Bol. Soc. Paran. Mat. (3s.) **v. 29** 2 (2011): 37–41.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v29i2.12190

The semi normed space defined by entire sequences

N. Subramanian, K. Chandrasekhara Rao and K. Balasubramanian

ABSTRACT: In this paper we introduce the sequence spaces $\Gamma(p, \sigma, q, s)$, $\Lambda(p, \sigma, q, s)$ and define a semi normed space (X, q), semi normed by q. We study some properties of these sequence spaces and obtain some inclusion relations.

Key Words: Entire sequence, Analytic sequence, Invariant mean, Semi norm.

Contents

1. Introduction

A complex sequence, whose k^{th} term is x_k, is denoted by $\{x_k\}$ or simply x. Let ϕ be the set of all finite sequences. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^{\frac{1}{k}} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called entire sequence if $\lim_{k\to\infty} |x_k|^{\frac{1}{k}} = 0$. The vector space of all entire sequences will be denoted by Γ. Let σ be a one-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n)), m = 1, 2, 3, \ldots$.

A continuous linear functional ϕ on Λ is said to be an invariant mean or a σ-mean if and only if (1) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n (2) $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$ and (3) $\phi(\{x_{\sigma}(n)\}) = \phi(\{x_n\})$ for all $x \in \Lambda$. For certain kinds of mappings σ, every invariant mean ϕ extends the limit functional on the space C of all real convergent sequences in the sense that $\phi(x) = lim\, x$ for all $x \in C$. Consequently $C \subset V_{\sigma}$,where $V_{\sigma}$is the set of analytic sequences all of those $\sigma-$means are equal .

If $x = (x_n)$, set $Tx = (Tx)^{1/n} = (x_{\sigma}(n))$.It can be shown that
$V_{\sigma} = \left\{ x = (x_n) : m \overset{lim}{\to} \infty\, t_{mn}(x_n)^{1/n} = L \text{ uniformly in } n, L = \sigma- \; n \overset{lim}{\to} \infty\, (x_n)^{1/n} \right\}$
where

$$t_{mn}(x) = \frac{(x_n + Tx_n + \cdots + T^m x_n)^{1/n}}{m+1} \tag{1}$$

Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, ...x_n, 0, 0, ...\}$, $\delta^{(n)} = (0, 0, ..., 1, 0, 0, \ldots)$, 1 in the n^{th} place and zeros elsewhere. An FK-space

2000 *Mathematics Subject Classification:* 46A45,46B45

(Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k (k = 1, 2, ...)$ are continuous.

2. Definitions and Preliminaries

Definition 2.1 *The space consisting of all those sequences x in w such that $\left(|x_k|^{1/k}\right) \to 0$ as $k \to \infty$ is denoted by Γ. In other words $\left(|x_k|^{1/k}\right)$ is a null sequence . Γ is called the space of entire sequences . The space Γ is a metric space with the metric $d(x,y) = \left\{\sup_k \left(|x_k - y_k|^{1/k}\right) : k = 1, 2, 3, \cdots\right\}$ for all $x = \{x_k\}$ and $y = \{y_k\}$ in Γ.*

Definition 2.2 *The space consisting of all those sequences x in w such that $\left(sup_k \left(|x_k|^{1/k}\right)\right) < \infty$ is denoted by Λ. In other words$\left(\sup_k \left(|x_k|^{1/k}\right)\right)$ is a bounded sequence.*

Definition 2.3 *Let p,q be semi norms on a vector space X. Then p is said to be stronger than q if whenever (x_n) is a sequence such that $p(x_n) \to 0$, then also $q(x_n) \to 0$. If each is stronger than the other, then p and q are said to be equivalent.*

Lemma 2.4 *Let p and q be semi norms on a linear space X. Then p is stronger than q if and only if there exists a constant M such that $q(x) \le Mp(x)$ for all $x \in X$.*

Definition 2.5 *A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \le 1$, for all $k \in N$.*

Definition 2.6 *A sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces .*

Remark 2.7 *From the above two definitions, it is clear that a sequence space E is solid implies that E is monotone .*

Definition 2.8 *A sequence E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies that $y_k = 0$.*

Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k < \sup p_k = G$. Let $D = Max(1, 2^{G-1})$. Then for $a_k, b_k \in C$, the set of complex numbers for all $k \in N$ we have

$$|a_k + b_k|^{1/k} \le D\left\{|a_k|^{1/k} + |b_k|^{1/k}\right\}. \tag{2}$$

Let (X, q) be a semi normed space over the field C of complex numbers with the semi norm q. The symbol $\Lambda(X)$ denotes the space of all analytic sequences defined over X. We define the following sequence spaces:

$$\Lambda(p,\sigma,q,s) = \left\{x \in \Lambda(X) : \sup_{n,k} k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{p_k} < \infty \text{uniformly in } n \ge 0, s \ge 0\right\}$$

$$\Gamma(p,\sigma,q,s)=\left\{x\in\Gamma(X):k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{p_k}\to 0,\ \text{as}\,k\to\infty\text{unif/ in } n\geq 0, s\geq 0\right\}.$$

3. Main Results

Theorem 3.1 *$\Gamma(p,\sigma,q,s)$ is a linear space over the set of complex numbers.*

Proof: The proof is easy, so omitted .

Theorem 3.2 *$\Gamma(p,\sigma,q,s)$ is a paranormed space with*

$$g^*(x)=\left\{\sup_{k\geq 1}k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right],\ \textit{uniformly in } n>0\right\}$$

where $H=max\,(1,\sup_k p_k)$.

Proof: Clearly $g(x)=g(-x)$ and $g(\theta)=0$, where θ is the zero sequence . It can be easily verified that $g(x+y)\leq g(x)+g(y)$. Next $x\to\theta,\lambda$ fixed implies $g(\lambda x)\to 0$. Also $x\to\theta$ and $\lambda\to 0$ implies $g(\lambda x)\to 0$. The case $\lambda\to 0$ and x fixed implies that $g(\lambda x)\to 0$ follows from the following expressions.

$$g(\lambda x)=\left\{(\sup_{k\geq 1}k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]\ \text{uniformly in } n,m\in N\right\}$$

$$g(\lambda x)=\left\{(|\lambda|\,r)^{p_m/H}:sup_{k\geq 1}k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right],r>0,\ \text{uniformly in } n,m\in N\right\}$$

where $r=\frac{1}{|\lambda|}$. Hence $\Gamma(p,\sigma,q,s)$ is a paranormed space . This completes the proof.

Theorem 3.3 $\Gamma(p,\sigma,q,s)\bigcap\Lambda(p,\sigma,q,s)\subseteq\Gamma(p,\sigma,q,s)$.

Proof: The proof is easy, so omitted .

Theorem 3.4 $\Gamma(p,\sigma,q,s)\subset\Lambda(p,\sigma,q,s)$.

Proof: The proof is easy, so omitted .

Remark 3.1 *Let q_1 and q_2 be two semi norms on X, we have*
(i) $\Gamma(p,\sigma,q_1,s)\bigcap\Gamma(p,\sigma,q_2,s)\subseteq\Gamma(p,\sigma,q_1+q_2,s)$;
(ii) If q_1 is stronger than q_2, then $\Gamma(p,\sigma,q_1,s)\subseteq\Gamma(p,\sigma,q_2,s)$;
(iii) If q_1 is equivalent to q_2, then $\Gamma(p,\sigma,q_1,s)=\Gamma(p,\sigma,q_2,s)$.

Theorem 3.5 *(i) Let $0\leq p_k\leq r_k$ and $\left\{\frac{r_k}{p_k}\right\}$ be bounded. Then* $\Gamma(r,\sigma,q,s)\subset\Gamma(p,\sigma,q,s)$;
(ii) $s_1\leq s_2$ implies $\Gamma(p,\sigma,q,s_1)\subset\Gamma(p,\sigma,q,s_2)$.

Proof of (i):
Let

$$x \in \Gamma(r,\sigma,q,s) \tag{3}$$

$$k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{r_k} \to 0 \quad as \quad k \to \infty \tag{4}$$

Let $t_k = k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{r_k}$ and $\lambda_k = \frac{p_k}{r_k}$. Since $p_k \le r_k$, we have $0 \le \lambda_k \le 1$. Take $0 < \lambda > \lambda_k$. Define $u_k = t_k(t_k \ge 1); u_k = 0(t_k < 1)$; and $v_k = 0(t_k \ge 1); v_k = t_k(t_k < 1)$; $t_k = u_k + v_k$. $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that

$$u_k^{\lambda_k} \le t_k \quad and \quad v_k^{\lambda_k} \le v_k^{\lambda} \tag{5}$$

(i.e) $t_k^{\lambda_k} \le t_k + v_k^{\lambda}$ by (5) $k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)^{r_k}\right]^{\lambda_k} \le k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{r_k}$ $k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)^{r_k}\right]^{p_k/r_k} \le k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{r_k}$ $k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{p_k} \le k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{r_k}$ But $k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{r_k} \to 0 \;\; as \;\; k \to \infty$ by (4) . $k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{p_k} \to 0 \;\; as \;\; k \to \infty$.

Hence

$$x \in \Gamma(p,\sigma,q,s). \tag{6}$$

From (3) and (6) we get $\Gamma(r,\sigma,q,s) \subset \Gamma(p,\sigma,q,s)$. This completes the proof .
Proof of (ii): The proof is easy, so omitted .

Theorem 3.6 *The space $\Gamma(p,\sigma,q,s)$ is solid and as such is monotone .*

Proof: Let $(x_k) \in \Gamma(p,\sigma,q,s)$ and (α_k) be a sequence of scalars such that $|\alpha_k| \le 1$ for all $k \in N$. Then $k^{-s}\left[q\left(\left|\alpha_k x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{p_k} \le k^{-s}\left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{p_k}$ for all $k \in N$. $\left[q\left(\left|\alpha_k x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{p_k} \le \left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{p_k}$ for all $k \in N$. This completes the proof.

Theorem 3.7 *The space $\Gamma(p,\sigma,q,s)$ are not convergence free in general.*

Proof: The proof follows from the following example.
Example: Let $s = 0; p_k = 1$ for k even and p_k=2 for k odd. Let $X = C, q(x) = |x|$ and $\sigma(n) = n+1$ for all $n \in N$. Then we have $\sigma^2(n) = \sigma(\sigma(n)) = \sigma(n+1) = (n+1)+1 = n+2$ and $\sigma^3(n) = \sigma(\sigma^2(n)) = \sigma(n+2) = (n+2)+1 = n+3$. Therefore $\sigma^k(n) = (n+k)$ for all $n, k \in N$. Consider the sequences (x_k) and (y_k) defined as $x_k = \left(\frac{1}{k}\right)^k$ and $(y_k) = k^k$ for all $k \in N$. (i.e)$|x_k|^{1/k} = \frac{1}{k}$ and $|y_k|^{1/k} = k$ for all $k \in N$.
Hence $\left|\left(\frac{1}{(n+k)}\right)^{n+k}\right|^{p_k} \to 0$ as $k \to \infty$. Therefore $(x_k) \in \Gamma(p,\sigma)$. But $\left|\left(\frac{1}{(n+k)}\right)^{n+k}\right|^{p_k}$ $r \not\to 0$ as $k \to \infty$. Hence $(y_k) \notin \Gamma(p,\sigma)$. Hence the space $\Gamma(p,\sigma,q,s)$ are not convergence free in general. This completes the proof.

Acknowledgments

I wish to thank the referees for their several remarks and valuable suggestions that improved the presentation of the paper.

References

1. Y.Altin and M.Et, Generalized difference sequence spaces defined by a modulus function in a locally convex space, *Soochow J. Math.* **31(2)**(2005), 233-243.
2. H.I.Brown, The summability filed of a perfect $\ell - \ell$ method of summation, *J. Anal. Math.* **20**(1967), 281-287.
3. R.Colak, M.Et, and E.Malkowsky, Some topics of sequence spaces, Lecture Notes in Mathematics, *Firat University Press, Elazig, Turkey*, **2004**.
4. C.Goffman and G.Pedrick, First Course in Functional Analysis, *Prentice Hall India, New Delhi*,**1974**.
5. P.K.Kamthan and M.Gupta, Sequence spaces and Series. Lecture Notes in Pure and Applied Mathematics, *65 Marcel Dekker, Inc., New York*, **1981**.
6. I.J.Maddox, Elements of Functional Analysis, *Cambridge Univ. Press*, **1970**.
7. I.J.Maddox, Sequence spaces defined by a modulus, *Math. Proc. Cambridge Philos. Soc.*(100) (1986), 161-166.
8. H.Nakano, "Concave modulars", *Journal of the Mathematical Society of Japan*, **Vol. 5, no.1**(1953), 29-49.
9. W.H.Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.*, **25**(1973), 973-978.
10. B.C.Tripathy, S.Mahanta and M.Et, On a class of generalized difference sequence space defined by modulus function, *Hakkaido Math. Jour.*, **XXX1V(3)**(2005), 667-677.
11. A.Wilansky, Functional Analysis, *Blaisdell publishing company, New York*, **1964**.
12. A.Wilansky, Summability through Functional Analysis, *North Holland Mathematics Studies, North-Holland Publishing, Amsterdam*, **Vol. 85** (1984).

N.Subramanian
Department of Mathematics, SASTRA University,
Thanjavur-613 401, India.
E-mail address: nsmaths@yahoo.com

and

K.Chandrasekhara Rao
Department of Mathematics, SASTRA University,
Thanjavur-613 401, India.
E-mail address: kchandrasekhara@rediffmail.com

and

K.Balasubramanian
Department of Mathematics, SASTRA University,
Thanjavur-613 401, India.
E-mail address: k_bala27@yahoo.co.in