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On an elliptic equation of p-Laplacian type with nonlinear boundary condition

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ABSTRACT: We consider elliptic equations of p-Laplacian type with the nonlinear boundary condition of the form

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda f_1(u) + \mu g_1(u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \lambda f_2(u) + \mu g_2(u) & \text{in } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial n}$ is the outer unit normal derivative, λ, μ are parameters. The functions f_i, $i = 1, 2$, are assumed to be $(p-1)$-sublinear while g_i, $i = 1, 2$, are $(p-1)$-assymptotically linear at infinity. Using variational techniques, an existence result is given.

Key Words: Elliptic equation, p-Laplacian type, $(p-1)$-sublinear, $(p-1)$-assymptotically linear, Nonlinear boundary condition.

Contents

1. Introduction

Consider the elliptic equation of p-Laplacian type with nonlinear boundary condition

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda f_1(u) + \mu g_1(u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \lambda f_2(u) + \mu g_2(u) & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial n}$ is the outer unit normal derivative, $1 < p < N$, λ, μ are parameters.

Problem (1.1) has been studied in many works, such as [1,2,3,4,5,9], in which the authors have used different methods to obtain the existence of solutions. In a recent paper [7], we have considered the situation: $g_i \equiv 0$ $(i = 1, 2)$, f_i, $i = 1, 2$, are $(p-1)$-sublinear at infinity. We then used the three critical point theorem of G. Bonanno [6] to obtain a multiplicity result for (1.1). A natural question is to see what happens if the problem in [7] is affected by a certain perturbation. For this purpose, in this note, we establish an existence result for (1.1) in the case when $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are $(p-1)$-sublinear and $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are

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$(p-1)$-assymptotically at infinity. The proof relies essentially on the minimum principle in [8, Theorem 2.1].

In order to state the main result of this work, we would introduce the following hypotheses

(**f**) f_i, $i = 1,2$ are continuous and $(p-1)$-sublinear at infinity, i.e.,

$$\lim_{|t|\to\infty} \frac{|f_i(t)|}{|t|^{p-1}} = 0;$$

(**g**) g_i, $i = 1,2$ are continuous and $(p-1)$-assymptotically at infinity, i.e.,

$$\lim_{|t|\to\infty} \frac{|g_i(t)|}{|t|^{p-1}} = l_i < +\infty.$$

Let $W^{1,p}(\Omega)$ be the usual Sobolev space with respect to the norm

$$\|u\|_{1,p}^p = \int_\Omega (|\nabla u|^p + |u|^p)dx$$

and $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. For any $1 < p < N$ and $1 \leqq q \leqq p^\star = \frac{Np}{N-p}$, we denote by $S_{q,\Omega}$ the best constant in the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ and for all $1 \leqq q \leqq p_\star = \frac{(N-1)p}{N-p}$, we also denote by $S_{q,\partial\Omega}$ the best constant in the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, i.e.

$$S_{q,\partial\Omega} = \inf_{u\in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega(|\nabla u|^p + |u|^p)dx}{\left(\int_{\partial\Omega}|u|^q d\sigma\right)^{\frac{p}{q}}}.$$

Moreover, if $1 \leqq q < p^\star$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact and if $1 \leqq q < p_\star$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact. As a consequence, we have the existence of extremals, i.e. functions where the infimum is attained (see [2,5]).

Definition 1.1. A function $u \in W^{1,p}(\Omega)$ is said to be a weak solution of problem (1.1) if and only if

$$\int_\Omega \Big[|\nabla u|^{p-2}\nabla u\cdot\nabla v + |u|^{p-2}uv\Big]dx - \lambda\int_\Omega f_1(u)vdx - \lambda\int_{\partial\Omega} f_2(u)vd\sigma$$
$$- \mu\int_\Omega g_1(u)vdx - \mu\int_{\partial\Omega} g_2(u)vd\sigma = 0$$

for all $v \in W^{1,p}(\Omega)$.

Theorem 1.2. *Assume conditions* (**f**) *and* (**g**) *are fulfilled. Moreover, there exists* $s_0 > 0$ *such that*

$$F_1(s_0) := \int_0^{s_0} f_1(t)dt > 0 \text{ and } F_2(s_0) := \int_0^{s_0} f_2(t)dt > 0.$$

Then for each $\lambda \in \mathbb{R}$ *large enough, there exists* $\overline{\mu} > 0$, *such that problem (1.1) has at least a non-trivial weak solution* u *in* $W^{1,p}(\Omega)$ *for every* $\mu \in (0,\overline{\mu})$.

2. Existence of solutions

For $\lambda, \mu \in \mathbb{R}$, let us define the functional $J_{\lambda,\mu} : W^{1,p}(\Omega) \to \mathbb{R}$ associated to problem (1.1) by the formula

$$\begin{aligned} J_{\lambda,\mu}(u) = \frac{1}{p}\int_{\Omega}\Big[|\nabla u|^p + |u|^p\Big]dx - \lambda\int_{\Omega} F_1(u)dx - \lambda\int_{\partial\Omega} F_2(u)d\sigma \\ - \mu\int_{\Omega} G_1(u)dx - \mu\int_{\partial\Omega} G_2(u)d\sigma \\ = \Lambda(u) - I_{\lambda,\mu}(u), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \Lambda(u) &= \frac{1}{p}\int_{\Omega}\Big[|\nabla u|^p + |u|^p\Big]dx, \\ I_{\lambda,\mu}(u) &= \lambda\int_{\Omega} F_1(u)dx + \lambda\int_{\partial\Omega} F_2(u)d\sigma + \mu\int_{\Omega} G_1(u)dx + \mu\int_{\partial\Omega} G_2(u)d\sigma \end{aligned} \tag{2.2}$$

for all $u \in W^{1,p}(\Omega)$. Then, a simple computation shows that $J_{\lambda,\mu}$ is of C^1 class and

$$\begin{aligned} DJ_{\lambda,\mu}(u)(v) = \int_{\Omega}\Big[|\nabla u|^{p-2}\nabla u\cdot\nabla v + |u|^{p-2}uv\Big]dx - \lambda\int_{\Omega} f_1(u)vdx - \lambda\int_{\partial\Omega} f_2(u)vd\sigma \\ - \mu\int_{\Omega} g_1(u)vdx - \mu\int_{\partial\Omega} g_2(u)vd\sigma = 0 \end{aligned}$$

for all $u, v \in W^{1,p}(\Omega)$. Thus, weak solutions of problem (1.1) are exactly the critical points of $J_{\lambda,\mu}$.

Lemma 2.1. *For every $\lambda \in \mathbb{R}$, there exists $\overline{\mu} > 0$, depending on λ, such that for every $\mu \in (0, \overline{\mu})$, the functional $J_{\lambda,\mu}$ is coercive.*

Proof. Firstly, we have

$$S_{p,\Omega}\|u\|_{L^p(\Omega)} \leqq \|u\|_{1,p} \text{ and } S_{p,\partial\Omega}\|u\|_{L^p(\partial\Omega)} \leqq \|u\|_{1,p}$$

for all $u \in W^{1,p}(\Omega)$.

Let us fix $\lambda \in \mathbb{R}$, arbitrary. By (**f**), there exist $\delta_i = \delta_i(\lambda)$, $i = 1, 2$, such that

$$|f_1(t)| \leqq S_{p,\Omega}^p \frac{1}{2(1+|\lambda|)}|t|^{p-1}, \quad \forall |t| \geqq \delta_1$$

and

$$|f_2(t)| \leqq S_{p,\partial\Omega}^p \frac{1}{2(1+|\lambda|)}|t|^{p-1}, \quad \forall |t| \geqq \delta_2.$$

Integrating the above inequalities, we have

$$|F_1(t)| \leqq S_{p,\Omega}^p \frac{1}{2p(1+|\lambda|)}|t|^p + \max_{|s|\leqq\delta_1}|f_1(s)||t|, \quad \forall t \in \mathbb{R} \tag{2.3}$$

and

$$|F_2(t)| \leqq S^p_{p,\partial\Omega}\frac{1}{2p(1+|\lambda|)}|t|^p + \max_{|s|\leqq\delta_2}|f_2(s)||t|, \quad \forall t\in\mathbb{R}. \tag{2.4}$$

Since g_i, $i=1,2$ are $(p-1)$-asymptotically linear at infinity, there exist two constants $m_i>0$, $i=1,2$, such that

$$|g_1(t)| \leqq m_1 p S^p_{p,\Omega}|t|^{p-1} + m_1,$$

$$|g_2(t)| \leqq m_2 p S^p_{p,\partial\Omega}|t|^{p-1} + m_2$$

for all $t\in\mathbb{R}$. It implies that

$$|G_1(t)| \leqq m_1 S^p_{p,\Omega}|t|^p + m_1|t|, \tag{2.5}$$

and

$$|G_2(t)| \leqq m_2 S^p_{p,\partial\Omega}|t|^p + m_2|t| \tag{2.6}$$

for all $t\in\mathbb{R}$.

Hence, for any $u\in W^{1,p}(\Omega)$, we deduce that

$$\begin{aligned}
J_{\lambda,\mu}(u) &\geqq \Lambda(u) - |I_{\lambda,\mu}(u)| \\
&\geqq \frac{1}{p}\|u\|^p_{1,p} - \frac{|\lambda|}{2p(1+|\lambda|)}\|u\|^p_{1,p} - \frac{|\lambda|}{S_{p,\Omega}}|\Omega|_N^{\frac{1}{p'}}\|u\|_{1,p}\max_{|s|\leqq\delta_1}|f_1(s)| \\
&\quad - \frac{|\lambda|}{2p(1+|\lambda|)}\|u\|^p_{1,p} - \frac{|\lambda|}{S_{p,\partial\Omega}}|\partial\Omega|_{N-1}^{\frac{1}{p'}}\|u\|_{1,p}\max_{|s|\leqq\delta_1}|f_2(s)| \\
&\quad - |\mu|m_1\|u\|^p_{1,p} - m_1\frac{|\mu|}{S_{p,\Omega}}|\Omega|_N^{\frac{1}{p'}}\|u\|_{1,p} \\
&\quad - |\mu|m_2\|u\|^p_{1,p} - m_2\frac{|\mu|}{S_{p,\partial\Omega}}|\partial\Omega|_{N-1}^{\frac{1}{p'}}\|u\|_{1,p} \\
&= \Big(\frac{1}{p(1+|\lambda|)} - |\mu|(m_1+m_2)\Big)\|u\|^p_{1,p} - \frac{|\lambda|}{S_{p,\Omega}}|\Omega|_N^{\frac{1}{p'}}\|u\|_{1,p}\max_{|s|\leqq\delta_1}|f_1(s)| \\
&\quad - \frac{|\lambda|}{S_{p,\partial\Omega}}|\partial\Omega|_{N-1}^{\frac{1}{p'}}\|u\|_{1,p}\max_{|s|\leqq\delta_1}|f_2(s)| - m_1\frac{|\mu|}{S_{p,\Omega}}|\Omega|_N^{\frac{1}{p'}}\|u\|_{1,p} \\
&\quad - m_2\frac{|\mu|}{S_{p,\partial\Omega}}|\partial\Omega|_{N-1}^{\frac{1}{p'}}\|u\|_{1,p},
\end{aligned}$$

where $p'=\frac{p}{p-1}$. Let $\overline{\mu}=\frac{1}{p(m_1+m_2)(1+|\lambda|)}$ and fix $\mu\in(0,\overline{\mu})$. Since $p>1$ we have $J_{\lambda,\mu}(u)\to+\infty$ as $\|u\|_{1,p}\to\infty$. Thus, the functional $J_{\lambda,\mu}$ is coercive. □

Lemma 2.2. *Let λ and $\overline{\mu}$ be chosen as in the previous lemma. Then for each $\mu\in(0,\overline{\mu})$, the functional $J_{\lambda,\mu}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_m\}$ be a sequence in $W^{1,p}(\Omega)$ such that

$$J_{\lambda,\mu}(u_m) \to \bar{c}, \quad DJ_{\lambda,\mu}(u_m) \to 0 \text{ in } W^{-1,p}(\Omega) \text{ as } m \to \infty. \tag{2.7}$$

Since the functional $J_{\lambda,\mu}$ is coercive, the sequence $\{u_m\}$ is bounded in $W^{1,p}(\Omega)$. Then, there exist a subsequence still denoted by $\{u_m\}$ and a function $u \in W^{1,p}(\Omega)$, such that $\{u_m\}$ converges weakly to u in $W^{1,p}(\Omega)$. Hence, $\{\|u_m - u\|_{1,p}\}$ is bounded and by (2.7), $DJ_{\lambda,\mu}(u_m)(u_m - u)$ converges to 0 as $m \to \infty$.

By (**f**), there exists a constant $C_1 > 0$ such that

$$|f_i(t)| \leqq C_1(1 + |t|^{p-1}), \quad i = 1, 2$$

for all $t \in \mathbb{R}$. Therefore,

$$\begin{aligned} 0 \leqq \int_\Omega |f_1(u_m)||u_m - u|dx &\leqq C_1 \int_\Omega |u_m - u|dx + C \int_\Omega |u_m|^{p-1}|u_m - u|dx \\ &\leqq C_1 \Big[|\Omega|_N^{\frac{1}{p'}} + \|u_m\|^{p-1}_{L^p(\Omega)}\Big] \|u_m - u\|_{L^p(\Omega)} \end{aligned}$$

and

$$\begin{aligned} 0 \leqq \int_{\partial\Omega} |f_2(u_m)||u_m - u|dx &\leqq C_1 \int_{\partial\Omega} |u_m - u|dx + C_1 \int_{\partial\Omega} |u_m|^{p-1}|u_m - u|dx \\ &\leqq C_1 \Big[|\partial\Omega|_{N-1}^{\frac{1}{p'}} + \|u_m\|^{p-1}_{L^p(\partial\Omega)}\Big] \|u_m - u\|_{L^p(\partial\Omega)}. \end{aligned}$$

Since $\{u_m\}$ converges strongly to u in the spaces $L^p(\Omega)$ and $L^p(\partial\Omega)$, the above inequalities imply that

$$\lim_{m\to\infty} \int_\Omega f_1(u_m)(u_m - u)dx = 0 \tag{2.8}$$

and

$$\lim_{m\to\infty} \int_{\partial\Omega} f_2(u_m)(u_m - u)dx = 0. \tag{2.9}$$

On the other hand, by (**g**), there exists a constant $C_2 > 0$ such that

$$|g_i(t)| \leqq C_2(1 + |t|^{p-1}), \quad i = 1, 2$$

for all $t \in \mathbb{R}$. Therefore, the similar arguments above show that

$$\lim_{m\to\infty} \int_\Omega g_1(u_m)(u_m - u)dx = 0 \tag{2.10}$$

and

$$\lim_{m\to\infty} \int_{\partial\Omega} g_2(u_m)(u_m - u)dx = 0. \tag{2.11}$$

By relations (2.8)-(2.11), we get

$$\lim_{m\to\infty} DI_{\lambda,\mu}(u_m)(u_m - u) = 0. \tag{2.12}$$

Combining (2.11) and (2.7), it follows that

$$\lim_{m\to\infty} \Lambda(u_m)(u_m - u) = 0. \tag{2.13}$$

Hence, standard arguments help us to show that the sequence $\{u_m\}$ converges strongly to u in $W^{1,p}(\Omega)$. Thus, the functional $J_{\lambda,\mu}$ satisfies the Palais-Smale condition in $W^{1,p}(\Omega)$. □

Proof Theorem 1.2. By Lemmas 2.1 and 2.2, using the minimum principle [8, Theorem 2.1], we deduce that for each $\lambda \in \mathbb{R}$, there exists $\overline{\mu} > 0$, such that for any $\mu \in (0, \overline{\mu})$, problem (1.1) has a weak solution $u \in W^{1,p}(\Omega)$. We will show that u is not trivial for λ large enough. Indeed, let s_0 be a real number such that

$$F_1(s_0) := \int_0^{s_0} f_1(t)dt > 0 \text{ and } F_2(s_0) := \int_0^{s_0} f_2(t)dt > 0$$

and let $\Omega_0 \subset \Omega$ be an open subset with $|\Omega_0|_N > 0$. Then, there exists $u_0 \in C_0^\infty(\Omega)$ such that $u_0(x) \equiv s_0$ on $\overline{\Omega}_0$ and $0 \leqq u_0(x) \leqq s_0$ in $\Omega\backslash\Omega_0$. We have

$$\begin{aligned} J_{\lambda,\mu}(u_0) &= \frac{1}{p}\int_\Omega \Big[|\nabla u_0|^p + |u_0|^p\Big]dx - \lambda\int_\Omega F_1(u_0)dx - \lambda\int_{\partial\Omega} F_2(u_0)dx \\ &\quad - \mu\int_\Omega G_1(u_0)dx - \mu\int_{\partial\Omega} G_2(u_0)dx \\ &\leqq \frac{1}{p}\int_\Omega \Big[|\nabla u_0|^p + |u_0|^p\Big]dx - \lambda\int_{\Omega_0} F_1(u_0)dx - \lambda\int_{\partial\Omega_0} F_2(u_0)dx \\ &\quad - \mu\int_{\Omega_0} G_1(u_0)dx - \mu\int_{\partial\Omega_0} G_2(u_0)dx \\ &= C - \lambda\Big(F_1(s_0)|\Omega_0|_N + F_2(s_0)|\Omega_0|_{N-1}\Big), \end{aligned}$$

where C is a positive constant (C depends on μ). Therefore, for $\lambda > 0$ large enough, we have $J_{\lambda,\mu}(u_0) < 0$. Thus, the solution u is not trivial. The proof of Theorem 1.2 is now completed. □

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