Bol. Soc. Paran. Mat. (3s.) **v. 29** 2 (2011): 9–14.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v29i2.11649

Some properties of semi-linear uniform spaces

Abdalla Tallafha

ABSTRACT: Semi-linear uniform space is a new space defined by Tallafha, A and Khalil, R in [3], the authors studied some cases of best approximation in such spaces, and gave some open problems in approximation theory in uniform spaces. Besides they defined a set valued map ρ on $X \times X$ and asked two questions about the properties of ρ. The purpose of this paper is to answer these questions. Besides we shall define another set valued map δ on $X \times X$ and give more properties of semi-linear uniform spaces using the maps ρ and δ. Also we shall give an example of a semi-linear uniform space which is not metrizable.

Key Words: metrizable spaces, Uniform spaces.

Contents

1. Introduction

Let X be a set and D_X be a collection of subsets of $X \times X$, such that each element V of D_X contains the diagonal $\Delta = \{(x,x) : x \in X\}$ and $V = V^{-1} = \{(y,x) : (x,y) \in V\}$, for all $V \in D_X$. D_X is called the family of all entourages of the diagonal. Let Γ be a sub-collection of D_X. Then

Definition 1.1. [1] The pair (X, Γ) is called a **uniform space** if:
(i) If V_1 and V_2 are in Γ then $V_1 \cap V_2 \in \Gamma$.
(ii) For every $V \in \Gamma$, there exists $U \in \Gamma$ such that $U \circ U \subset V$.
(iii) $\bigcap\{V : V \in \Gamma\} = \Delta$.
(vi) If $V \in \Gamma$ and $V \subseteq W \in D_X$, then $W \in \Gamma$.

Uniform spaces had been studied extensively through years. We refer the reader to [1], and [2], for the basic structure of uniform spaces. In [3], the authors define a set valued map ρ, called metric type, on semi-linear uniform spaces that enables one to study analytical concepts on uniform type spaces. They asked two questions about the properties of ρ. Besides they studied some cases of best approximation in such spaces, and gave some open problems in approximation theory in uniform

2000 *Mathematics Subject Classification:* Primary: 54E35, Secondary: 41A65.

spaces. The object of this paper is to answer the first natural question that one should ask: "is there a semi-linear uniform space which is not metrizable?". Besides we solve question 1 and 2 in [3]. Also we shall define another set valued map δ on $X \times X$, which is used with ρ to give more properties of semi-linear uniform spaces. Also we shall use the set valued map δ and ρ, to defined a new semi-linear uniform spaces. Finally we study the relation between ρ and δ, and we shall show that, $\rho(x,y) = \rho(s,t)$ if and only if $\delta(x,y) = \delta(s,t)$.

2. Semi-linear uniform spaces.

Let (X,Γ) be a uniform space. By a **chain** in $X \times X$ we mean a totally (or linearly) ordered collection of subsets of $X \times X$, where $V_1 \leq V_2$ means $V_1 \subseteq V_2$, [3].

Definition 2.1.[3]. **A semi-linear uniform space** is a uniform space (X,Γ), where Γ is a chain and condition (vi) in definition 1.1 is replaced by $\bigcup \{V : V \in \Gamma\} = X \times X$.

Remark: 1- The condition Γ is a chain implies the condition,
(i) V_1 and V_2 are in Γ then $V_1 \cap V_2 \in \Gamma$.
2) We may assume that $X \times X \notin \Gamma$ and $\Delta \notin \Gamma$, since $X \times X \in \Gamma$, does not change the structure of the semi-linear uniform space, even the topology induced on X by (X,Γ). Also if $\Delta \in \Gamma$, then the topology induced on X by (X,Γ) is the discreet one which is metrizable.

Throughout the rest of this paper, (X,Γ) will be assumed semi-linear uniform space, which satisfied $X \times X \notin \Gamma$ and $\Delta \notin \Gamma$.

Definition 2.2.[3] Let (X ,Γ) be a semi-linear uniform space. For $(x,y) \in X \times X$, let $\Gamma_{(x,y)} = \{V \in \Gamma : (x,y) \in V\}$. Then, the set valued map ρ on $X \times X$ is defined by $\rho(x,y) = \bigcap \left\{V : V \in \Gamma_{(x,y)}\right\}$. The map ρ will be called a set metric on (X,Γ).

Clearly $\rho(x,y) = \rho(y,x)$ for all $(x,y) \in X \times X$, and $\Delta \subseteq \rho(x,y)$ for all $(x,y) \in X \times X$, and $\Gamma \backslash \Gamma_{(x,y)} = \left(\Gamma_{(x,y)} \right)^c = \{V \in \Gamma : (x,y) \notin V \}$, so we shall denote $\Gamma \backslash \Gamma_{(x,y)}$ by $\Gamma^c_{(x,y)}$.

Now we shall define another function δ on $X \times X$.

Definition 2.3. Let (X,Γ) be a semi-linear uniform space. Then, the set valued map δ on $X \times X$ is defined by

$$\delta(x,y) = \begin{cases} \bigcup\{V : V \in \Gamma^c_{(x,y)}\}, & \text{if } x \neq y \\ \phi, & \text{if } x = y. \end{cases}$$

Clearly, if $x = y$ then $\Gamma^c_{(x,y)}$ is the empty set so we define $\delta(x,x)$ to be the empty set. Also $\delta(x,y) = \delta(y,x)$ for all $(x,y) \in X \times X$. and $\Delta \subseteq \delta(x,y)$ for all $x \neq y$. As in uniform spaces, the topology induced on X by Γ is defined by a local base $B(x,V)$.

Definition 2.4 [1] . For $x \in X$ and $V \in \Gamma$, we define the open ball of center x and radius V to be $B(x,V) = \{y \in X : (x,y) \in V\}$. Equivalently using our notation $B(x,V) = \{y : \rho(x,y) \subseteq V\}$. Clearly if $y \in B(x,V)$, then there is a $W \in \Gamma$ such that $B(y,W) \subseteq B(x,V)$, so $\beta_x = \{B(x,V) : V \in \Gamma\}$ is a local base at x.

3. Properties of the maps ρ, δ.

In this section we shall give some important properties of the maps ρ, δ. Also we shall give an example of a semi-linear uniform space which is not metrizable.

Proposition 3.1. Let (X,Γ) be a semi-linear uniform space. Then,

i) If $V \in \Gamma^c_{(x,y)}$, then $V \subsetneqq \rho(x,y)$.

ii) $\delta(x,y) \subseteq \rho(x,y)$ for all $(x,y) \in X \times X$.

iii) If $V \in \Gamma_{(x,y)}$,then $\delta(x,y) \subseteq V$.

iv) If $(x,y) \in \rho(s,t)$ then $\rho(x,y) \subseteq \rho(s,t)$.

v) If $(x,y) \in \delta(s,t)$ then $\delta(x,y) \subseteq \delta(s,t)$.

Proof: i) Suppose $V \in \Gamma^c_{(x,y)}$. Then $V \subseteq U$, for all $U \in \Gamma_{(x,y)}$, so $(x,y) \notin V \subseteq \rho(x,y)$. Since $(x,y) \in \rho(x,y)$, the result follows.

ii) If $x = y$, the result follows, If not by (**i**) $\delta(x,y) \subseteq \rho(x,y)$.

iii) Since $\rho(x,y) \subseteq V$, for all $V \in \Gamma_{(x,y)}$ so the result follows by (**ii**)

iv) $(x,y) \in \rho(s,t)$ implies that $\Gamma_{(s,t)} \subseteq \Gamma_{(x,y)}$ so $\rho(x,y) \subseteq \rho(s,t)$.

v) Since $(x,y) \in \delta(s,t)$, then $s \neq t$ and there exist $U \in \Gamma_{(x,y)} \cap \Gamma^c_{(s,t)}$. If $V \in \Gamma^c_{(x,y)}$,then $V \subseteq U \subseteq \delta(s,t)$, and hence $\delta(x,y) \subseteq \delta(s,t)$.

Proposition 3.2. Let (X,Γ) be a semi-linear uniform space. Then,

I) If $U \in \Gamma$ satisfies $U \subsetneqq \rho(x,y)$, then $U \subseteq \delta(x,y)$.

II) If $U \in \Gamma$ satisfies $\delta(x,y) \subsetneqq U$, then $\rho(x,y) \subseteq U$.

iii) If $U \in \Gamma$ satisfies $\delta(x,y) \subseteq U \subseteq \rho(x,y)$, then $U = \delta(x,y)$ or $U = \rho(x,y)$.

iv) If $(s,t) \notin \delta(x,y)$ then $\delta(x,y) \subseteq \delta(s,t)$.

v) If $(s,t) \notin \rho(x,y)$ then $\rho(x,y) \subseteq \delta(s,t)$.

vi) If $\delta(x,y) \subsetneqq \delta(s,t)$, the there exist $U \in \Gamma$, such that $\delta(x,y) \subsetneqq U \subseteq \delta(s,t)$.

vii) If $\rho(x,y) \subsetneqq \rho(s,t)$, the there exist $U \in \Gamma$, such that $\rho(x,y) \subseteq U \subsetneqq \rho(s,t)$.

Proof: i) If $U \subsetneqq \rho(x,y)$, then $(x,y) \notin U$ and $x \neq y$. So $U \subseteq \delta(x,y)$.

ii) If $x = y$ then $\rho(x,y) \subseteq U$ for all $U \in \Gamma$. If not, since $\delta(x,y) \subsetneqq U$, then $(x,y) \in U$.

iii) The assumption implies that $x \neq y$, so the result is obvious from (i) or (ii).

iv) If $s = t$, then $x = y$ and the result follows. If $s \neq t$, then the result follows if $x \neq y$. Other wise, let $U \in \Gamma_{(s,t)}$, then $V \subseteq U$, for all $V \in \Gamma^c_{(x,y)}$, so $\delta(x,y) \subseteq \delta(s,t)$.

v) If $(s,t) \notin \rho(x,y)$, then $s \neq t$ and, there exist $V \in \Gamma_{(x,y)}$, such that $(s,t) \notin V$, so $\rho(x,y) \subseteq V \subseteq \rho(s,t)$.

vi) The assumption implies that $s \neq t$. If $x = y$ then any $U \in \Gamma^c_{(s,t)}$ satisfies the result. Suppose that $x \neq y$, so $\delta(x,y) \subsetneqq \delta(s,t)$, implies the existence of a point (a,b) such that$(a,b) \in \delta(s,t)$ and $(a,b) \notin \delta(x,y)$. So there exist $U \in \Gamma_{(a,b)}$, $U \subseteq \delta(s,t)$, and $\delta(x,y) \subsetneqq U$.

vii) Suppose $\rho(x,y) \subsetneqq \rho(s,t)$. Then there exist a point (a,b) such that $(a,b) \in \rho(s,t)$ and $(a,b) \notin \rho(x,y)$. Hence there exist $U \in \Gamma_{(x,y)}$ such that $(a,b) \notin U$, and $U \subsetneqq \rho(s,t)$.

In the following examples we shall show that in Proposition 2.2, we can not replace ($\subseteq$ *by* $\subsetneqq$) or ($\subsetneqq$ *by* $\subseteq$) in (i) and (ii).More precisely,

1) $U \subsetneqq \rho(x,y) \nrightarrow U \subsetneqq \delta(x,y)$. 2) $U \subseteq \rho(x,y) \nrightarrow U \subseteq \delta(x,y)$.

3) $\delta(x,y) \subsetneqq U \nrightarrow \rho(x,y) \subsetneqq U$. 4) $\delta(x,y) \subseteq U \nrightarrow \rho(x,y) \subseteq U$.

Example 3.3. Let $t \in (0,\infty)$. Let $V_t = \{(x,y) : y - t < x < y + t,\ y \in \mathbb{R}\ \}$, and $\Gamma = \{V_t : 0 < t < \infty, \}$. Then $(\mathbb{R},\Gamma)$, is a semi-linear uniform space. It follows $\delta(1,0) = \{(x,y) : y-1 < x < y+1,\ y \in \mathbb{R}\ \}$, $\rho(1,0) = \{(x,y) : y-1 \le x \le y+1, y \in \mathbb{R}\}$, and so,$V_1 \subsetneqq \rho(1,0)$ and $V_1 = \delta(1,0)$.

Example 3.4. Let $t \in (0,\infty)$, Let $V_t = \{(x,y) : y-t \le x \le y+t,\ y \in \mathbb{R}\ \}$, and $\Gamma = \{V_t : 0 < t < \infty, \}$. Then $(\mathbb{R},\Gamma)$, is a semi-linear uniform space. So $\delta(1,0) = \{(x,y) : y-1 < x < y+1,\ y \in \mathbb{R}\ \}$ and $\rho(1,0) = \{(x,y) : y-1 \le x \le y+1, y \in \mathbb{R}\ \}$, so

$V_1 = \rho(1,0)$ and $\delta(1,0) \subsetneqq V_1$.

In [3] , the authors asked the following questions in semi-linear uniform spaces.
Question1: Is $\rho(x,y) \subseteq \rho(x,z) \cap \rho(z,y)$?.
Question 2. If $\rho(x,z) = \rho(x,w)$,for some $x \in X$, must $w = z$?.
Question 3. If E is compact, must E be proximinal?.

The answer of question 1 is negative, even if $\cap$ is replaced by $\cup$, since in Example 2.3. $\rho(1,0) = \{(x,y) : y-1 \le x \le y+1, y \in \mathbb{R}\}$, $\rho(1,\frac{1}{2}) = \{(x,y) : y-\frac{1}{2} \le x \le y+\frac{1}{2}, y \in \mathbb{R}\ \} = \rho(\frac{1}{2},0)$. So $\rho(1,0) \nsubseteq \rho(1,\frac{1}{2}) \cup \rho(\frac{1}{2},0)$.

The following example is a semi-linear uniform space which is not metrizable. Also this example answers Question 2, negatively.

Example 3.5. Let $U_t = \{(x,y) : x^2 + y^2 < t\ \} \cup \{(x,x) : x \in \mathbb{R}\}$. Then $(\mathbb{R},\Gamma)$,is a semi-linear uniform space which is not metrizable, where, $\Gamma = \{U_t : 0 < t < \infty\}$, and $\rho(3,4) = \{(x,y) : x^2 + y^2 \le 25\} = \rho(3,-4)$.

4. New semi-linear uniform spaces.

In this section we shall define a new semi-linear uniform space using old one.

Theorem 4.1. Let (X,Γ) be a semi-linear uniform space. Then,

i) $\{\rho(x,y) : (x,y) \in X \times X\}$ is a chain.

ii) $\{\delta(x,y) : (x,y) \in X \times X,\ x \ne y\}$ is a chain.

Proof: i) Clearly $\{\rho(x,y) : (x,y) \in X \times X\}$ is a partially ordered set under the set inclusion. Let $\rho(x,y), \rho(s,t)$ be two different elements. Suppose $\rho(x,y) \nsubseteq \rho(s,t)$.

From (iii), Proposition 2.1, $(x,y) \notin \rho(s,t)$ which implies the existence of $U \in \Gamma_{(s,t)} \cap \Gamma^{c}_{(x,y)}$. Hence by (i), same Proposition, $U \subseteq \rho(x,y)$, so $(s,t) \in U \subseteq \rho(x,y)$, and the result follows from (iii) in Proposition 2.1.

ii) Similarly $\{\delta(x,y) : (x,y) \in X \times X,\ x \neq y\}$ is partially ordered by set inclusion. Let $\delta(x,y), \delta(s,t)$ be two different elements. Suppose $\delta(x,y) \nsubseteq \delta(s,t)$. From (iv), Proposition 2.1, $(x,y) \notin \delta(s,t)$. So $\Gamma^{c}_{(s,t)} \subseteq \Gamma^{c}_{(x,y)}$.

Let $\boldsymbol{\rho} = \{\rho(x,y) : (x,y) \in X \times X,\ x \neq y\}$, and $\boldsymbol{\delta} = \{\delta(x,y) : (x,y) \in X \times X, x \neq y\}$. Then we have

Theorem 4.2: Let (X,Γ) be a semi-linear uniform space. Then,

i) $(X,\boldsymbol{\rho})$ is a semi-linear uniform space.

ii) $(X,\boldsymbol{\delta})$ is a semi-linear uniform space.

Proof: Clearly each element in $(X,\boldsymbol{\rho}), (X,\boldsymbol{\delta})$ is symmetric, contains the diagonal, moreover by Theorem 3.1 $\boldsymbol{\rho}, \boldsymbol{\delta}$ are chains. Let $(x,y) \in X \times X$. Then there exist $U \in \Gamma$, such that $(x,y) \in U$, so $(x,y) \in \rho(x,y)$, let $(s,t) \notin U$, then $(x,y) \in \delta(s,t)$ and $\bigcup \{\rho(x,y) : \rho(x,y) \in \boldsymbol{\rho}\} = X \times X = \bigcup \{\delta(x,y) : \delta(x,y) \in \boldsymbol{\delta}\}$. To complete the proof we need the following

1- $\bigcap \{\delta(x,y) : \delta(x,y) \in \boldsymbol{\delta}\} = \Delta$. Clearly $\Delta \subseteq \bigcap \{\delta(x,y) : \delta(x,y) \in \boldsymbol{\delta}\}$. Suppose $\Delta \subsetneqq \bigcap \{\delta(x,y) : \delta(x,y) \in \boldsymbol{\delta}\}$. Let $(s,t) \in \bigcap \{\delta(x,y) : \delta(x,y) \in \boldsymbol{\delta}\}$, $s \neq t$,so by $(\mathbf{v})$, Proposition 2.1, $\bigcap \{\delta(x,y) : (x,y) \in X \times X\} = \delta(s,t)$, which is impossible sine $(s,t) \notin \delta(s,t)$.

2- Let $\delta(x,y) \in \boldsymbol{\delta}$, we want to find (s,t) such that $\delta(s,t) \circ \delta(s,t) \subseteq \delta(x,y)$. Since $x \neq y$, let $U \in \Gamma^{c}_{(x,y)}$, so there exist V such that $V \circ V \subseteq U$. Let $(s,t) \in V$. Thus by $(\mathbf{iii})$ Proposition 2.1 $\delta(s,t) \subseteq V$, and $\delta(s,t) \circ \delta(s,t) \subseteq V \circ V \subseteq U \subseteq \delta(x,y)$.

3- $\bigcap \{\rho(x,y) : \rho(x,y) \in \boldsymbol{\rho}\} = \Delta$. Clearly $\Delta \subseteq \bigcap \{\rho(x,y) : \rho(x,y) \in \boldsymbol{\rho}\}$. Suppose $\Delta \subsetneqq \bigcap \{\rho(x,y) : \rho(x,y) \in \boldsymbol{\rho}\}$, and let $(s,t) \in \bigcap \{\rho(x,y) : \rho(x,y) \in \boldsymbol{\rho}\}$, $s \neq t$. Then by $(\mathbf{iv})$ and Proposition 2.1, we get $\bigcap \{\rho(x,y) : \rho(x,y) \in \boldsymbol{\rho}\} = \rho(s,t)$, Since $s \neq t$, there exist $U \in \Gamma^{c}_{(s,t)}$,so by $(\mathbf{i})$ Proposition 2.1, $V \subsetneqq \rho(s,t)$. Let $(a,b) \in V$, $a \neq b$. Then $\rho(a,b) \subseteq V \subsetneqq \rho(s,t) \subseteq \rho(a,b)$.

4- Let $\rho(x,y) \in \boldsymbol{\rho}$. We want to find (s,t) such that $\rho(s,t) \circ \rho(s,t) \subseteq \rho(x,y)$. Since $x \neq y$, let $U \in \Gamma^{c}_{(x,y)}$. So there exists V such that $V \circ V \subseteq U$, Let $(s,t) \in V$, so $\rho(s,t) \circ \rho(s,t) \subseteq V \circ V \subseteq U \subseteq \rho(x,y)$.

Clearly $X \times X$ and Δ not an elements of $\boldsymbol{\rho}, \boldsymbol{\delta}$, respectively.

Theorem 4.3. Let (X,Γ) be a semi-linear uniform space. Then, $\Theta = \boldsymbol{\rho} \cup \boldsymbol{\delta} \cup \Gamma$ is a chain.

Proof: Clearly Θ is well ordered by set inclusion. Let θ_1, θ_2 be two different elements in Θ. Then we have the following cases,

1) $\theta_1 = \rho(x,y)$ for some $(x,y) \in X \times X$, and $\theta_2 = \delta(s,t)$ fore some $(s,t) \in X \times X$. So if $(s,t) \in \theta_1$, then the result follows by $(\mathbf{iv})$ Proposition 2.1. Otherwise, if $(s,t) \notin \theta_1$, then the result follows by $(\mathbf{v})$ Proposition 2.2.)

2) $\theta_1 = \rho(x,y)$ for some $(x,y) \in X \times X$, and $\theta_2 = U$ for some $U \in \Gamma$. Clearly if $(x,y) \in U$, then $\rho(x,y) \subseteq U$. On the other hand the result follows by $(\mathbf{i})$ Proposition 2.1.

3) $\theta_1 = \delta(x, y)$ for some $(x, y) \in X \times X$, and $\theta_2 = U$ for some $U \in \Gamma$, as in (2), if $(x, y) \in U$, then $\delta(x, y) \subseteq U$, otherwise $U \subseteq \delta(x, y)$. The other cases are trivial.

Clearly if $(X, \Gamma_1), (X, \Gamma_2)$ are two semi-linear uniform spaces and $\Gamma_1 \cup \Gamma_2$, is a chain, then $(X, \Gamma_1 \cup \Gamma_2)$ is a semi-linear uniform space. So we have,

Corollary 4.4. (X, Θ) is a semi-linear uniform space.

To study a property in a semi-linear uniform space, some times it is easier to deal with the map ρ rather than δ and visa versa. Now we shall show that it doesn't make a difference which map you work with.

Lemma 4.5. Let (X, Γ) be a semi-linear uniform space. Then, $\rho(x, y) \subseteq \rho(s, t)$ if and only if $\delta(x, y) \subseteq \delta(s, t)$.

Proof: Suppose $\rho(x, y) \subseteq \rho(s, t)$, let $(a, b) \in \delta(x, y)$. So there exist $U \in \Gamma^c_{(x,y)} \cap \Gamma_{(a,b)}$. By (i) Proposition 2.1. $U \subsetneqq \rho(x, y)$. Hence $U \subsetneqq \rho(s, t)$, so by (i) Proposition 2.2, $U \subseteq \delta(s, t)$. Therefor $(a, b) \in \delta(s, t)$.For the converse if $\delta(x, y) \subseteq \delta(s, t)$, then $\Gamma^c_{(x,y)} \subseteq \Gamma^c_{(s,t)}$, so $\Gamma_{(s,t)} \subseteq \Gamma_{(x,y)}$.

Lemma 4.6 gives the following nice result.

Theorem 4.6. Let (X, Γ) be a semi-linear uniform space. Then, $\rho(x, y) = \rho(s, t)$ if and only if $\delta(x, y) = \delta(s, t)$.

References

[1] Engelking, R. Outline of General Topology, North-Holand,Amsterdam, 1968.

[2] James, I.M., Topological and Uniform Spaces. Undergraduate Texts in Mathematics. Springer-Verlag 1987.

[3] Tallafha, A. and Khalil, R., Best Approximation in Uniformity type spaces. European Journal of Pure and Applied Mathematics, Vol. 2, No. 2, 2009,(231-238).

Abdalla Tallafha
Department of Mathematics, Jordan University
Amman, Jordan
E-mail address: a.tallafha@ju.edu.jo