac{1}{3}v^3\Big)(t)\ dx \quad \text{for all} \quad t > T'. \tag{2.21}$$

(It suffices to choose $\varepsilon \leq (6k - 6k')/(3k - 2k')$.)

Thus, choosing a sufficiently small ε we deduce from (2.11), (2.20) and (2.21) that

$$\begin{aligned} \frac{1}{3}\int v_x^2(t)\ dx &\leq \int \Big(v_x^2 - \frac{1}{3}v^3\Big)(t)\ dx \\ &\leq \left(\int \Big(v_x^2 - \frac{1}{3}v^3\Big)(0)\ dx\right) e^{-k'(t-T')} \\ &=: C' e^{-k't} \quad \text{for all} \quad t > T' \end{aligned}$$

which implies (2.18) for all $t > T'$. The left-hand side of (2.18) being continuous, the estimate (2.18) remains valid for all $t \geq 0$ with some bigger constant C'.

Next we show similarly that for any fixed $k' < k$ there exists a positive constant C'' such that

$$\|v_{xx}\|_2 = C'' e^{-k't} \quad \text{for all} \quad t \geq 0. \tag{2.22}$$

Using (2.15)–(2.18)) we have

$$\begin{aligned} \|v^4(t)\|_1 &\leq \|v(t)\|_\infty^2 \|v(t)\|_2^2 \\ &\leq \|v_x(t)\|_2^2 \|v(t)\|_2^2 \\ &\leq \|v_{xx}(t)\|_2^2 \|v^0\|_2^2 e^{-2kt} \end{aligned}$$

and

$$\Big|\int (v_x^2 v)(t)\ dx\Big| \le \|v_x(t)\|_2^2\|v(t)\|_\infty$$
$$\le \|v_x(t)\|t_2^3$$
$$\le \|v_{xx}(t)\|_2^2 C' e^{-k't}.$$

It follows that for any fixed $\varepsilon > 0$ (to be chosen later) there exists $T'' > 0$ such that

$$\int \big|(v_x^2 v)(t)\big| + v(t)^4\ dx \le \varepsilon \int v_{xx}(t)^2\ dx, \quad \text{for all} \quad t > T''. \tag{2.23}$$

Choosing $\varepsilon > 0$ sufficiently small we conclude from (2.23) that

$$\int \Big(v_{xx}^2 - \frac{5}{3}v_x^2 v + \frac{5}{36}v^4\Big)(t)\ dx \ge \frac{1}{3}\int v_{xx}(t)^2\ dx \ge 0 \tag{2.24}$$

and

$$-2k\int \Big(v_{xx}^2 - \frac{5}{2}v_x^2 v + \frac{5}{18}v^4\Big)(t)\ dx$$
$$\le -2k''\int \Big(v_{xx}^2 - \frac{5}{3}v_x^2 v + \frac{5}{36}v^4\Big)(t)\ dx \tag{2.25}$$

for all $t > T''$. We deduce from (2.13), (2.24) and (2.25) that

$$\frac{1}{3}\int v_{xx}(t)^2\ dx \le \int \Big(v_{xx}^2 - \frac{5}{3}v_x^2 v + \frac{5}{36}v^4\Big)(t)\ dx$$
$$\le \Big(\int \Big(v_{xx}^2 - \frac{5}{3}v_x^2 v + \frac{5}{36}v^4\Big)(0)\ dx\Big)\ e^{-2k'(t-T')}$$
$$=: (C')^2 e^{-2k't},$$

proving (2.22) for all $t > T'$. The left-hand side of (2.22) being continuous, the estimate (2.22) remains valid for every $t \ge 0$ if we choose some larger constant C''.

Now we may easily complete the proof of the theorem. By (2.17), (2.18) and (2.22) for every fixed $k' < k$ there exists a positive constant $C_1 > 0$ such that

$$\|v(t)\|_{H_p^2} \le C_1 e^{-k't} \quad \text{for all} \quad t \ge 0. \tag{2.26}$$

Using the equation (2.6) hence we conclude easily that

$$\|w'(t)\|_{H_p^{-1}} \le C_2 e^{-k't} \quad \text{for all} \quad t \ge 0 \tag{2.27}$$

with some constant $C_2 > 0$. The estimate (2.8) follows from (2.26) and (2.27).

3. Proof of Theorem 1.2 for $m \geq 3$

The proof is constructive. It is based on an infinite sequence of polynomial conservation laws for the KdV equation obtained in [6]. We begin by recalling four important properties concerning these laws, established in [4] and [6].

(i) There exists a sequence of polynomials $P_n = P_n(v_0, \ldots, v_n)$ of $n+1$ variables, $n = 0, 1, \ldots$, such that setting also

$$P_{-1} = P_{-1}(v_0) := v_0$$

we have
$$\frac{\partial P_n}{\partial v_0} = (n+1)P_{n-1}, \quad n = 0, 1, \ldots.$$

(ii) The highest order term of P_n with respect to v_n is v_n^2.

(iii) Defining the *rank* of a product $v_0^{a_0} \cdots v_k^{a_k}$ by

$$\operatorname{rank}(v_0^{a_0} \cdots v_k^{a_k}) = \sum_{j=0}^{k} a_j \left(1 + \frac{j}{2}\right),$$

each term of P_n has rank $n+2$, $n = 0, 1, \ldots$.

(iv) Given a function $w \in C^\infty((0,1) \times (0,\infty))$ let us compute each partial derivative
$$\frac{\partial P_n}{\partial t}\left(w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\right)$$
by the Leibniz rule and then replace each factor of the form

$$\frac{\partial^{j+1} w}{\partial t \partial x^j} \quad \text{by} \quad \frac{\partial^j w}{\partial x^j}\left(-w\frac{\partial w}{\partial x} - \frac{\partial^3 w}{\partial x^3}\right), \quad j = 0, \ldots, n.$$

Then the result may be written in the form

$$\frac{\partial P_n}{\partial t}\left(w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\right) = \frac{\partial Y_n}{\partial t}\left(w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^{n+2} w}{\partial x^{n+2}}\right)$$

where Y_n is a suitable polynomial of $n+3$ variables, independent of the choice of the function w, $n = -1, 0, 1, \ldots$. (We remark that our notation differs from that of [4] and [6]: with their notation we have $P_n = T_{n-2}$ and $Y_n = X_{n-2}$; we follow [5].)

In order to simplify the notation in the sequel we shall denote the partial derivative $\frac{\partial^j w}{\partial x^j}$ of the solution function w by $w_{,j}$; in particular, $w_{,0} = w$. Furthermore, all integrals will be taken on the interval $(0,1)$, i.e., $\int \cdot \, dx = \int_0^1 \cdot \, dx$. Finally, we write $\|\cdot\|_p$ for the norm in the space $L^p(0,1)$, $1 \leq p \leq \infty$.

We need the following important lemma.

Lemma 3.1 *Consider the solution w of (1.4) for some $w_0 \in H_p^m$. Then the following properties hold true:*

(a) We have in $(0, \infty)$ the identity

$$\frac{d}{dt} \int (w - [w])^2 \; dx = -2K \int (w - [w])^2 \; dx. \tag{3.1}$$

(b) For each $n = 1, \dots, m$ there exist two polynomials

$$Q_{n-1} = Q_{n-1}(v_0, \dots, v_{n-1}) \quad \text{and} \quad R_{n-1} = R_{n-1}(v_0, \dots, v_{n-1})$$

of n variables such that the following identity is satisfied in $(0, \infty)$:

$$\begin{aligned} \frac{d}{dt} \int w_{,n}^2 + Q_{n-1}(w - [w], w_{,1}, \dots, w_{,n-1}) \; dx \\ = -2K \int w_{,n}^2 + R_{n-1}(w - [w], w_{,1}, \dots, w_{,n-1}) \; dx. \end{aligned} \tag{3.2}$$

(c) Each term of $Q_{n-1}(v_0, \dots, v_{n-1})$ and of $R_{n-1}(v_0, \dots, v_{n-1})$ is the product of at least three, not necessarily different, factors, the exponent of v_{n-1} being always less than four.

(a) As we have seen in the introduction, $[w(t)]$ does not depend on t:

$$[w(t)] = [w_0], \quad t \geq 0 \tag{3.3}$$

and the identity (3.1) is satisfied: the formal proofs given before are justified by the regularity (1.5) of the solution.

(b) Using properties (ii) and (iii) we see that P_n has the form

$$P_n(v_0, \dots, v_n) = v_n^2 + bv_n + c$$

where b is a polynomial in $v_0, \dots, v_{n-2}$ (constant if $n = 1$) and c is a polynomial in $v_0, \dots, v_{n-1}$. Using the periodic boundary conditions it follows that

$$\int P_n(w - [w], w_{,1}, \dots, w_{,n}) \; dx = \int w_{,n}^2 + Q_{n-1}(w - [w], w_{,1}, \dots, w_{,n-1}) \; dx \tag{3.4}$$

where

$$Q_{n-1}(w - [w], w_{,1}, \dots, w_{,n-1}) = \begin{cases} c(w - [w]) & \text{for } n = 1, \\ -b_x w_{,n-1} + c & \text{for } n \geq 2. \end{cases}$$

Now we compute

$$\frac{d}{dt} \int P_n(w - [w], w_{,1}, \dots, w_{,n}) \; dx. \tag{3.5}$$

We observe that, as a consequence of (i) and (3.3), $P_n(w_0 - [w], w_1, \dots, w_n)$ is a linear combination of $P_j(w_0 - [w], w_1, \dots, w_j)$ for $j = 0, \dots, n$ (we apply Taylor's formula with respect to w). Therefore, using (1.4) and (iv), we may compute (3.5) in the following way:

- we compute (3.5) formally, using the Leibniz rule;
- we replace the factors $\frac{\partial}{\partial t}(w-[w])$ by $-K(w-[w])$ and the factors $\frac{\partial}{\partial t}w_{,j}$ by $-Kw_{,j}$, $j=1,\dots,n$;
- we integrate over $(0,1)$.

This computation transforms $w_{,n}^2$ into $-2Kw_{,n}^2$ and $Q_{n-1}(w-[w],w_{,1},\dots,w_{,n-1})$ into $-2KR_{n-1}(w-[w],w_{,1},\dots,w_{,n-1})$ where R_{n-1} is a polynomial in n variables. Therefore we have

$$\frac{d}{dt}\int P_n(w-[w],w_{,1},\dots,w_{,n})\,dx = -2K\int w_{,n}^2+R_{n-1}(w-[w],w_{,1},\dots,w_{,n-1})\,dx, \quad (3.6)$$

and (3.2) follows from (3.4) and (3.6).

(c) One can readily verify that each term of

$$Q_{n-1}(v_0,\dots,v_{n-1})$$

and of

$$R_{n-1}(v_0,\dots,v_{n-1})$$

has the same rank as $P_n(v_0,\dots,v_n)$. Hence, if $cv_0^{a_0}\cdots v_k^{a_k}$ is a term of one of these two polynomials (with some nonzero constant c), then

$$\sum_{j=0}^{k} a_j\left(1+\frac{j}{2}\right)=n+2. \quad (3.7)$$

Since $k\le n-1$, we deduce from (3.7) the inequality

$$\sum_{j=0}^{k} a_j\left(1+\frac{n}{2}\right)>n+2,\ \text{ i.e., }\ \sum_{j=0}^{k} a_j>2;$$

this proves the first statement. On the other hand, (3.7) implies that

$$\sum_{j=0}^{k} a_j\left(1+\frac{n-1}{2}\right)\le n+2,$$

whence, since $n\ge 1$,

$$a_{n-1}\le 2+\frac{2}{n+1}\le 3.$$

This proves the second statement.

Now we turn to the proof of Theorem 1.2. First we observe that (3.1) immediately implies

$$\|w(\cdot,t)-[w]\|_2=e^{-Kt}\|w_0-[w_0]\|_2,\quad t\ge 0. \quad (3.8)$$

We will prove by induction on j that

$$\|w_{,j}(\cdot,t)\|_2 \le ce^{-K't}, \quad t \ge 0 \tag{3.9}$$

for $j = 1, \ldots, m$.

Let $n \le m$ be a positive integer and assume that (3.9) is satisfied for $j = 1, \ldots, n-1$. We will prove that it is satisfied for $j = n$, too. Using the trivial inequalities

$$\|v - [v]\|_\infty \le \|v_{,1}\|_1, \quad v \in H^1(0,1)$$

and

$$\|v\|_1 \le \|v\|_p, \quad v \in L^p(0,1), \quad p \in (1,\infty]$$

and applying part (c) of Lemma 3.1 we obtain the estimates

$$\|Q_{n-1}(w-[w], w_{,1}, \ldots, w_{,n-1})\|_1 \le C\|w_{,n-1}\|_2^N \|w_{,n}\|_2^2$$

and

$$\|R_{n-1}(w-[w], w_{,1}, \ldots, w_{,n-1})\|_1 \le C\|w_{,n-1}\|_2^N \|w_{,n}\|_2^2$$

where C and N are positive integers, independent of $t \ge 0$. Applying (3.8) if $n = 1$ and (3.9) for $j = n-1$ if $n \ge 2$, there follows the existence of a positive number T such that for all $t > T$ the following inequalities hold:

$$\int Q_{n-1}(w-[w], w_{,1}, \ldots, w_{,n-1})\, dx \le \frac{1}{2}\|w_{,n}\|_2^2 \tag{3.10}$$

and

$$\begin{aligned} &-2K \int w_{,n}^2 + R_{n-1}(w-[w], w_{,1}, \ldots, w_{,n-1})\, dx \\ &\qquad \le -2K' \int w_{,n}^2 + Q_{n-1}(w-[w], w_{,1}, \ldots, w_{,n-1})\, dx. \end{aligned} \tag{3.11}$$

Majorizing the right side of (3.2) by use of (3.11) we see with the use of (3.10) that there is a constant c_1 such that

$$\int Q_{n-1}(w-[w], w_{,1}, \ldots, w_{,n-1})\, dx \le c_1 e^{-2K't}, \quad t \ge T.$$

Hence, using (3.10) again,

$$\|w_{,n}(\cdot,t)\|_2 \le \sqrt{2c_1} e^{-K't}, \quad t \ge T. \tag{3.12}$$

Since $\|w_{,n}(\cdot,t)\|_2$ is continuous on $[0,T]$, it is also bounded there:

$$\|w_{,n}(\cdot,t)\|_2 \le c_2, \quad t \in [0,T]. \tag{3.13}$$

The validity of (3.9) for $j = n$ now follows from (3.12) and (3.13) by taking

$$C := \max\left\{\sqrt{2c_1}, c_2 e^{K't}\right\}.$$

We have thus proved (3.8) and (3.9) for $j = 1, \ldots, m$. Equivalently, we have established that

$$\|w(\cdot,t) - [w]\|_{H_p^m} \le Ce^{-K't}, \quad t \ge 0. \tag{3.14}$$

To complete the proof of the theorem it suffices to prove the estimate

$$\left\| \frac{\partial w}{\partial t}(\cdot,t) \right\|_{H_p^{m-3}} \le C\|w(\cdot,t) - [w]\|_{H_p^m}$$

for some constant C, independent of $t \ge 0$. Since

$$w_t = -K(w - [w]) - ww_x - w_{xxx} = K(w - [w]) - (w - [w])w_x - [w]w_x - w_{xxx},$$

in view of (3.14) it is enough to show that

$$\|(w(\cdot,t) - [w])w_{,1}\|_{H_p^{m-3}} \le Ce^{-K't}, \quad t \ge 0. \tag{3.15}$$

For any nonnegative integers $j \le m-1$ and $k \le m-1$, using the definition of H_p^m we have

$$\begin{aligned}
\left\| \frac{\partial^j}{\partial x^j}(w - [w]) \frac{\partial^k}{\partial x^k} w_{,1} \right\|_2 &\le \left\| \frac{\partial^j}{\partial x^j}(w - [w]) \right\|_\infty \left\| \frac{\partial^k}{\partial x^k} w_{,1} \right\|_2 \\
&\le \|w_{,j+1}\|_1 \|w_{,k+1}\|_2 \\
&\le \|w_{,j+1}\|_2 \|w_{,k+1}\|_2 \\
&\le \|w - [w]\|_{H_p^m}^2.
\end{aligned}$$

Using the Leibniz formula it follows that

$$\|(w - [w])w_{,1}\|_{H_p^{m-1}} \le C\|w - [w]\|_{H_p^m}^2. \tag{3.16}$$

Since $K' > 0$, (3.14) and (3.16) imply

$$\|(w(\cdot,t) - [w])w_{,1}\|_{H_p^{m-1}} \le Ce^{-K't}, \quad t \ge 0 \tag{3.17}$$

which is even stronger than (3.15).

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Vilmos Komornik
Département de Mathématique, Université de Strasbourg,
7 rue René Descartes, 67084 Strasbourg Cedex, France
E-mail address: vilmos.komornik@math.unistra.fr