Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/spm

(3s.) **v. 29** 1 (2011): 67–70.
ISSN-00378712 in press
doi:10.5269/bspm.v29i1.11224

Maximal Divisible Subgroups in Modular Group Rings over a Finite Ring or a Field

Peter Danchev

abstract: We describe up to an isomorphism the algebraic structure of the maximal divisible subgroup $dVR[G]$ of the group $VR[G]$ of normalized units in a group ring $R[G]$, provided that G is an abelian group such that G_t/G_p is (infinite) bounded and R is a field of prime characteristic p. This supplies recent author's results in Rad. Mat. (2004), Commun. Algebra (2011), Bull. Braz. Math. Soc. (2010) and J. Alg. Numb. Th. Acad. (2010).

Key Words: abelian groups, divisible subgroups, commutative rings, fields, binomial extensions, normalized units, cardinalities.

Contents

1. Introduction

Throughout this short paper, let G be a multiplicative abelian group as is the custom when discussing group rings and R a commutative unitary ring of prime characteristic p. As usual, suppose $R[G]$ is the group ring of G over R with unit group $UR[G]$ and its normalized component $VR[G]$; note that the direct decomposition $UR[G] = VR[G] \times R^*$ holds where R^* is the unit group of R. Moreover, let dG and G_t to denote the maximal divisible subgroup and the maximal torsion subgroup of G, respectively; notice that the direct decomposition $G_t = \coprod_p G_p$ is true whenever G_p is the p-primary component of G. Likewise, let $N(R)$ be the nil-radical of R.

Traditionally, for any set S, we let $|S|$ denote its cardinality and for any natural number n, we let ζ_n denote the primitive nth root of unity. Moreover, as usual, if R is a field, $R(\zeta_n)$ denotes the binomial extension of R by adding ζ_n with dimension equal to $(R(\zeta_n) : R)$ but if R is a ring, $R[\zeta_n]$ denotes the free R-module algebraically generated as a ring by ζ_n with dimension equal to $[R[\zeta_n] : R]$. Denote by $L^{(p)}$ the maximal (p-)perfect subring of a ring L with characteristic p, and by $G^{(p)}$ the maximal p-divisible subgroup of a group G. Also, $id(L) = \{e \in L : e^2 = e\}$ is the set of all idempotents in L.

All other unstated explicitly notions and notations are standard and follow those from [8].

2000 *Mathematics Subject Classification:* 16S34, 16U60, 20K10, 20K21

A problem of major interest in the theory of commutative modular group rings is to characterize the maximal divisible subgroup. This subject is motivated in accordance to the direct decomposition $VR[G] = dVR[G] \times K$ for some subgroup K of $VR[G]$.

In [9], $dV_pR[G]$ was described up to isomorphism in terms of R and G. Furthermore, we extended in [2] (see also [3]) this result describing the isomorphism class of $dVR[G]$, provided R is a field and G is a group such that $G_t = G_p$. Next, we improved our technique in [4] and [5] and, as a result, we obtained a comprehensive description of $dVR[G]$ assuming only that $G_t = G_p$. Finally, in [6] a satisfactory characterization of $dVR[G]$ was given uniquely in terms associated with R and G and their sections, provided that R is an indecomposable ring (or even more, R is a direct product of finitely many indecomposable subrings) and G is a group with the restriction that G_t/G_p is finite.

So, the goal of this brief article is to strengthen this last achievement establishing the isomorphism structure of $dVR[G]$ for infinite quotient G_t/G_p. However, we shall restrict our attention only when R is a field or R is a finite ring.

2. Main Results

First, one simple but very useful reduction lemma.

Lemma 1. *Suppose G is a group such that G_t/G_p is bounded. Then G is the direct sum of a bounded group and a p-mixed group.*

Proof. Since $G_t/G_p \cong \coprod_{q\neq p} G_q$ is bounded and $\coprod_{q\neq p} G_q$ is pure in G (this is true because it is pure in G_t being its direct factor and G_t is pure in G), it is a folklore fact that $\coprod_{q\neq p} G_q$ is a direct factor of G too, $G = (\coprod_{q\neq p} G_q) \times M$ say for some $M \leq G$. Clearly $M \cong G/\coprod_{q\neq p} G_q$ is p-mixed owing to the fact that $M_t \cong (G/\coprod_{q\neq p} G_q)_t = G_t/\coprod_{q\neq p} G_q \cong G_p$. $\triangle$

We now recall one crucial assertion from [4] and [5] that will be used in the sequel.

Theorem 2. *Suppose A is a p-mixed group and L a ring of prime characteristic p. Then the following isomorphism is valid*:

$$dVL[A] \cong \coprod_\lambda \mathbf{Z}(p^\infty) \times \coprod_\mu (dA/dA_p)$$

where $\lambda = \max(|L^{(p)}|, |A^{(p)}|)$ if $dA_p \neq 1$, or $\lambda = \max(|N(L^{(p)})|, |A^{(p)}|)$ if $dA_p = 1$, $A^{(p)} \neq 1$ and $N(L^{(p)}) \neq 0$, or $\lambda = 0$ if either $dA_p = 1$ or $A^{(p)} = 1$ and $N(L^{(p)}) = 0$, whereas $\mu = |\mathrm{id}(L)| \geq \aleph_0$ or $\mu = \log_2|\mathrm{id}(L)|$ if $|\mathrm{id}(L)| < \aleph_0$.

So, we have at our disposal all the information needed to prove the first main result.

Theorem 3. *Suppose that R is a field and that G is a group such that G_t/G_p is infinite bounded. Then the following isomorphism formula is fulfilled:*

$$dUR[G] \cong \coprod_\lambda \mathbf{Z}(p^\infty) \times \coprod_\mu (dG/dG_p) \times \coprod_{n=0}^{\infty} \coprod_{a(n)} dR(\zeta_n)^*$$

where $\lambda = max(|R^{p^\omega}|, |G^{(p)}|)$ if $dG_p \neq 1$ or $\lambda = 0$ if either $dG_p = 1$ or $G^{(p)} = \coprod_{q\neq p} G_q$ whereas $\mu = |G_t/G_p|$, and $a(n) = \frac{|\{g\in G_t/G_p : order(g)=n\}|}{(R(\zeta_n):R)}$.

Proof. Applying Lemma 1, one can write $G = B \times M$ where $B = \coprod_{q \neq p} G_q \cong G_t/G_p$ is bounded and M is p-mixed. Consequently, $R[G] \cong (R[B])[M]$ and hence $UR[G] \cong U(R[B])[M] = V(R[B])[M] \times UR[B]$. Thus, $dUR[G] \cong dV(R[B])[M] \times dUR[B]$.

Furthermore, we shall describe these two direct factors separately:

For the characterization of first factor $dV(R[B])[M]$ we employ Theorem 2 substituting $L = R[B]$ and $A = M$; observe that $R[B]$ is a commutative unitary ring of $char(R[B]) = p$. Taking into account [7], especially that $id(R[B]) = |B| = |G_t/G_p|$, the classical fact that $R^{(p)} = R^{p^\omega}$ and some other well-known arguments like $dG = dM$ whence $dG_p = dM_p$, $M^{(p)} \cong G^{(p)}/\coprod_{q\neq p} G_q$, $L^{(p)} = R^{p^\omega}[B]$ and hence $N(L^{(p)}) = N(R^{p^\omega})[B] = 0$ (see Proposition 4 below), the desired equalities of λ and μ are obtained.

For the description of the second factor we appeal to [1] to infer that $UR[\coprod_{q\neq p} G_q] \cong \coprod_{n=0}^{\infty} \coprod_{a(n)} R(\zeta_n)^*$ where $a(n)$ is given as above. Therefore, it follows at once that $dUR[\coprod_{q\neq p} G_q] \cong \coprod_{n=0}^{\infty} \coprod_{a(n)} dR(\zeta_n)^*$, and we are done. △

Proposition 4. *If C is an abelian group whose $C_p = 1$, then $N(R[C]) = N(R)[C]$.*

Proof. Assume $x = r_1c_1 + \cdots + r_sc_s \in N(R[C])$. Hence there is an $m \in \mathbb{N}$ such that $(r_1c_1 + \cdots + r_sc_s)^{p^m} = r_1^{p^m} c_1^{p^m} + \cdots + r_s^{p^m} c_s^{p^m} = 0$. Since x is written in canonical form and $C_p = 1$, it follows that $r_1^{p^m} c_1^{p^m} + \cdots + r_s^{p^m} c_s^{p^m}$ is in canonical record as well. Consequently, $r_1^{p^m} = \cdots = r_s^{p^m} = 0$ and thus $r_1, \cdots, r_s \in N(R)$. Finally x obviously lies in $N(R)[C]$ as required. This proves that $N(R[C]) \subseteq N(R)[C]$. The converse inclusion is elementary, so that it is equivalent to the desired equality. △

The second chief result is the following.

Theorem 5. *Let R be finite and G a group for which G_t/G_p is finite. Then the following isomorphism formula is fulfilled:*

$$dVR[G] \cong \coprod_{\lambda} \mathbf{Z}(p^\infty) \times \coprod_{\mu} (dG/dG_p)$$

where $\lambda = |G^{(p)}|$ if $dG_p \neq 1$ or $dG_p = 1$, $G^{(p)} \neq \coprod_{q\neq p} G_q$ and $N(R^{(p)}) \neq 0$ as well as $\lambda = 0$ if either $dG_p = 1$ and $N(R^{(p)}) = 0$, or $G^{(p)} = \coprod_{q\neq p} G_q$ and $N(R^{(p)}) = 0$, whereas $\mu = \sum_{d/exp(G_t/G_p)} \sum_{1 \leq i \leq log_2|id(R)|} a_i(d)$ with $a_i(d) = \frac{|\{g \in G_t/G_p : order(g) = d\}|}{[R_i[\zeta_d] : R_i]}$.

Proof. As above, Lemma 1 implies that $G = B \times M$ where $B = \coprod_{q\neq p} G_q \cong G_t/G_p$ is finite and M is p-mixed. Therefore, $R[G] \cong R[B][M]$ whence $VR[G] \times U(R) = UR[G] \cong U(R[B])[M] = V(R[B])[M] \times UR[B]$ and thus $dVR[G] \cong dV(R[B])[M]$ since both $U(R)$ and $UR[B]$ are finite and thereby their maximal divisible subgroups are equal to 1. We next apply Theorem 2 to $L = R[B]$ and $A = M$. Observe that L is a finite commutative unitary ring of $char(L) = p$. So, $id(L)$ is finite as computed in [7]. It easily follows as in the previous theorem that $L^{(p)} = R^{(p)}[B]$ since B is p-divisible and, moreover, in view of Proposition 4 we have $N(L^{(p)}) = N(R^{(p)})[B]$, hence $N(L^{(p)}) = 0$ exactly when $N(R^{(p)}) = 0$. On the other hand, $dG = dM$ whence $dG_p = dM_p$, $M^{(p)} \cong G^{(p)}/\coprod_{q\neq p} G_q$ with

$|M^{(p)}| = |G^{(p)}|$ whenever $|M^{(p)}| \geq \aleph_0$ since $\coprod_{q\neq p} G_q$ is finite. $\triangle$

Remark. The last statement can also be derived from Corollary 4 of [6]. Nevertheless, the above proof is slightly more conceptual and easy than that of the original source [6].

A problem which immediately arises is the following:

Problem. Extend the preceding theorems to the case when G_t/G_p is unbounded and R is a field, or to the case when G_t/G_p is infinite bounded and R is a finite ring.

References

1. P. V. Danchev, *Normed units in abelian group rings*, Glasgow Math. J. **43** (2001), 365-373.
2. P. V. Danchev, *Maximal divisible subgroups in modular group algebras of p-mixed and p-splitting abelian groups*, Rad. Mat. **13** (2004), 23-32.
3. P. V. Danchev, *Maximal divisible subgroups in modular group algebras of torsion abelian groups over certain commutative rings*, An. St. Univ. "Al. I. Cuza", Iasi - Mat. **53** (2007), 325-329.
4. P. V. Danchev, *Maximal divisible subgroups in p-mixed modular abelian group rings*, Commun. Algebra **39** (2011).
5. P. V. Danchev, *Maximal divisible subgroups in modular group rings of p-mixed abelian groups*, Bull. Braz. Math. Soc. **41** (2010), 63-72.
6. P. V. Danchev, *Maximal divisible subgroups in abelian group algebras over indecomposable rings*, J. Alg. Numb. Th. Acad., to appear.
7. P. V. Danchev, *The number of idempotents in commutative group rings of prime characteristic*, to appear.
8. G. Karpilovsky, Unit Groups of Group Rings, Longman Sci. and Techn., Harlow (1989).
9. N. A. Nachev, *Invariants of the Sylow p-subgroup of the unit group of a commutative group ring of characteristic p*, Commun. Algebra **23** (1995), 2469-2489.

Peter Danchev
13, General Kutuzov Str.
bl. 7, fl. 2, ap. 4
4003 Plovdiv, Bulgaria
E-mail address: pvdanchev@yahoo.com