Bol. Soc. Paran. Mat. (3s.) **v. 29** 1 (2011): 25–39.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/spm doi:10.5269/bspm.v29i1.11491

Positive solutions with changing sign energy to a nonhomogeneous elliptic problem of fourth order

M.Talbi and N.Tsouli

abstract: In this paper, we study the existence for two positive solutions to a nonhomogeneous elliptic equation of fourth order with a parameter λ such that $0 < \lambda < \hat{\lambda}$. The first solution has a negative energy while the energy of the second one is positive for $0 < \lambda < \lambda_0$ and negative for $\lambda_0 < \lambda < \hat{\lambda}$. The values λ_0 and $\hat{\lambda}$ are given under variational form and we show that every corresponding critical point is solution of the nonlinear elliptic problem (with a suitable multiplicative term).

Key Words:: Ekeland's principle, p-Laplacian operator, Palais-Smale condition.

Contents

1. Introduction

We consider the problem with Navier boundary conditions

$$(P_\lambda) \qquad \begin{cases} \Delta_p^2 u &= \lambda |u|^{q-2}u + |u|^{r-2}u \quad \text{in} \quad \Omega \\ u &= \Delta u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

Here Ω is a smooth domain in $\mathbb{R}^N$ $(N \geq 1)$, Δ_p^2 is the p-biharmonic operator defined by $\Delta_p^2 u = \Delta(|\ \Delta u\ |^{p-2}\ \Delta u)$, λ is a positive parameter, p, q and r are reals such that

$$1 < q < p < r < p_2^*, \text{ where } \begin{cases} p_2^* = \dfrac{Np}{N-2p} & \text{if } p < N/2, \\ p_2^* = +\infty & \text{if } p \geq N/2. \end{cases}$$

Such kind of problems with combined concave and convex nonlinearities were studied recently by several authors [2,3,4,5,6,7,9,10,11,17] in the case of operator Δ_p. Our main results here can be summarized as follows:
Let us put $X = W_0^{2,p}(\Omega) \cap W^{2,p}(\Omega)$. We find two characteristic values λ_0 and $\hat{\lambda}$ $(\lambda_0 < \hat{\lambda})$ under variational form, i.e.

$$(V) \qquad \lambda_0 = C_0(p,q,r) \inf_{u \in X\setminus\{0\}} F(u) \quad \text{and} \quad \hat{\lambda} = \hat{C}(p,q,r) \inf_{u \in X\setminus\{0\}} F(u),$$

2000 *Mathematics Subject Classification:* 35J60, 35J25, 35J65

such that two branches of positive solutions to (P_λ) exist for $\lambda \in]0, \hat{\lambda}[$ (the functional F will be given below). Moreover, the energy of the first positive solution is negative for $\lambda \in]0, \hat{\lambda}[$ while the energy of the second positive solution changes sign at λ_0, i.e. it is positive for$\lambda \in]0, \lambda_0[$ and negative for $\lambda \in]\lambda_0, \hat{\lambda}[$. Notice that these two positive solutions are found simultaneously and that our approach does not use the mountain-pass lemma.
On the other hand, we show that every solution of (V) is a solution of the problem (P_λ) (with a suitable multiplicative term). This second point lets expect that the first nonlinear eigenvalue ζ of (V), i.e.

$$\zeta = sup\{\lambda > 0 : (P_\lambda) \quad \text{has a nonnegative solution}\}$$

may satisfy a variational problem similar to (V) (see [4] for $p = 2$). Let us precise that $\hat{\lambda}$ coincides with ζ when $q \to p$ and that $\hat{\lambda}$ constitutes a good minoration of ζ in the general case $1 < q < p$.

We consider the transformation of Poisson problem used by P.Drábek and M.Ôtani (cf. [12]):
We recall some properties of the Dirichlet problem for the Poisson equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

It is well known that (1.1) is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $f \in L^p(\Omega)$ and for any $p \in]1, +\infty[$.
We denote by :
$X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$,
$\|u\|_p = (\int_\Omega |u|^p dx)^{1/p}$ the norm in $L^p(\Omega)$,
$\|u\|_{2,p} = (\|\Delta u\|_p^p + \|u\|_p^p)^{1/p}$ the norm in X,
$\|u\|_\infty$ the norm in $L^\infty(\Omega)$,
and $< ., . >$ is the duality bracket between $L^p(\Omega)$ and $L^{p'}(\Omega)$, where $p' = p/(p-1)$.
Denote by Λ the inverse operator of $-\Delta : X \to L^p(\Omega)$.
The following lemma gives us some properties of the operator Λ (cf. [12], [16])

Lemma 1.1 *(i) (Continuity): There exists a constant $c_p > 0$ such that*

$$\|\Lambda f\|_{2,p} \leq c_p \|f\|_p$$

holds for all $p \in]1, +\infty[$ and $f \in L^p(\Omega)$.

(ii) (Continuity) Given $k \in \mathbb{N}^*$, there exists a constant $c_{p,k} > 0$ such that

$$\|\Lambda f\|_{W^{k+2,p}} \leq c_{p,k} \|f\|_{W^{k,p}}$$

holds for all $p \in]1, +\infty[$ and $f \in W^{k,p}(\Omega)$.

(iii) *(Symmetry) The following identity:*

$$\int_\Omega \Lambda u \cdot v dx = \int_\Omega u \cdot \Lambda v dx$$

holds for all $u \in L^p(\Omega)$ *and* $v \in L^{p'}(\Omega)$ *with* $p \in]1, +\infty[$.

(iv) *(Regularity) Given* $f \in L^\infty(\Omega)$, *we have* $\Lambda f \in C^{1,\alpha}(\bar\Omega)$ *for all* $\alpha \in]0,1[$; *moreover, there exists* $c_\alpha > 0$ *such that*

$$\|\Lambda f\|_{C^{1,\alpha}} \leq c_\alpha \|f\|_\infty.$$

(v) *(Regularity and Hopf-type maximum principle) Let* $f \in C(\bar\Omega)$ *and* $f \geq 0$ *then* $w = \Lambda f \in C^{1,\alpha}(\bar\Omega)$, *for all* $\alpha \in]0,1[$ *and* w *satisfies:* $w > 0$ *in* Ω, $\frac{\partial w}{\partial n} < 0$ *on* $\partial\Omega$.

(vi) *(Order preserving property) Given* $f, g \in L^p(\Omega)$ *if* $f \leq g$ *in* Ω, *then* $\Lambda f < \Lambda g$ *in* Ω.

Remark 1.1 $(\forall u \in X)(\forall v \in L^p(\Omega)) \quad v = -\Delta u \Longleftrightarrow u = \Lambda v.$
Let us denote N_p the Nemytskii operator defined by

$$\begin{cases} N_p(v)(x) = |v(x)|^{p-2}v(x) & \text{if } v(x) \neq 0 \\ N_p(v)(x) = 0 & \text{if } v(x) = 0, \end{cases}$$

and we have $\forall v \in L^p(\Omega), \forall w \in L^{p'}(\Omega)$:

$$N_p(v) = w \Longleftrightarrow v = N_{p'}(w).$$

We define the functionals $P, Q, R : L^p(\Omega) \to \mathbb{R}$ as follows:

$$P(v) = \| v \|_p^p, \quad Q(v) = ||\Lambda v||_q^q \quad \text{and} \quad R(v) = ||\Lambda v||_r^r.$$

The operator Λ enables us to transform problem (P_λ) to an other problem which we will study in the space $L^p(\Omega)$.

Definition 1.1 We say that $u \in X \setminus \{o\}$ is a solution of problem (P_λ), if $v = -\Delta u$ is a solution of the following problem

$$(P'_\lambda) \qquad \begin{cases} \text{Find } v \in L^p(\Omega) \setminus \{o\}, \text{ such that} \\ N_p(v) = \lambda\Lambda(N_q(\Lambda v)) + \Lambda(N_r(\Lambda v)) \quad \text{in} \quad L^{p'}(\Omega). \end{cases}$$

For solutions of (P_λ) we understand critical points of the associated Euler-Lagrange functional $E_\lambda \in \mathcal{C}^1(L^p(\Omega))$, given by

$$E_\lambda(v) = \frac{1}{p}P(v) - \lambda\frac{1}{q}Q(v) - \frac{1}{r}R(v).$$

As in (cf. [13,19]), we introduce the modified Euler-Lagrange functional defined on $\mathbb{R} \times L^p(\Omega)$ by $\tilde{E}_\lambda(t, v) = E - \lambda(tv)$. If v is an arbitrary element of $L^p(\Omega)$, $\partial_t \tilde{E}_\lambda(., v)$ (resp. $\partial_{tt}\tilde{E}_\lambda(., v)$)are the first (resp. second) derivative of the real valued function: $t \mapsto \tilde{E}_\lambda(t, v)$.

2. Preliminary results

Since the functional $\tilde{E}_\lambda$ is even in t and that we are interested by the positive solutions, we limit our study for $t > 0$.

Lemma 2.1 *For every $v \in L^p(\Omega) \setminus \{0\}$, There is a unique $\lambda(v) > 0$ such that the real valued function $t \mapsto \partial\tilde{E}_\lambda(t, v)$ has exactly two positive zeros (resp. one positive zero) if $0 < \lambda < \lambda(v)$ (resp. $\lambda = \lambda(v)$). This function has no zero for $\lambda > \lambda(v)$.*

Proof: Let v be an arbitrary element of $L^p(\Omega) \setminus \{0\}$ and let us write

$$\partial_t\tilde{E}_\lambda(t, v) = t^{q-1}\tilde{F}_\lambda(t, v), \quad \text{where } \tilde{F}_\lambda(t, v) = t^{p-q}P(v) - \lambda Q(v) - t^{r-q}R(v).$$

Then

$$\partial_{tt}\tilde{E}_\lambda(t, v) = (q-1)t^{q-2}\tilde{F}_\lambda(t, v) + t^{q-1}\partial_t\tilde{F}_\lambda(t, v),$$

holds true, with

$$\partial_t\tilde{F}_\lambda(t, v) = t^{p-q-1}[(p-q)P(v) - (r-q)t^{r-p}R(v)].$$

It is clair that the real valued function $t \mapsto \tilde{F}_\lambda(t, v)$ is increasing on $]0, t(v)[$, decreasing on $]t(v), +\infty[$ and attains its unique maximum for $t = t(v)$, where

$$t(v) = \left(\frac{p-q}{r-q}\frac{P(v)}{R(v)}\right)^{\frac{1}{r-p}}. \tag{2.1}$$

Thus, if $\tilde{F}_\lambda(t(v), v) > 0$ (resp. $\tilde{F}_\lambda(t(v), v) = 0$), the function $t \mapsto \tilde{F}_\lambda(t, v)$ has two positive zeros (resp. one positive zero) and has no zero if $\tilde{F}_\lambda(t(v), v) < 0$. On the other hand, a direct computation gives

$$\tilde{F}_\lambda(t(v), v) = \frac{r-p}{p-q}\left(\frac{p-q}{r-q}\frac{P(v)}{R(v)}\right)^{\frac{r-q}{r-p}}R(v) - \lambda Q(v).$$

We deduce that $\tilde{F}_\lambda(t(v), v) > 0$ (resp. $\tilde{F}_\lambda(t(v), v) < 0$) for $\lambda < \lambda(v)$ (resp. $\lambda > \lambda(v)$) and $\tilde{F}_{\lambda(v)}(t(v), v) = 0$, where

$$\lambda(v) = \hat{c}\frac{P^{\frac{r-q}{r-p}}(v)}{Q(v)R^{\frac{p-q}{r-p}}(v)}, \tag{2.2}$$

with

$$\hat{c} = \frac{r-p}{p-q}\left(\frac{p-q}{r-q}\right)^{\frac{r-q}{r-p}}.$$

Hence, if $\lambda \in]0, \lambda(v)[$, the real valued function $t \mapsto \partial_t\tilde{E}_\lambda(t, v)$ has two positive zeros, denoted by $t_1(v, \lambda)$ and $t_2(v, \lambda)$, verifying $0 < t_1(v, \lambda) < t(v) < t_2(v, \lambda)$. Since $\tilde{F}_\lambda(t_1(v, \lambda), v) = \tilde{F}_\lambda(t_2(v, \lambda), v) = 0$, $\partial_t\tilde{F}_\lambda(t, v) > 0$ for $t < t(v)$ and $\partial_t\tilde{F}_\lambda(t, v) < 0$ for $t > t(v)$, it follows that

$$\partial_{tt}\tilde{E}_\lambda(t_1(v, \lambda), v) > 0 \quad \text{and} \quad \partial_{tt}\tilde{E}_\lambda(t_2(v, \lambda), v) < 0. \tag{2.3}$$

This means that the real valued function $t \mapsto \tilde{E}_\lambda(t, v)$, $(t > 0)$ achieves its unique local minimum at $t = t_1(v, \lambda)$ and its global maximum at $t = t_2(v, \lambda)$.□

Lemma 2.2 *If we put $\hat{\lambda} = \inf_{v \in L^p(\Omega)\setminus\{0\}} \lambda(v)$, then $\hat{\lambda} > 0$.*

Proof: By Sobolev injection theorem, we have $X \hookrightarrow L^q(\Omega)$ and $X \hookrightarrow L^r(\Omega)$. Thus there exists two positive constants c_1 and c_2 such that

$$||\Lambda v||_q \leq c_1||v||_p \quad \text{et} \quad ||\Lambda v||_r \leq c_2||v||_p.$$

Then (2.2) implies for every $v \in L^p(\Omega) \setminus \{0\}$

$$\lambda(v) \geq \frac{\hat{c}}{c_1^q c_2^{\frac{r(p-q)}{r-p}}} > 0.$$

□

Consider $\lambda \in]0, \hat{\lambda}[$ and let (v_n) be minimizing sequence of $v \mapsto \tilde{E}_\lambda(t_1(v,\lambda), v)$ in $L^p(\Omega) \setminus \{0\}$ (resp. of $v \mapsto \tilde{E}_\lambda(t_2(v,\lambda), v)$).
Put $V_n = t_1(v_n,\lambda)v_n$ and $W_n = t_2(v_n,\lambda)v_n$.

Lemma 2.3 *The sequences (V_n) and (W_n) verify :*

$$(i) \qquad \limsup_{n\to+\infty} ||V_n||_p < +\infty \quad (\textit{resp.} \limsup_{n\to+\infty} ||W_n||_p < +\infty\ )$$
$$(ii) \qquad \liminf_{n\to+\infty} ||V_n||_p > 0 \quad (\textit{resp.} \liminf_{n\to+\infty} ||W_n||_p > 0)$$

Proof: (i) We know that $\partial_t \tilde{E}_\lambda[t_1(v_n,\lambda), v_n) = 0$.
Hence

$$||V_n||_p^p = \lambda||\Lambda V_n||_q^q + ||\Lambda V_n||_r^r. \tag{2.4}$$

Suppose that there is a subsequence of (V_n), still denoted by (V_n) such that $\lim_{n\to+\infty} ||V_n||_p = +\infty$. Us $r > q$, there exist a constant $c > 0$ such that $||\Lambda V_n||_q \leq c||\Lambda V_n||_r$. Then the relation (2.4) implies that $\lim_{n\to+\infty} ||\Lambda V_n||_r = +\infty$. The fact that $0 < q < r$ enables us to deduce: $||\Lambda V_n||_q^q = o_n(||\Lambda V_n||_r^r)$. Then

$$||V_n||_p^p = ||\Lambda V_n||_r^r(1 + o_n(1)),$$

and

$$E_\lambda(V_n) = ||\Lambda V_n||_r^r(\frac{1}{p} - \frac{1}{r} + o_n(1)).$$

which implies that $E_\lambda(V_n)$ tends to $+\infty$ as n goes to $+\infty$ and this is impossible. The same arguments with a minimizing sequence (v_n) of $v \mapsto \tilde{E}_\lambda(t_2(v,\lambda), v)$ show that $\limsup_{n\to+\infty} ||W_n||_p < +\infty$.
(ii) Relation (2.4) and the fact that $\partial_{tt}\tilde{E}_\lambda[t_1(v_n,\lambda), v_n) > 0$, implies

$$(p-1)||V_n||_p^p - \lambda(q-1)||\Lambda V_n||_q^q - (r-1)||\Lambda V_n||_r^r > 0. \tag{2.5}$$

If we combine (2.4) and (2.5), we obtain for every $n \in \mathbb{N}$

$$\lambda(p-q)||\Lambda V_n||_q^q + (p-r)||\Lambda V_n||_r^r > 0.$$

So

$$\begin{aligned} E_\lambda(V_n) &= \lambda\frac{q-p}{pq}Q(V_n)+\frac{r-p}{pr}R(V_n) \\ &\leq \frac{-1}{pq}(\lambda(p-q)Q(V_n)+(p-r)R(v_n)) \\ &< 0. \end{aligned}$$

suppose that there is a subsequence of (V_n), still denoted by (V_n) such that $\lim\limits_{n\to+\infty}||V_n||_p=0$. By Sobolev injection theorem we deduce that $\lim\limits_{n\to+\infty}||\Lambda V_n||_q=0$ and $\lim\limits_{n\to+\infty}||\Lambda V_n||_r=0$. It follows that $\lim\limits_{n\to+\infty}E_\lambda(V_n)=0$, *i.e* $\inf\limits_{v\in L^p(\Omega)\setminus\{0\}}\tilde{E}_\lambda(t_1(v,\lambda),v)=0$, which is impossible since $\tilde{E}_\lambda(t_1(v_n,\lambda),v_n)<0$ for every n.

Let (v_n) be a minimizing sequence of $v\mapsto\tilde{E}_\lambda(t_2(v),v)$ in $L^p(\Omega)\setminus\{0\}$. Sinse $\partial_t\tilde{E}_\lambda(t_2(v_n),v_n)=0$ and $\partial_{tt}\tilde{E}_\lambda(t_2(v_n),v_n)<0$, it follows that

$$\begin{cases} ||W_n||_p^p & -\lambda||\Lambda W_n||_q^q-||\Lambda W_n||_r^r=0, \\ (p-1) & ||W_n||_p^p-\lambda(q-1)||\Lambda W_n||_q^q-(r-1)||\Lambda W_n||_r^r<0. \end{cases}$$

Combining the two last inequalities and by Sobolev injection theorem there exist a constant c' such that for every n we have

$$(p-q)||W_n||_p^p<(r-q)||\Lambda W_n||_r^r\leq c'||W_n||_p^r.$$

Hence

$$(p-q)\leq c'||W_n||_p^{r-p}.$$

Now, suppose that there is a subsequence of (W_n), still denoted by (W_n) such that $\lim\limits_{n\to+\infty}||W_n||_p=0$. This implies that $p-q\leq 0$. which is impossible since $p>q$.□

Lemma 2.4 *The functionals* $v\mapsto\tilde{E}_\lambda(t_1(v,\lambda),v)$ *and* $v\mapsto\tilde{E}_\lambda(t_2(v,\lambda),v)$ *are bonded bellow in* $L^p(\Omega)$.

Proof: Let (v_n) be a minimizing sequence of the functional $v\mapsto\tilde{E}_\lambda(t_1(v,\lambda),v)$. We know that $\partial_t\tilde{E}_\lambda(t_1(v_n,\lambda),v_n)=0$, then

$$[t_1(v_n,\lambda)]^p||v_n||_p^p=\lambda[t_1(v_n,\lambda)]^q||\Lambda v_n||_q^q+[t_1(v_n,\lambda)]^r||\Lambda v_n||_r^r.$$

Hence

$$\tilde{E}_\lambda(t_1(v_n,\lambda),v_n)=\lambda(\frac{1}{p}-\frac{1}{q})[t_1(v_n,\lambda)]^q||\Lambda v_n||_q^q+(\frac{1}{p}-\frac{1}{r})[t_1(v_n,\lambda)]^r||\Lambda v_n||_r^r.$$

As $p<r$, we conclude that

$$\tilde{E}_\lambda(t_1(v_n,\lambda),v_n)\geq\lambda(\frac{1}{p}-\frac{1}{q})[t_1(v_n,\lambda)]^q||\Lambda v_n||_q^q. \tag{2.6}$$

Sobolev injection of X in $L^q(\Omega)$ and the fact that $\limsup_{n\to+\infty} ||V_n||_p < +\infty$, implies that there exists c and k positive such that for every n in $\mathbb{N}$, we have $||V_n||_p < k$. and $||\Lambda V_n||_q \leq c||V_n||_p < kc$. As $q < p$, the inequality (2.6) implies

$$\tilde{E}_\lambda(t_1(v_n,\lambda),v_n) > (\frac{1}{p}-\frac{1}{q})\lambda k^q c^q.$$

We show by the same method that the functional $v \mapsto \tilde{E}_\lambda(t_2(v,\lambda),v)$ is bonded bellow. □

Put

$$\alpha_1(\lambda) = \inf_{v\in L^p(\Omega)\setminus\{0\}} \tilde{E}_\lambda(t_1(v,\lambda),v). \tag{2.7}$$

$$\alpha_2(\lambda) = \inf_{v\in L^p(\Omega)\setminus\{0\}} \tilde{E}_\lambda(t_2(v,\lambda),v). \tag{2.8}$$

We have the following lemma:

Lemma 2.5 *If $\lambda \in]0,\hat{\lambda}[$, then*

$$\alpha_1(\lambda) = \inf_{v\in S, v\geq 0} \tilde{E}_\lambda(t_1(v,\lambda),v) \quad \textit{and} \quad \alpha_2(\lambda) = \inf_{v\in S, v\geq 0} \tilde{E}_\lambda(t_2(v,\lambda),v),$$

where S is the unit sphere of $L^p(\Omega)$.

Proof: Let $t > 0$. If $\partial_t\tilde{E}_\lambda(t,v) > 0$, then $t \in]t_1(v,\lambda),t_2(v,\lambda)[$.
Since $|\Lambda v| \leq \Lambda|v|$, we deduce that

$$\partial_t\tilde{E}_\lambda(t_i(|v|,\lambda),v) \geq \partial_t\tilde{E}_\lambda(t_i(|v|,\lambda),|v|) = 0, \quad i = 1,\ 2.$$

It follows that $]t_1(|v|,\lambda),t_2(|v|,\lambda)[\subseteq]t_1(v,\lambda),t_2(v,\lambda)[$.
Hence, $t_1(|v|,\lambda) \geq t_1(v,\lambda)$.
Using the fact that $t \mapsto \tilde{E}_\lambda(t,|v|)$ is decreasing on $]0,t_1(|v|,\lambda)]$, we get

$$\tilde{E}_\lambda(t_1((v,\lambda),|v|) \geq \tilde{E}_\lambda(t_1(|v|,\lambda),|v|)$$

and since $|\Lambda v| \leq \Lambda|v|$, we get

$$\tilde{E}_\lambda(t_1(v,\lambda),v) \geq \tilde{E}_\lambda(t_1(v,\lambda),|v|).$$

Hence we conclude that

$$\tilde{E}_\lambda(t_1(|v|,\lambda),|v|) \leq \tilde{E}_\lambda(t_1(v,\lambda),v).$$

Since $|\Lambda v| \leq \Lambda|v|$ and the function $t \mapsto \tilde{E}_\lambda(t,v)$ is creasing on $[t_1(v,\lambda),t_2(v,\lambda)]$, we obtain

$$\begin{aligned}\tilde{E}_\lambda(t_2(|v|,\lambda),|v|) &\leq \tilde{E}_\lambda(t_2(|v|,\lambda),v).\\ &\leq \tilde{E}_\lambda(t_2(v,\lambda),v).\end{aligned}$$

Finally, we have showed that for every $v \in L^p(\Omega) \setminus \{0\}$

$$\tilde{E}_\lambda(t_i(|v|,\lambda),|v|) \leq \tilde{E}_\lambda(t_i(v,\lambda),v), \quad \text{where} \quad i = 1,\ 2. \tag{2.9}$$

Moreover, for every $\gamma > 0$, we get

$$\tilde{E}_\lambda(\gamma t, \frac{v}{\gamma}) = \tilde{E}_\lambda(t,v),$$

$$\partial_t \tilde{E}_\lambda(\gamma t, \frac{v}{\gamma}) = \frac{1}{\gamma}\partial_t \tilde{E}_\lambda(t,v),$$

$$\partial_{tt} \tilde{E}_\lambda(\gamma t, \frac{v}{\gamma}) = \frac{1}{\gamma^2}\partial_{tt} \tilde{E}_\lambda(t,v).$$

It follows that

$$t_1(v,\lambda) = \frac{1}{\gamma} t_1(\frac{v}{\gamma},\lambda), \tag{2.10}$$

$$t_2(v,\lambda) = \frac{1}{\gamma} t_2(\frac{v}{\gamma},\lambda). \tag{2.11}$$

By the virtu of (2.9), (2.10) and (2.11), we conclude that

$$\alpha_1(\lambda) = \inf_{v\in S, v\geq 0} \tilde{E}_\lambda(t_1(v,\lambda),v), \tag{2.12}$$

$$\alpha_2(\lambda) = \inf_{v\in S, v\geq 0} \tilde{E}_\lambda(t_2(v,\lambda),v), \tag{2.13}$$

where S is the unit sphere of $L^p(\Omega)$. □

Lemma 2.6 *Let $(v_n) \subset S$ be a minimizing sequence of (2.12) (resp. of (2.13)). Then, $(V_n) := (t_1(v_n,\lambda)v_n)$ (resp. $(W_n) := (t_2(v_n,\lambda)v_n)$) are Palais-Smale sequences for the functional E_λ.*

Proof: We will show this lemma only for the sequence (V_n), the proof for (W_n) can be done in the same way.
Let $\lambda \in]0, \hat{\lambda}[$. Then $\lim_{n\to+\infty} E_\lambda(V_n) = \alpha_1(\lambda)$.
Now we show that $\lim_{n\to+\infty} E'_\lambda(V_n) = 0$.
Notice that for every $v \in L^p(\Omega)\backslash\{0\}$, we have $\partial_t \tilde{E}_\lambda(t_1(v,\lambda),v) = 0$ and $\partial_{tt} \tilde{E}_\lambda(t_1(v,\lambda),v) \neq 0$. The implicit function theorem implies that the functional $v \mapsto t_1(v,\lambda)$ is C^1 since $\tilde{E}_\lambda$ is. Let us introduce the C^1 functional $f_{1,\lambda}$ defined on S by

$$f_{1,\lambda}(v) = \tilde{E}_\lambda(t_1(v,\lambda),v) = E_\lambda(t_1(v,\lambda)v).$$

Hence

$$\alpha_1(\lambda) = \inf_{v\in S} f_{1,\lambda}(v) = \inf_{v\in S, v\geq 0} f_{1,\lambda}(v) \quad \text{and} \quad \lim_{n\to+\infty} f_{1,\lambda}(v_n) = \alpha_1(\lambda).$$

Using the Ekeland variational principle on the complete manifold $(S, ||\ ||_p)$ to the functional $f_{1,\lambda}$, we conclude that

$$|f'_{1,\lambda}(v_n)(\varphi)| \leq \frac{1}{n}||\varphi||_p, \quad \text{for every} \quad \varphi \in T_{v_n}S,$$

where $T_{v_n}S$ is the tangent space to S at the point v_n.
Moreoever, since $\partial_t \tilde{E}_\lambda(t_1(v_n, \lambda), v_n) \equiv 0$, then for every $\varphi \in T_{v_n}S$, one has

$$\begin{aligned} f'_{1,\lambda}(v_n)(\varphi) &= \partial_t \tilde{E}_\lambda(t_1(v_n, \lambda), v_n)\partial_v t_1(v_n, \lambda)(\varphi) \\ &\quad + \partial_v \tilde{E}_\lambda(t_1(v_n, \lambda), v_n)(\varphi) \\ &= \partial_v \tilde{E}_\lambda(t_1(v_n, \lambda), v_n)(\varphi), \end{aligned}$$

where $\partial_v t_1(v_n, \lambda)$ denotes the derivative of $t_1(., \lambda)$ with respect to its first variable at the point (v_n, λ).
Furthermore, let

$$\begin{aligned} P : L^p(\Omega)\backslash\{0\} &\to \mathbb{R} \times S \\ v &\mapsto (P_1(v), P_2(v)) = (\| v \|_p, \frac{v}{\| v \|_p}). \end{aligned}$$

Applying Hölder's inequality, we get for every $(v,\ \varphi) \in L^p(\Omega)\backslash\{0\} \times L^p(\Omega)$:

$$\|P'_2(v)(\varphi)\|_p \leq 2\frac{\|\varphi\|_p}{\|v\|_p}.$$

From lemma 2.3 and by the fact that $\|V_n\|_p = t(v_n, \lambda)$, there exists positive constant C such that

$$t_1(v_n, \lambda) \geq C, \forall n \in \mathbb{N}.$$

Hence for every $\varphi \in L^p(\Omega)$, we obtain

$$\begin{aligned} |E' - \lambda(V_n)(\varphi)| &= |\partial_t \tilde{E}_\lambda(P_1(V_n), P_2(V_n))P'_1(V_n)(\varphi) \\ &\quad + \partial_v \tilde{E}_\lambda(P_1(V_n), P_2(V_n))P'_2(V_n)(\varphi)| \\ &= |\partial_v \tilde{E}_\lambda(t(v_n), v_n)P'_2(V_n)(\varphi)| \\ &= |f'_{1,\lambda}(v_n)P'_2(V_n)(\varphi)| \\ &\leq \frac{1}{n} \| P'_2(V_n)(\varphi) \|_p \\ &\leq \frac{2}{n}\frac{\| \varphi \|_p}{C} \end{aligned}$$

We easily conclude that

$$\lim_{n\to+\infty} E' - \lambda(V_n) = 0 \quad \text{in} \quad L^{p'}(\Omega).$$

□

Remark 2.1 Until now, the minimizing sequences we consider are in S and are nonnegative.

3. Existence results

Theorem 3.1 *Let $1 < q < p < r < p_2^*$ and $\lambda \in]0,\ \hat{\lambda}[$. Then the problem (P_λ) has at least two positive solutions.*

Proof: We will use the notations of the previous lemmas.
Since the sequences (V_n) and (W_n) are Palais-Smale for the functional E_λ, it suffices to show that E_λ $(0 < \lambda < \hat{\lambda})$ satisfy Palais-Smale condition.
By lemma 2.3, we deduce that (V_n) is bonded in $L^p(\Omega)$. Passing if necessary to a subsequence, we get

$$\left\{ \begin{array}{rcll} V_n & \rightharpoonup & V_1 & \text{in} \quad L^p(\Omega), \\ \Lambda V_n & \rightharpoonup & \Lambda V_1 & \text{in} \quad X, \\ \Lambda V_n & \to & \Lambda V_1 & \text{in} \quad L^r(\Omega), \quad (\text{and in} \quad L^q(\Omega)). \end{array} \right. \tag{3.1}$$

On the other hand we have,

$$\begin{aligned} \langle N_p(V_n), V_n - V_1 \rangle = \quad & \langle \quad E'_\lambda(V_n), V_n - V_1 \rangle + \lambda \int_\Omega N_q(\Lambda V_n)(\Lambda V_n - \Lambda V_1)dx \\ & + \quad \int_\Omega N_r(\Lambda V_n)(\Lambda V_n - \Lambda V)dx. \end{aligned}$$

Moreover, $E'_\lambda(V_n) \to 0$, $N_q(\Lambda V_n) \to N_q(\Lambda V_1)$ and $N_r(\Lambda V_n) \to N_r(\Lambda V_1)$.
Then $\langle N_p(V_n), V_n - V_1 \rangle \to 0$.
The fact that N_p is $(S+)$ type implies that $V_n \to V_1$ dans $L^p(\Omega)$.
We know that for any minimizing sequence (v_n) of (2.12), there is a subsequence still denoted by (v_n) such that $V_n = t_1(v_n, \lambda)v_n$ and $t_1(v_n, \lambda) = ||V_n||_p$. Hence

$$t_1(v_n, \lambda) \to ||V_1||_p = t_1,$$

which implies that

$$v_n \to V_1/t_1 = v_1, \quad \text{and} \quad t_1 = t_1(v_1, \lambda),$$

where $v_1 \in S$.
In the same way, for any minimizing sequence $(v_n) \subset S$ of (2.13), passing if necessary to a subsequence, there is $t_2 \in]0, +\infty[$ such that

$$\left\{ \begin{array}{rcll} t_2(v_n, \lambda)v_n & \to & t_2 & \text{in} \quad \mathbb{R}, \\ v_n & \to & v_2 = V_2/t_2, & \end{array} \right.$$

where V_2 is the limit of the sequence $(W_n) := (t_2(v_n, \lambda)v_n)$ in $L^p(\Omega)$ and $t_2 = ||V_2||_p = t_2(v_2, \lambda)$.
At this stage, it is easy to see that $V_1 \neq V_2$. Indeed, since $\partial_{tt}\tilde{E}_\lambda(t_1(v_1, \lambda), v_1) > 0$ and $\partial_{tt}\tilde{E}_\lambda(t_2(v_2, \lambda), v_2) < 0$, it follows that $\partial_{tt}E_\lambda(t_1, V_1/t_1) > 0$ and
$\partial_{tt}E_\lambda(t_2, V_2/t_2) < 0$. This achieves the proof. □

In the sequel the solutions V_1 and V_2 of (P'_λ), for $\lambda \in]0, \hat{\lambda}[$, will be denoted by $V_{1,\lambda}$ and $V_{2,\lambda}$. Also, $t_{1,\lambda}$, $t_{2,\lambda}$, $v_{1,\lambda}$ and $v_{2,\lambda}$ will stand for $t_1(v_1, \lambda)$, $t_2(v_2, \lambda)$, v_1 and v_2 respectively.

Theorem 3.2 *Let* $1 < q < p < r < p_2^*$. *Then*

$$(i) \qquad E_\lambda(V_{1,\lambda}) < 0 \quad for \quad \lambda \in]0, \hat{\lambda}[,$$

$$(ii) \quad \begin{cases} E_\lambda(V_{2,\lambda}) > 0 & for\ \lambda \in]0, \lambda_0[, \\ E_\lambda(V_{2,\lambda}) < 0 & for\ \lambda \in]\lambda_0, \hat{\lambda}[, \end{cases}$$

where

$$\lambda_0 = \frac{q}{r}(\frac{r}{p})^{\frac{r-q}{r-p}}\hat{\lambda}.$$

Proof: (*i*) Let us recall that $\partial_t \tilde{E}_\lambda(t_{1,\lambda}, v_{1,\lambda}) = 0$ and $\partial_{tt}\tilde{E}_\lambda(t_{1,\lambda}, v_{1,\lambda}) > 0$. Then

$$\begin{cases} P(V_{1,\lambda}) - \lambda Q(V_{1,\lambda}) - R(V_{1,\lambda}) = 0, \\ (p-1)P(V_{1,\lambda}) - \lambda(q-1)Q(V_{1,\lambda}) - (r \quad -1)R(V_{1,\lambda}) > 0. \end{cases}$$

Using the fact that $1 < q < p < r$, we get

$$\lambda(p-q)Q(V_{1,\lambda}) + (p-r)R(V_{1,\lambda}) > 0.$$

Hence

$$\begin{aligned} E_\lambda(V_{1,\lambda}) &= \lambda\frac{q-p}{pq}Q(V_{1,\lambda}) + \frac{r-p}{pr}R(V_{1,\lambda}) \\ &\leq \frac{-1}{pq}(\lambda(p-q)Q(V_{1,\lambda}) + (p-r)R(v_{1,\lambda})) \\ &< 0. \end{aligned}$$

(*ii*) Let v be an arbitrary element of $L^p(\Omega) \setminus \{0\}$ and let us write

$$\tilde{E}_\lambda(t, v) = t^q \tilde{G}_\lambda(t, v), \quad \text{where} \quad \tilde{G}_\lambda(t, v) = \frac{t^{p-q}}{p}P(v) - \frac{\lambda}{q}Q(v) - \frac{t^{r-q}}{r}R(v).$$

It follows that

$$\partial_t \tilde{E}_\lambda(t, v) = qt^{q-1}\tilde{G}_\lambda(t, v) + t^q \partial \tilde{G}_\lambda(t, v),$$

with

$$\partial_t \tilde{G}_\lambda(t, v) = t^{p-q-1}(\frac{p-q}{p}P(v) - \frac{r-q}{r}t^{r-p}R(v)).$$

It is clear that the real valued function $t \to \tilde{G}_\lambda(t, v)$ is increasing on $]0, t_0(v)[$, decreasing on $]t_0(v), +\infty[$ and attains its unique maximum for $t = t_0(v)$, where

$$t_0(v) = (\frac{r}{p})^{\frac{1}{r-p}}t(v), \tag{3.2}$$

and $t(v)$ is defined by the relation (2.1).
On the other hand, a direct computation gives

$$\tilde{G}_\lambda(t_0(v), v) = \frac{1}{r}(\frac{r}{p})^{\frac{r-q}{r-p}}\frac{r-p}{p-q}(\frac{p-q}{r-q}\frac{P(v)}{R(v)})^{\frac{r-q}{r-p}}R(v) - \lambda\frac{Q(v)}{q}.$$

Similarly, $\tilde{G}_\lambda(t_0(v),v) > 0$ (resp. $\tilde{G}_\lambda(t_0(v),v) < 0$) if $\lambda < \lambda_0(v)$ (resp. $\lambda > \lambda_0(v)$) and
$\tilde{G}_{\lambda_0(v)}(t_0(v),v) = 0$, where

$$\lambda_0(v) = \frac{q}{r}(\frac{r}{p})^{\frac{r-q}{r-p}}\lambda(v), \tag{3.3}$$

with $\lambda(v)$ given by (2.2). Thus, we get

$$\begin{cases} \tilde{E}_\lambda(t_0(v),v) & > & 0 & \text{if} & \lambda < \lambda_0(v), \\ \tilde{E}_\lambda(t_0(v),v) & = & 0 & \text{if} & \lambda = \lambda_0(v), \\ \tilde{E}_\lambda(t_0(v),v) & < & 0 & \text{if} & \lambda > \lambda_0(v). \end{cases} \tag{3.4}$$

First, since the function

$$\begin{array}{rcl}]0,1[& \to & \mathbb{R} \\ t & \to & \dfrac{\ln t}{1-t} \end{array}$$

is increasing, then for every real numbers x and y such that $0 < x < y$, one has

$$\ln(\frac{1}{x}) > \frac{1-x}{1-y}\ln(\frac{1}{y}) = \ln(\frac{1}{y})^{\dfrac{1-x}{1-y}},$$

and consequently

$$0 < x(1/y)^{\frac{1-x}{1-y}} < 1.$$

In the particular case $x = \dfrac{q}{r}$ and $y = \dfrac{p}{r}$, we get

$$0 < \frac{q}{r}(\frac{r}{p})^{\frac{r-q}{r-p}} < 1,$$

and therfore $0 < \lambda_0(v) < \lambda(v)$.
Moreover, for every $v \in L^p(\Omega)\backslash\{0\}$, one has $\tilde{G}_{\lambda_0(v)}(t,v) < 0$ for $t \in]0,+\infty[\backslash\{t_0(v)\}$ and $\tilde{G}_{\lambda_0(v)}(t_0(v),v) = 0$. Hence, the real valued function $t \to \tilde{E}_{\lambda_0(v)}(t,v), (t > 0)$, attains its unique maximum at $t = t_0(v)$ and we obtain the following interesting identity

$$t_2(v,\lambda_0(v)) = t_0(v). \tag{3.5}$$

On the other hand, let

$$\lambda_0 = \inf_{v \in L^p(\Omega)\backslash\{0\}} \lambda_0(v). \tag{3.6}$$

(2.2) et (3.2) implies that

$$\lambda_0(v) = \frac{p}{q}(\frac{r}{p})^{\frac{r-q}{r-p}}\hat{c}\frac{P^{\frac{r-q}{r-p}}(v)}{Q(v)R^{\frac{p-q}{r-p}}(v)}.$$

Let us put

$$M = \{v \in L^p(\Omega), Q(v)R^{\frac{p-q}{r-p}}(v) = 1\}.$$

It is clair that M is weakly closed.
Moreover the functional $v \mapsto P^{\frac{r-q}{r-p}}(v)$ is weakly lower semi-continuous and coercive on M. Thus this functional attaints its minimum on M. The homogeneities of $v \mapsto P^{\frac{r-q}{r-p}}(v)$ and $v \mapsto Q(v)R^{\frac{p-q}{r-p}}(v)$ enables us to conclude that there is $v^* \in S$ such that

$$\inf_{v\in M} \lambda_0(v) = \inf_{v\in L^p(\Omega)\setminus\{0\}} \lambda_0(v) = \inf_{v\in S} \lambda_0(v) = \lambda_0(v^*) = \lambda_0.$$

Now, let $\lambda \in]0, \lambda_0[$, Then, for every $v \in L^p(\Omega) \setminus \{0\}$ one has $\lambda < \lambda_0(v)$ and consequently, $\tilde{E}_\lambda(t_0(v), v) > 0$ holds from (3.4), . Then the function $t \mapsto \tilde{E}_\lambda(t, v)$, $(t > 0)$ attains its maximum at $t_2(v, \lambda)$ such that $\tilde{E}_\lambda(t_2(v, \lambda), v) > 0$ for every $v \in L^p(\Omega) \setminus \{0\}$. In particular, we have $\tilde{E}_\lambda(t_2(v_{2,\lambda}, \lambda), v_{2,\lambda}) > 0$, *i.e.* $E_\lambda(V_{2,\lambda}) > 0$.
If $\lambda = \lambda_0$, then

$$\begin{aligned} E_{\lambda_0}(V_{2,\lambda_0}) &= \tilde{E}_{\lambda_0}(t_2(v_{2,\lambda_0}), v_{2,\lambda_0}) \\ &= \inf_{v\in S} \tilde{E}_{\lambda_0}(t_2(v, \lambda_0), v) \\ &\le \tilde{E}_{\lambda_0}(t_2(v^*, \lambda_0(v^*)), v^*) \\ &= \tilde{E}_{\lambda_0(v^*)}(t_0(v^*), v^*) \\ &= 0, \end{aligned}$$

which implies that $E_{\lambda_0}(V_{2,\lambda_0}) \le 0$. In addition, it is known from (3.4) that $\tilde{E}_{\lambda_0}(t_0(v), v) \ge 0$, for every $v \in L^p(\Omega) \setminus \{0\}$. Then, since $\tilde{E}_{\lambda_0}(t_2(v_{2,\lambda_0}, \lambda_0), v_{2,\lambda_0})$ is a global maximum of the function $t \mapsto \tilde{E}_{\lambda_0}(t, v_{2,\lambda_0})$, $(t > 0)$, we have

$$\tilde{E}_{\lambda_0}(t_2(v_{2,\lambda_0}, \lambda_0), v_{2,\lambda_0}) \ge \tilde{E}_{\lambda_0}(t_0(v_{2,\lambda_0}), v_{2,\lambda_0}) \ge 0.$$

We conclude that

$$E_{\lambda_0}(V_{2,\lambda_0}) = \tilde{E}_{\lambda_0}(t_2(v_{2,\lambda_0}, \lambda_0), v_{2,\lambda_0}) = 0.$$

Finally, suppose that $\lambda_0 < \lambda < \hat{\lambda}$.
We know that for every $(t, v) \in]0, +\infty[\times L^p(\Omega) \setminus \{0\}$, the real valued function $\lambda \mapsto \tilde{E}_\lambda(t, v)$ is decreasing on $[\lambda_0, \hat{\lambda}]$, hence we deduce

$$\begin{aligned} \tilde{E}_\lambda(t_2(v_{2,\lambda}, \lambda), v_{2,\lambda}) &= \inf_{v\in S} \tilde{E}_\lambda(t_2(v, \lambda), v) \\ &\le \tilde{E}_\lambda(t_2(v^*, \lambda), v^*) \\ &< \tilde{E}_{\lambda_0}(t_2(v, \lambda), v). \end{aligned}$$

Moreover, the real valued function $t \mapsto \tilde{E}_{\lambda_0}(t, v^*)$, $(t > 0)$, attains its unique maximum for $t = t_0(v^*)$.Then

$$\begin{aligned} \tilde{E}_{\lambda_0}(t_2(v^*, \lambda), v^*) &\le \tilde{E}_{\lambda_0}(t_0(v^*), v^*) \\ &= \tilde{E}_{\lambda_0(v^*)}(t_0(v^*), v^*) \\ &= 0. \end{aligned}$$

Hence $\tilde{E}_\lambda(t_2(v_{2,\lambda}, \lambda), v_{2,\lambda}) < 0$, which achieves this proof. □

Theorem 3.3 *if v^* is a solution of (3.6), then $t_0(v^*)v^*$ is a solution of (P'_{λ_0}).*

Proof: Let v^* be a solution of (3.6), then $\lambda_0 = \lambda_0(v^*)$ and for every $h \in L^p(\Omega)$, we have

$$\begin{aligned} E'_{\lambda_0}(t_0(v^*)v^*)(h) &= \frac{1}{p}t_0^{p-1}(v^*)\langle P'(v^*), h\rangle - \frac{\lambda_0}{q}t_0^{q-1}(v^*)\langle Q'(v^*), h\rangle \\ &\quad -\frac{1}{r}t_0^{r-1}(v^*)\langle R'(v^*), h\rangle \\ &= \frac{P(v^*)(t_0(v^*))^{p-1}}{p}(\frac{\langle P'(v^*), h\rangle}{P(v^*)} \\ &\quad -\frac{p\lambda_0}{q}t_0^{q-p}\frac{\langle Q'(v^*), h\rangle}{P(v^*)} - \frac{p}{r}t_0^{r-p}\frac{\langle R'(v^*), h\rangle}{P(v^*)}). \end{aligned}$$

By the virtu of relations (2.1), (2.2), (3.2) and (3.3), a direct computation gives for every $h \in L^p(\Omega)$

$$\frac{p\lambda_0}{q}t_0^{q-p}\frac{\langle Q'(v^*), h\rangle}{P(v^*)} = \frac{r-p}{r-q}\frac{\langle Q'(v^*), h\rangle}{Q(v^*)},$$

and

$$\frac{p}{r}t_0^{r-p}\frac{\langle R'(v^*), h\rangle}{P(v^*)}) = \frac{p-q}{r-q}\frac{\langle R'^*(v^*), h\rangle}{R(v^*)}.$$

Then

$$E'_{\lambda_0}(t_0(v^*)v^*)(h) = K(\frac{r-q}{r-p}\frac{\langle P'(v^*), h\rangle}{P(v^*)} - \frac{\langle Q'(v^*), h\rangle}{Q(v^*)} - \frac{p-q}{r-p}\frac{\langle R'(v^*), h\rangle}{R(v^*)}),$$

where

$$K = \frac{r-p}{r-q}\frac{P(v^*)}{p}[t_0(v^*)]^{p-1}.$$

In the other hand, the relations (2.2) and (3.3) implies that for every $h \in L^p(\Omega)$

$$\langle \lambda_0'(v^*), h\rangle = \lambda_0(v^*)(\frac{r-q}{r-p}\frac{\langle P'(v^*), h\rangle}{P(v^*)} - \frac{\langle Q'(v^*), h\rangle}{Q(v^*)} - \frac{p-q}{r-p}\frac{\langle R'(v^*), h\rangle}{R(v^*)}).$$

Since $\langle \lambda_0'(v^*), h\rangle = 0$ for every $h \in L^p(\Omega)$, we deduce that

$$\langle E'_{\lambda_0}(t_0(v^*)v^*), h\rangle = \frac{K}{\lambda_0}\langle \lambda_0'(v^*), h\rangle = 0,$$

for every $h \in L^p(\Omega)$.
Which implies that $t_0(v^*)v^*$ is a solution of (P'_{λ_0}).

Remark 3.1 It is very interesting to notice that in the case of homogeneous Dirichlet boundary condition, we have

$$\lim_{q\to p}\hat{\lambda} = \inf_{v\in L^p(\Omega)\setminus\{0\}}\frac{\int_\Omega |v(x)|^p dx}{\int_\Omega |\Lambda v(x)|^p dx},$$

Hence, in the case where $p = q$, $\hat{\lambda}$ is the first eigenvalue of the problem (P'_λ), *i.e.* the problem (P'_λ) has positive solutions for $\lambda \in]0, \hat{\lambda}]$ and has no positive solution for $\lambda > \hat{\lambda}$.

References

1. R. A. Adams, *Sobolev Spaces,* Academic Press, New York, (1975).

2. C. O. Alves, *Existence of Positive Solutions for a Problem with Lack of Compactness Involving the p-Laplacian,* Nonlinear Analysis- TMA, 51 (2002), 1187-1206.

3. C. O. Alves, P. C. Carrião and O. H. Miyagaki, *Signed solution for a class of quasilinear elliptic problem with critical growth,* Commun. Pure Appl. Anal. 1 (2002), no. 4, 531-545.

4. A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems,* J. Funct. Anal. 122 (1994), no. 2, 519-543.

5. A. Ambrosetti, J. Garcia Azorero and I.Peral, *Multiplicity results for some nonlinear elliptic equations,* J. Funct. Anal. 137 (1996) no. 1, 219-242.

6. A. Ambrosetti, J. Garcia Azorero and I.Peral, *Existence and multiplicity results for some nonlinear elliptic equations: a survey,* Rend. Mat. Appl. (7) 20 (2000), 167-198.

7. T. Bartsch and M. Willem, *On an elliptic equation with concave and convex nonlinearities,* Proc. Amer. Math. Soc. 123 (1995), no. 11, 3555-3561.

8. J.Berkovits and V.Mustonen, *Nonlinear mappings of monotone type (classification and degree theory),* 1988, Math. Univer. Oulu, Linnanmaa, Oulu, Finland.

9. L. Boccardo, M. Escobedo and I. Peral, *A Dirichlet problem involving critical exponents,* Nonlinear Anal. 24 (1995), 1639-1648.

10. J. F. Bonder and J. D. Rossi, *Existence results for the p-Laplacian with nonlinear boundary conditions,* J. Math. Anal. Appl. 263 (2001) 195-223.

11. J. F. Bonder and J. D. Rossi, *Asymptotic behavior of the best Sobolev trace constant in expanding and contracting domains,* Commun. Pure Appl. Anal. 1 (2002), no. 3, 359-378.

12. P. Dràbek and M. Ôtani, *Global Bifurcation Result for the p-Biharmonic Operator,* Electronic Journal Differential Equations 2001(2001), No 48, pp. 1-19.

13. P. Dràbek and S. Pohozaev, *Positive Solutions for the p-Laplacian: Application of the fibering method,* Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 703-726.

14. I. Ekeland, *On the Variational Principle,* J. Math. Anal. Appl. 47 (1974) 324-353.

15. A. El Hamidi, *Multiple Solutions with Changing Sign Energy to a Nonlinear Elliptic Equations,* Comm. Pure Appl. Anal. 3 (2004) 253-265.

16. D. Gilbar and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Second ed., Springer New York Tokyo (1983).

17. P. Korman, *On uniqueness of positive solutions for a class of semilinear equations,* Discrete Contin. Dyn. Syst. 8 (2002), no. 4, 865-871.

18. M. Talbi and N. Tso uli, *Existence of Solutions for a Nonlinear Elliptic Problem of Fourth Order with Weight,* Mediterr. j. math. 3 (2006), 87-96.

19. M. Willem, *Minimax Theorems,* Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, (1996).

M.Talbi and N.Tsouli
Département de Mathématiques et Informatique
Faculté des Sciences,
Université Mohamed 1, Oujda, Maroc
talbi_md@yahoo.fr
tsouli@sciences.univ-oujda.ac.ma