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On Ψ_*-operator in ideal m-spaces

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ABSTRACT: An ideal on a set X is a nonempty collection of subsets of X with heredity property which is also closed finite unions. The concept of ideal m-spaces was introduced by Al-Omari and Noiri [1]. In this paper, we introduce and study an operator $\Psi_* : \mathcal{P}(X) \to \mathcal{M}$ defined as follows for every $A \in X$, $\Psi_*(A) = \{x \in X :$ there exists a $U \in \mathcal{M}(x)$ such that $U - A \in \mathcal{I}\}$, and observes that $\Psi_*(A) = X - (X - A)_*$.

Key Words: ideal, Ψ_*-operator, ideal m-space

Contents

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1. Introduction

The notion of ideal topological spaces was first studied by Kuratowski [6] and Vaidyanathaswamy [15]. Compatibility of the topology τ with an ideal $\mathcal{I}$ was first defined by Njåstad [11]. In 1990, Jankovic and Hamlett [4,3] investigated further properties of ideal topological spaces and another operator called Ψ-operator is defined as $\Psi(A) = X - (X - A)^*$. In 2007 Modak and Bandyopadhyay [8] used the Ψ-operator to define interesting generalized open sets. In 2009, Ozbakir and Yildirim [12] defined the minimal local function by using minimal structures and deduced some results. Quite recently, Al-Omari and Noiri [1] defined the ideal m-spaces and obtained some interesting results. In this paper, we introduce and study an operator $\Psi_* : \mathcal{P}(X) \to \mathcal{M}$ defined as follows for every $A \in X$, $\Psi_*(A) =$

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$\{x \in X$: there exists a $U \in \mathcal{M}(x)$ such that $U - A \in \mathcal{I}\}$, and observes that $\Psi_*(A) = X - (X - A)_*$. We utilize the Ψ_*-operator to define interesting generalized m-open sets and study their properties.

2. Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ), respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following properties:

1. $A \in \mathcal{I}$ and $B \subseteq A$ implies that $B \in \mathcal{I}$.
2. $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every open set U containing $x\}$ is called the local function of A with respect to $\mathcal{I}$ and τ (see [4,6]). We simply write A^* instead of $A^*(\mathcal{I}, \tau)$ in case there is no chance for confusion. For every ideal topological space $(X, \tau, \mathcal{I})$, there exists a topology $\tau^*(\mathcal{I})$, finer than τ, generating by the base $\beta(\mathcal{I}, \tau) = \{U - J : U \in \tau$ and $J \in \mathcal{I}\}$. It is known in [4] that $\beta(\mathcal{I}, \tau)$ is not always a topology. When there is no ambiguity, $\tau^*(\mathcal{I})$ is denoted by τ^*. Recall that A is said to be $*$-dense in itself (resp. τ^*-closed, $*$-perfect) if $A \subseteq A^*$ (resp. $A^* \subseteq A$, $A = A^*$). For a subset $A \subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will denote the closure and the interior of A in (X, τ^*), respectively.

Definition 2.1 [1] A subfamily $\mathcal{M}$ of the power set $\mathcal{P}(X)$ of a nonempty set X is called an m-structure on X if $\mathcal{M}$ satisfies the following conditions:

1. $\mathcal{M}$ contains ϕ and X,
2. $\mathcal{M}$ is closed under the finite intersection.

The pair $(X, \mathcal{M})$ is called an m-space. An m-space $(X, \mathcal{M})$ with an ideal $\mathcal{I}$ on X is called an ideal m-space and is denoted by $(X, \mathcal{M}, \mathcal{I})$.

Definition 2.2 [1] A set $A \in \mathcal{P}(X)$ is called an m-open set if $A \in \mathcal{M}$. $B \in \mathcal{P}(X)$ is called an m-closed set if $X - B \in \mathcal{M}$. We set $mInt(A) = \cup\{U : U \subseteq A, U \in \mathcal{M}\}$ and $mCl(A) = \cap\{F : A \subseteq F, X - F \in \mathcal{M}\}$.

Definition 2.3 [1] Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. For a subset A of X, we define the following set: $A_*(\mathcal{I}, \mathcal{M}) = \{x \in X : A \cap U \notin \mathcal{I}$ for every $U \in \mathcal{M}(x)\}$, where $\mathcal{M}(x) = \{U \in \mathcal{M} : x \in U\}$. In case there is no confusion $A_*(\mathcal{I}, \mathcal{M})$ is briefly denoted by A_* and is called the $\mathcal{M}$-local function of A with respect to $\mathcal{I}$ and $\mathcal{M}$.

Lemma 2.1 *[1] Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A, B any subsets of X. Then the following properties hold:*

1. $(\phi)_* = \phi$,
2. $(A_*)_* \subset A_*$,
3. $A_* \cup B_* = (A \cup B)_*$.

Theorem 2.1 *[1] Let $(X, \mathcal{M})$ be an m-space, $\mathcal{I}$ and $\mathcal{J}$ be ideals on X, and let A and B be subsets of X. Then the following properties hold:*

1. *If $A \subseteq B$, then $A_* \subseteq B_*$.*
2. *If $\mathcal{I} \subseteq \mathcal{J}$, then $A_*(\mathcal{I}) \supseteq A_*(\mathcal{J})$.*
3. $A_* = mCl(A_*) \subseteq mCl(A)$ *(i.e.A_* is an m-closed subset of $mCl(A)$).*
4. *If $A \subseteq A_*$, then $A_* = mCl(A_*) = mCl(A)$.*
5. *If $A \in \mathcal{I}$, then $A_* = \phi$.*

Corollary 2.1A *[1] Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A, I subsets of X with $I \in \mathcal{I}$. Then $(A \cup I)_* = A_* = (A - I)_*$.*

Remark 2.1 In [1] Al-Omari and Noiri obtained that $Cl_*(A) = A \cup A_*$ is a Kuratowski closure operator. We will denote by $\mathcal{M}_*$ the topology generated by Cl_*, that is, $\mathcal{M}_* = \{U \subseteq X : Cl_*(X - U) = X - U\}$.

Theorem 2.2 *[1] Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. Then $\beta(\mathcal{M}, \mathcal{I}) = \{V - I : V \in \mathcal{M}, I \in \mathcal{I}\}$ is a basis for $\mathcal{M}_*$.*

Theorem 2.3 *[1] Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space, then the following properties are equivalent:*

1. *$\mathcal{M} \cap \mathcal{I} = \phi$;*
2. *If $I \in \mathcal{I}$, then $mInt(I) = \phi$;*
3. *For every $G \in \mathcal{M}$, $G \subseteq G_*$;*
4. *$X = X_*$.*

3. Ψ_*-operator in ideal m-spaces

Definition 3.1 Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. An operator $\Psi_* : \mathcal{P}(X) \to \mathcal{M}$ is defined as follows for every $A \in X$, $\Psi_*(A) = \{x \in X$: there exists a $U \in \mathcal{M}(x)$ such that $U - A \in \mathcal{I}\}$ and observes that $\Psi_*(A) = X - (X - A)_*$.

Several basic facts concerning the behavior of the operator Ψ_* are included in the following theorem.

Theorem 3.1 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. Then the following properties hold:*

1. *If $A \subseteq X$, then $\Psi_*(A)$ is m-open.*
2. *If $A \subseteq B$, then $\Psi_*(A) \subseteq \Psi_*(B)$.*
3. *If $A, B \in \mathcal{P}(X)$, then $\Psi_*(A \cap B) = \Psi_*(A) \cap \Psi_*(B)$.*
4. *If $U \in \mathcal{M}_*$, then $U \subseteq \Psi_*(U)$.*
5. *If $A \subseteq X$, then $\Psi_*(A) \subseteq \Psi_*(\Psi_*(A))$.*
6. *If $A \subseteq X$, then $\Psi_*(A) = \Psi_*(\Psi_*(A))$ if and only if $(X - A)_* = ((X - A)_*)_*$.*
7. *If $A \in \mathcal{I}$, then $\Psi_*(A) = X - X_*$.*
8. *If $A \subseteq X$, then $A \cap \Psi_*(A) = Int_*(A)$.*
9. *If $A \subseteq X$, $I \in \mathcal{I}$, then $\Psi_*(A - I) = \Psi_*(A)$.*
10. *If $A \subseteq X$, $I \in \mathcal{I}$, then $\Psi_*(A \cup I) = \Psi_*(A)$.*

11. *If* $(A - B) \cup (B - A) \in \mathcal{I}$*, then* $\Psi_*(A) = \Psi_*(B)$*.*

Proof. (1) This follows from Theorem 2.1 (3).
(2) This follows from Theorem 2.1 (1).
(3) It follows from (2) that $\Psi_*(A \cap B) \subseteq \Psi_*(A)$ and $\Psi_*(A \cap B) \subseteq \Psi_*(B)$. Hence $\Psi_*(A \cap B) \subseteq \Psi_*(A) \cap \Psi_*(B)$. Now let $x \in \Psi_*(A) \cap \Psi_*(B)$. There exist $U, V \in \mathcal{M}(x)$ such that $U - A \in \mathcal{I}$ and $V - B \in \mathcal{I}$. Let $G = U \cap V \in \mathcal{M}(x)$ and we have $G - A \in \mathcal{I}$ and $G - B \in \mathcal{I}$ by heredity. Thus $G - (A \cap B) = (G - A) \cup (G - B) \in \mathcal{I}$ by additivity, and hence $x \in \Psi_*(A \cap B)$. We have shown $\Psi_*(A) \cap \Psi_*(B) \subseteq \Psi_*(A \cap B)$ and the proof is complete.
(4) If $U \in \mathcal{M}_*$, then $X - U$ is $\mathcal{M}_*$-closed which implies $(X - U)_* \subseteq X - U$ and hence $U \subseteq X - (X - U)_* = \Psi_*(U)$.
(5) This follows from (1) and (4).
(6) This follows from the facts:

1. $\Psi_*(A) = X - (X - A)_*$.
2. $\Psi_*(\Psi_*(A)) = X - [X - (X - (X - A)_*)]_* = X - ((X - A)_*)_*$.

(7) By Corollary 2.1A we obtain that $(X - A)_* = X_*$ if $A \in \mathcal{I}$.
(8) If $x \in A \cap \Psi_*(A)$, then $x \in A$ and there exists a $U_x \in \mathcal{M}(x)$ such that $U_x - A \in \mathcal{I}$. Then by Theorem 2.2, $U_x - (U_x - A)$ is an $\mathcal{M}_*$-open neighborhood of x and $x \in Int_*(A)$. On the other hand, if $x \in Int_*(A)$, there exists a basic $\mathcal{M}_*$-open neighborhood $V_x - I$ of x, where $V_x \in \mathcal{M}$ and $I \in \mathcal{I}$, such that $x \in V_x - I \subseteq A$ which implies $V_x - A \subseteq I$ and hence $V_x - A \in \mathcal{I}$. Hence $x \in A \cap \Psi_*(A)$.
(9) This follows from Corollary 2.1A and $\Psi_*(A - I) = X - [X - (A - I)]_* = X - [(X - A) \cup I]_* = X - (X - A)_* = \Psi_*(A)$.
(10) This follows from Corollary 2.1A and $\Psi_*(A \cup I) = X - [X - (A \cup I)]_* = X - [(X - A) - I]_* = X - (X - A)_* = \Psi_*(A)$.
(11) Assume $(A - B) \cup (B - A) \in \mathcal{I}$. Let $A - B = I$ and $B - A = J$. Observe that $I, J \in \mathcal{I}$ by heredity. Also observe that $B = (A - I) \cup J$. Thus $\Psi_*(A) = \Psi_*(A - I) = \Psi[(A - I) \cup J] = \Psi_*(B)$ by (9) and (10).

Corollary 3.1A *Let* $(X, \mathcal{M}, \mathcal{I})$ *be an ideal* m*-space. Then* $U \subseteq \Psi_*(U)$ *for every* m*-open set* $U \in \mathcal{M}$*.*

Proof. We know that $\Psi_*(U) = X - (X - U)_*$. Now $(X - U)_* \subseteq mCl(X - U) = X - U$, since $X - U$ is m-closed. Therefore, $U = X - (X - U) \subseteq X - (X - U)_* = \Psi_*(U)$.

Now we give an example of a set A which is not m-open but satisfies $A \subseteq \Psi_*(A)$.

Example 3.1 *Let $X = \{a, b, c, d\}$, $\mathcal{M} = \{\phi, X, \{a, c\}, \{d\}\}$, and $\mathcal{I} = \{\phi, \{c\}\}$. Let $A = \{a\}$. Then $\Psi_*(\{a\}) = X - (X - \{a\})_* = X - \{b, c, d\}_* = X - \{b, d\} = \{a, c\}$. Therefore, $A \subseteq \Psi_*(A)$, but A is not m-open.*

Theorem 3.2 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and $A \subseteq X$. Then the following properties hold:*

1. $\Psi_*(A) = \cup\{U \in \mathcal{M} : U - A \in \mathcal{I}\}$.
2. $\Psi_*(A) \supseteq \cup\{U \in \mathcal{M} : (U - A) \cup (A - U) \in \mathcal{I}\}$.

Proof. (1) This follows immediately from the definition of Ψ_*-operator.
(2) Since $\mathcal{I}$ is heredity, it is obvious that $\cup\{U \in \mathcal{M} : (U - A) \cup (A - U) \in \mathcal{I}\} \subseteq \cup\{U \in \mathcal{M} : U - A \in \mathcal{I}\} = \Psi_*(A)$ for every $A \subseteq X$.

Theorem 3.3 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. If $\sigma = \{A \subseteq X : A \subseteq \Psi_*(A)\}$. Then σ is a topology for X and $\sigma = \mathcal{M}_*$.*

Proof. Let $\sigma = \{A \subseteq X : A \subseteq \Psi_*(A)\}$. First, we show that σ is a topology. Observe that $\phi \subseteq \Psi_*(\phi)$ and $X \subseteq \Psi_*(X) = X$, and thus ϕ and $X \in \sigma$. Now if $A, B \in \sigma$, then $A \cap B \subseteq \Psi_*(A) \cap \Psi_*(B) = \Psi_*(A \cap B)$ which implies that $A \cap B \in \sigma$. If $\{A_\alpha : \alpha \in \Delta\} \subseteq \sigma$, then $A_\alpha \subseteq \Psi_*(A_\alpha) \subseteq \Psi_*(\cup A_\alpha)$ for every α and hence $\cup A_\alpha \subseteq \Psi_*(\cup A_\alpha)$. This shows that σ is a topology. Now if $U \in \mathcal{M}_*$ and $x \in U$, then by Theorem 2.2 there exist $V \in \mathcal{M}(x)$ and $I \in \mathcal{I}$ such that $x \in V - I \subseteq U$. Clearly $V - U \subseteq I$ so that $V - U \in \mathcal{I}$ by heredity and hence $x \in \Psi_*(U)$. Thus $U \subseteq \Psi_*(U)$ and we have shown $\mathcal{M}_* \subseteq \sigma$. Now let $A \in \sigma$, then we have $A \subseteq \Psi_*(A)$, that is, $A \subseteq X - (X - A)_*$ and $(X - A)_* \subseteq X - A$. This shows that $X - A$ is $\mathcal{M}_*$-closed and hence $A \in \mathcal{M}_*$. Thus $\sigma \subseteq \mathcal{M}_*$ and hence $\sigma = \mathcal{M}_*$.

Definition 3.2 [1] Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. We say the m-structure $\mathcal{M}$ is m-compatible with the ideal $\mathcal{I}$, denoted $\mathcal{M} \sim_* \mathcal{I}$, if the following holds for every $A \subseteq X$, if for every $x \in A$ there exists $U \in \mathcal{M}(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Theorem 3.4 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. Then $\mathcal{M} \sim_* \mathcal{I}$ if and only if $\Psi_*(A) - A \in \mathcal{I}$ for every $A \subseteq X$.*

Proof. Necessity. Assume $\mathcal{M} \sim_* \mathcal{I}$ and let $A \subseteq X$. Observe that $x \in \Psi_*(A) - A \in \mathcal{I}$ if and only if $x \notin A$ and $x \notin (X - A)_*$ if and only if $x \notin A$ and there exists $U_x \in \mathcal{M}(x)$ such that $U_x - A \in \mathcal{I}$ if and only if there exists $U_x \in \mathcal{M}(x)$ such that $x \in U_x - A \in \mathcal{I}$. Now, for each $x \in \Psi_*(A) - A$ and $U_x \in \mathcal{M}(x)$, $U_x \cap (\Psi_*(A) - A) \in \mathcal{I}$ by heredity and hence $\Psi_*(A) - A \in \mathcal{I}$ by assumption that $\mathcal{M} \sim_* \mathcal{I}$.

Sufficiency. Let $A \subseteq X$ and assume that for each $x \in A$ there exists $U_x \in \mathcal{M}(x)$ such that $U_x \cap A \in \mathcal{I}$. Observe that $\Psi_*(X - A) - (X - A) = \{x :$ there exists $U_x \in \mathcal{M}(x)$ such that $x \in U_x \cap A \in \mathcal{I}\}$. Thus we have $A \subseteq \Psi_*(X - A) - (X - A) \in \mathcal{I}$ and hence $A \in \mathcal{I}$ by heredity of $\mathcal{I}$.

Proposition 3.1 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space with $\mathcal{M} \sim_* \mathcal{I}$, $A \subseteq X$. If N is a nonempty m-open subset of $A_* \cap \Psi_*(A)$, then $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.*

Proof. If $N \subseteq A_* \cap \Psi_*(A)$, then $N - A \subseteq \Psi_*(A) - A \in \mathcal{I}$ by Theorem 3.4 and hence $N - A \in \mathcal{I}$ by heredity. Since $N \in \mathcal{M} - \{\phi\}$ and $N \subseteq A_*$, we have $N \cap A \notin \mathcal{I}$ by the definition of A_*.

As a consequence of the above theorem, we have the following.

Corollary 3.4B *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space with $\mathcal{M} \sim_* \mathcal{I}$. Then $\Psi_*(\Psi_*(A)) = \Psi_*(A)$ for every $A \subseteq X$.*

Proof. $\Psi_*(A) \subseteq \Psi_*(\Psi_*(A))$ follows from Theorem 3.1 (5). Since $\mathcal{M} \sim_* \mathcal{I}$, it follows from Theorem 3.4 that $\Psi_*(A) \subseteq A \cup I$ for some $I \in \mathcal{I}$ and hence $\Psi_*(\Psi_*(A)) = \Psi_*(A)$ by Theorem 3.1 (10).

Theorem 3.5 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space with $\mathcal{M} \sim_* \mathcal{I}$. Then $\Psi_*(A) = \cup\{\Psi_*(U) : U \in \mathcal{M}, \Psi_*(U) - A \in \mathcal{I}\}$.*

Proof. Let $\Phi(A) = \cup\{\Psi_*(U) : U \in \mathcal{M}, \Psi_*(U) - A \in \mathcal{I}\}$. Clearly, $\Phi(A) \subseteq \Psi_*(A)$. Now let $x \in \Psi_*(A)$. Then there exists $U \in \mathcal{M}(x)$ such that $U - A \in \mathcal{I}$. By Corollary 3.1A, $U \subseteq \Psi_*(U)$ and $\Psi_*(U) - A \subseteq [\Psi_*(U) - U] \cup [U - A]$. By Theorem 3.4, $\Psi_*(U) - U \in \mathcal{I}$ and hence $\Psi_*(U) - A \in \mathcal{I}$. Hence $x \in \Phi(A)$ and $\Phi(A) \supseteq \Psi_*(A)$. Consequently, we obtain $\Phi(A) = \Psi_*(A)$.

In [9], Newcomb defines $A = B$ [mod $\mathcal{I}$] if $(A - B) \cup (B - A) \in \mathcal{I}$ and observes that $=$ [mod $\mathcal{I}$] is an equivalence relation. By Theorem 3.1 (11), we have that if $A = B$ [mod $\mathcal{I}$], then $\Psi_*(A) = \Psi_*(B)$.

Definition 3.3 Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. A subset A of X is called a Baire set with respect to $\mathcal{M}$ and $\mathcal{I}$, denoted $A \in \mathcal{B}_r(X, \mathcal{M}, \mathcal{I})$, if there exists an m-open set $U \in \mathcal{M}$ such that $A = U$ [mod $\mathcal{I}$].

Lemma 3.1 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space with $\mathcal{M} \sim_* \mathcal{I}$. If $U, V \in \mathcal{M}$ and $\Psi_*(U) = \Psi_*(V)$, then $U = V$ [mod $\mathcal{I}$].*

Proof. Since $U \in \mathcal{M}$, we have $U \subseteq \Psi_*(U)$ and hence $U - V \subseteq \Psi_*(U) - V = \Psi_*(V) - V \in \mathcal{I}$ by Theorem 3.4. Similarly $V - U \in \mathcal{I}$. Now $(U - V) \cup (V - U) \in \mathcal{I}$ by additivity. Hence $U = V$ [mod $\mathcal{I}$].

Theorem 3.6 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space with $\mathcal{M} \sim_* \mathcal{I}$. If $A, B \in \mathcal{B}_r(X, \mathcal{M}, \mathcal{I})$, and $\Psi_*(A) = \Psi_*(B)$, then $A = B$ [mod $\mathcal{I}$].*

Proof. Let $U, V \in \mathcal{M}$ such that $A = U$ [mod $\mathcal{I}$] and $B = V$ [mod $\mathcal{I}$]. Now $\Psi_*(A) = \Psi_*(B)$ and $\Psi_*(B) = \Psi_*(V)$ by Theorem 3.1(11). Since $\Psi_*(A) = \Psi_*(U)$ implies that $\Psi_*(U) = \Psi_*(V)$, hence $U = V$ [mod $\mathcal{I}$] by Lemma 3.1. Hence $A = B$ [mod $\mathcal{I}$] by transitivity.

Proposition 3.2 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space.*

1. *If $B \in \mathcal{B}_r(X, \mathcal{M}, \mathcal{I}) - \mathcal{I}$, then there exists $A \in \mathcal{M} - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$].*

2. *Let $\mathcal{M} \cap \mathcal{I} = \phi$, then $B \in \mathcal{B}_r(X, \mathcal{M}, \mathcal{I}) - \mathcal{I}$ if and only if there exists $A \in \mathcal{M} - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$].*

Proof. (1) Assume $B \in \mathcal{B}_r(X, \mathcal{M}, \mathcal{I}) - \mathcal{I}$, then $B \in \mathcal{B}_r(X, \mathcal{M}, \mathcal{I})$. Now if there does not exist $A \in \mathcal{M} - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$], we have $B = \phi$ [mod $\mathcal{I}$]. This implies that $B \in \mathcal{I}$ which is a contradiction.
(2) Assume there exists $A \in \mathcal{M} - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$]. Then $A = (B - J) \cup I$, where $J = B - A, I = A - B \in \mathcal{I}$. If $B \in \mathcal{I}$, then $A \in \mathcal{I}$ by heredity and additivity, which contradicts that $\mathcal{M} \cap \mathcal{I} = \phi$.

Proposition 3.3 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space with $\mathcal{M} \cap \mathcal{I} = \phi$. If $B \in \mathcal{B}_r(X, \mathcal{M}, \mathcal{I}) - \mathcal{I}$, then $\Psi_*(B) \cap mInt(B_*) \neq \phi$.*

Proof. Assume $B \in \mathcal{B}_r(X, \mathcal{M}, \mathcal{I}) - \mathcal{I}$, then by Proposition 3.2(1), there exists $A \in \mathcal{M} - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$]. This implies that $\phi \neq A \subseteq A_* = ((B - J) \cup I)_* = B_*$, where $J = B - A, I = A - B \in \mathcal{I}$ by Theorem 2.3 and Corollary 2.1A. Also $\phi \neq A \subseteq \Psi_*(A) = \Psi_*(B)$ by Theorem 3.1 (11), so that $A \subseteq \Psi_*(B) \cap mInt(B_*)$.

Given an ideal m-space $(X, \mathcal{M}, \mathcal{I})$, let $\mathcal{U}(X, \mathcal{M}, \mathcal{I})$ denote $\{A \subseteq X :$ there exists $B \in \mathcal{B}_r(X, \mathcal{M}, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A\}$.

Proposition 3.4 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space with $\mathcal{M} \cap \mathcal{I} = \phi$. The following properties are equivalent:*

1. $A \in \mathcal{U}(X, \mathcal{M}, \mathcal{I})$;
2. $\Psi_*(A) \cap mInt(A_*) \neq \phi$;
3. $\Psi_*(A) \cap A_* \neq \phi$;
4. $\Psi_*(A) \neq \phi$;
5. $Int_*(A) \neq \phi$;
6. *There exists $N \in \mathcal{M} - \{\phi\}$ such that $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2): Let $B \in \mathcal{B}_r(X, \mathcal{M}, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A$. Then $mInt(B_*) \subseteq mInt(A_*)$ and $\Psi_*(B) \subseteq \Psi_*(A)$ and hence $mInt(B_*) \cap \Psi_*(B) \subseteq mInt(A_*) \cap \Psi_*(A)$. By Proposition 3.3, we have $\Psi_*(A) \cap mInt(A_*) \neq \phi$.
(2) $\Rightarrow$ (3): The proof is obvious.
(3) $\Rightarrow$ (4): The proof is obvious.
(4) $\Rightarrow$ (5): If $\Psi_*(A) \neq \phi$, then there exists $U \in \mathcal{M} - \{\phi\}$ such that $U - A \in \mathcal{I}$. Since $U \notin \mathcal{I}$ and $U = (U - A) \cup (U \cap A)$, we have $U \cap A \notin \mathcal{I}$. By Theorem 3.1, $\phi \neq (U \cap A) \subseteq \Psi_*(U) \cap A = \Psi_*((U - A) \cup (U \cap A)) \cap A = \Psi_*(U \cap A) \cap A \subseteq \Psi_*(A) \cap A = Int_*(A)$. Hence $Int_*(A) \neq \phi$.
(5) $\Rightarrow$ (6): If $Int_*(A) \neq \phi$, then by Theorem 2.2 there exists $N \in \mathcal{M} - \{\phi\}$ and $I \in \mathcal{I}$ such that $\phi \neq N - I \subseteq A$. We have $N - A \in \mathcal{I}$, $N = (N - A) \cup (N \cap A)$ and $N \notin \mathcal{I}$. This implies that $N \cap A \notin \mathcal{I}$.
(6) $\Rightarrow$ (1): Let $B = N \cap A \notin \mathcal{I}$ with $N \in \mathcal{M} - \{\phi\}$ and $N - A \in \mathcal{I}$. Then $B \in \mathcal{B}_r(X, \mathcal{M}, \mathcal{I}) - \mathcal{I}$ since $B \notin \mathcal{I}$ and $(B - N) \cup (N - B) = N - A \in \mathcal{I}$.

Theorem 3.7 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space, where $\mathcal{M} \cap \mathcal{I} = \phi$. Then for $A \subseteq X$, $\Psi_*(A) \subseteq A_*$.*

Proof. Suppose $x \in \Psi_*(A)$ and $x \notin A_*$. Then there exists a nonempty neighborhood $U_x \in \mathcal{M}(x)$ such that $U_x \cap A \in \mathcal{I}$. Since $x \in \Psi_*(A)$, by Theorem 3.2 $x \in \cup\{U \in \mathcal{M} : U - A \in \mathcal{I}\}$ and there exists $V \in \mathcal{M}$ such that $x \in V$ and $V - A \in \mathcal{I}$. Now we have $U_x \cap V \in \mathcal{M}(x)$, $U_x \cap V \cap A \in \mathcal{I}$ and $(U_x \cap V) - A \in \mathcal{I}$ by heredity. Hence by finite additivity we have $(U_x \cap V \cap A) \cup (U_x \cap V - A) = (U_x \cap V) \in \mathcal{I}$. Since $(U_x \cap V) \in \mathcal{M}(x)$, this is contrary to $\mathcal{M} \cap \mathcal{I} = \phi$. Therefore, $x \in A_*$. This implies that $\Psi_*(A) \subseteq A_*$.

Corollary 3.7C *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space, where $\mathcal{M} \cap \mathcal{I} = \phi$. Then for $A \subseteq X$, $\Psi_*(A) \subseteq mCl(A_*)$.*

Theorem 3.8 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. Then the following properties are equivalent:*

1. *$\mathcal{M} \cap \mathcal{I} = \phi$;*
2. *$\Psi_*(\phi) = \phi$;*
3. *If $A \subseteq X$ is m-closed, then $\Psi_*(A) - A = \phi$;*
4. *If $I \in \mathcal{I}$, then $\Psi_*(I) = \phi$.*

Proof. (1) $\Rightarrow$ (2) Since $\mathcal{M} \cap \mathcal{I} = \phi$, by Theorem 3.2 we have $\Psi_*(\phi) = \cup\{U \in \mathcal{M} : U \in \mathcal{I}\} = \phi$.
(2) $\Rightarrow$ (3) Suppose $x \in \Psi_*(A) - A$, then there exists a $U_x \in \mathcal{M}(x)$ such that $x \in U_x - A \in \mathcal{I}$ and $U_x - A \in \mathcal{M}$. But $U_x - A \in \{U \in \mathcal{M} : U \in \mathcal{I}\} = \Psi_*(\phi)$ which implies that $\Psi_*(\phi) \neq \phi$. Hence $\Psi_*(A) - A = \phi$.
(3) $\Rightarrow$ (4) Let $I \in \mathcal{I}$ and since ϕ is m-closed, then $\Psi_*(I) = \Psi_*(I \cup \phi) = \Psi_*(\phi) = \phi$.
(4) $\Rightarrow$ (1) Suppose $A \in \mathcal{M} \cap \mathcal{I}$, then $A \in \mathcal{I}$ and by (4) $\Psi_*(A) = \phi$. Since $A \in \mathcal{M}$, by Corollary 3.1A we have $A \subseteq \Psi_*(A) = \phi$. Hence $\mathcal{M} \cap \mathcal{I} = \phi$.

Definition 3.4 A subset A in an ideal m-space $(X, \mathcal{M}, \mathcal{I})$ is said to be $\mathcal{IM}$-dense if $A_* = X$.

The collection of all $\mathcal{IM}$-dense in $(X, \mathcal{M}, \mathcal{I})$ is denoted by $\mathcal{IM}D(X, \mathcal{M})$. The collection of all dense sets in (X, τ) is denoted by $D(X, \tau)$. Now we show that the collection of dense sets in a topological space $(X, \mathcal{M}_*)$ and the collection of $\mathcal{IM}$-dense sets in the ideal m-space $(X, \mathcal{M}, \mathcal{I})$ are equal if $\mathcal{M} \cap \mathcal{I} = \phi$.

Theorem 3.9 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. If $\mathcal{M} \cap \mathcal{I} = \phi$, then $\mathcal{IM}D(X, \mathcal{M}) = D(X, \mathcal{M}_*)$.*

Proof. Let $D \in \mathcal{IM}D(X,\mathcal{M})$. Then $Cl_*(D) = D \cup D_* = X$, i.e. $D \in D(X,\mathcal{M}_*)$. Therefore, $\mathcal{IM}D(X,\mathcal{M}) \subseteq D(X,\mathcal{M}_*)$.

Conversely, let $D \in D(X,\mathcal{M}_*)$. Then $Cl_*(D) = D \cup D_* = X$. We prove that $D_* = X$. Let $x \in X$ such that $x \notin D_*$. Therefore there exists $\phi \neq U \in \mathcal{M}$ such that $U \cap D \in \mathcal{I}$. Since $U \notin \mathcal{I}$, $U \cap (X - D) \notin \mathcal{I}$ and hence $U \cap (X - D) \neq \phi$. Let $x_0 \in U \cap (X - D)$. Then $x_0 \notin D$ and also $x_0 \notin D_*$. Because $x_0 \in D_*$ implies that $U \cap D \notin \mathcal{I}$ which is contrary to $U \cap D \in \mathcal{I}$. Thus $x_0 \notin D \cup D_* = Cl_*(D) = X$. This is a contradiction. Therefore, we obtain $D \in \mathcal{IM}D(X,\mathcal{M})$. Therefore, $D(X,\mathcal{M}_*) \subseteq \mathcal{IM}D(X,\mathcal{M})$. Hence $\mathcal{IM}D(X,\mathcal{M}) = D(X,\mathcal{M}_*)$.

Theorem 3.10 *Let $(X,\mathcal{M},\mathcal{I})$ be an ideal m-space. Then for $x \in X$, $X - \{x\}$ is $\mathcal{IM}$-dense if and only if $\Psi_*(\{x\}) = \phi$.*

Proof. The proof follows from the definition of $\mathcal{IM}$-dense sets, since $\Psi_*(\{x\}) = X - (X - \{x\})_* = \phi$ if and only if $X = (X - \{x\})_*$.

Proposition 3.5 *Let $(X,\mathcal{M},\mathcal{I})$ be an ideal m-space with $\mathcal{M} \cap \mathcal{I} = \phi$. Then $\Psi_*(A) \neq \phi$ if and only if A contains a nonempty $\mathcal{M}_*$-interior.*

Proof. Let $\Psi_*(A) \neq \phi$. By Theorem 3.2 (1), $\Psi_*(A) = \cup\{U \in \mathcal{M} : U - A \in \mathcal{I}\}$ and there exists a nonempty set $U \in \mathcal{M}$ such that $U - A \in \mathcal{I}$. Let $U - A = P$, where $P \in \mathcal{I}$. Now $U - P \subseteq A$. By Theorem 2.2, $U - P \in \mathcal{M}_*$ and A contains a nonempty $\mathcal{M}_*$-interior.

Conversely, suppose that A contains a nonempty $\mathcal{M}_*$-interior. Hence there exists $U \in \mathcal{M}$ and $P \in \mathcal{I}$ such that $U - P \subseteq A$. So $U - A \subseteq P$. Let $H = U - A \subseteq P$, then $H \in \mathcal{I}$. Hence $\cup\{U \in \mathcal{M} : U - A \in \mathcal{I}\} = \Psi_*(A) \neq \phi$.

4. $\widetilde{\Psi}_*$-Sets

Definition 4.1 Let $(X,\mathcal{M},\mathcal{I})$ be an ideal m-space. A subset A of X is called a $\widetilde{\Psi}_*$-set if $A \subseteq mCl(\Psi_*(A))$.

The collection of all $\widetilde{\Psi}_*$-sets in $(X,\mathcal{M},\mathcal{I})$ is denoted by $\widetilde{\Psi}_*(X,\mathcal{M})$.

Proposition 4.1 *Let $(X,\mathcal{M},\mathcal{I})$ be an ideal m-space. If $A \in \mathcal{M}$, then $A \in \widetilde{\Psi}_*(X,\mathcal{M})$.*

Proof. From Corollary 3.1A it follows that $\mathcal{M} \subseteq \widetilde{\Psi}_*(X,\mathcal{M})$.

Now we give an example which shows that the reverse inclusion is not true.

Remark 4.1 By Example 3.1, the reverse of Proposition 4.1 is not true even though $A \subseteq \Psi_*(A)$.

In the following examples, by $C(\mathcal{M})$ we denote the family of all m-closed sets in $(X, \mathcal{M})$.

Example 4.1 *Let $X = \{a,b,c,d\}$, $\mathcal{M} = \{\phi, X, \{a\}, \{c,d\}\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Then $C(\mathcal{M}) = \{\phi, X, \{b,c,d\}, \{a,b\}\}$. Let $A = \{a,d\}$, then $\Psi_*(A) = X - \{b,c\}_* = X - \{b\} = \{a,c,d\}$. Thus $mCl(\Psi_*(A)) = X$. Therefore, $A \subseteq mCl(\Psi_*(A))$ and hence $A \in \widetilde{\Psi}_*(X,\mathcal{M})$ but $A \notin \mathcal{M}$.*

We give an example which shows that any m-closed set in $(X, \mathcal{M}, \mathcal{I})$ may not be a $\widetilde{\Psi}_*$-set.

Example 4.2 *Let $X = \{a,b,c\}$, $\mathcal{M} = \{\phi, X, \{a\}, \{b\}, \{b,c\}\}$ and $\mathcal{I} = \{\phi, \{a\}\}$. Then $C(\mathcal{M}) = \{\phi, X, \{a,c\}, \{b,c\}, \{a\}\}$. Let $A = \{a,c\}$, then $\Psi_*(A) = X - \{b\}_* = X - \{b,c\} = \{a\}$. Thus $mCl(\Psi_*(A)) = \{a\}$. Therefore, $A \nsubseteq mCl(\Psi_*(A))$ and hence A is not a $\widetilde{\Psi}_*$-set but $A \in C(\mathcal{M})$.*

Proposition 4.2 *Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of nonempty $\widetilde{\Psi}_*$-sets in an ideal m-space $(X, \mathcal{M}, \mathcal{I})$, then $\cup_{\alpha\in\Delta} A_\alpha \in \widetilde{\Psi}_*(X,\mathcal{M})$.*

Proof. For each $\alpha \in \Delta$, $A_\alpha \subseteq mCl(\Psi_*(A_\alpha)) \subseteq mCl(\Psi_*(\cup_{\alpha\in\Delta} A_\alpha))$. This implies that $\cup_{\alpha\in\Delta} A_\alpha \subseteq mCl(\Psi_*(\cup_{\alpha\in\Delta} A_\alpha))$. Thus $\cup_{\alpha\in\Delta} A_\alpha \in \widetilde{\Psi}_*(X,\mathcal{M})$.

The following example shows that the intersection of two $\widetilde{\Psi}_*$-sets in $(X, \mathcal{M}, \mathcal{I})$ may not be a $\widetilde{\Psi}_*$-set.

Example 4.3 *Let $X = \{a,b,c,d\}$, $\mathcal{M} = \{\phi, X, \{a\}, \{b,c\}\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Then $C(\mathcal{M}) = \{\phi, X, \{b,c,d\}, \{a,d\}\}$. Let $A = \{a,d\}$, then $\Psi_*(A) = X - \{c,b\}_* = X - \{b,c,d\} = \{a\}$. Thus $mCl(\Psi_*(A)) = \{a,d\}$ and hence A is a $\widetilde{\Psi}_*$-set. Also let $B = \{b,c,d\}$, then $\Psi_*(B) = X - \{a\}_* = X - \{a,d\} = \{b,c\}$. Thus $mCl(\Psi_*(B)) = \{b,c,d\}$ and hence B is a $\widetilde{\Psi}_*$-set. Now $A \cap B = \{d\}$ and $\Psi_*(\{d\}) = X - \{a,c,b\}_* = X - \{a,b,c,d\} = \phi$. Thus $\{d\} \nsubseteq mCl(\Psi_*(\{d\})) = \phi$ and hence $A \cap B$ is not a $\widetilde{\Psi}_*$-set.*

Theorem 4.1 *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and $A \in \widetilde{\Psi}_*(X,\mathcal{M})$. If $U \in \mathcal{M}$, then $U \cap A \in \widetilde{\Psi}_*(X,\mathcal{M})$.*

Proof. Let $U \in \mathcal{M}$ and $A \in \widetilde{\Psi}_*(X, \mathcal{M})$. By Theorem 3.1 and Corollary 3.1A, we have

$$\begin{aligned} U \cap A &\subseteq U \cap mCl(\Psi_*(A)) \\ &\subseteq mCl(U \cap \Psi_*(A)) \\ &\subseteq mCl(\Psi_*(U) \cap \Psi_*(A)) \\ &= mCl(\Psi_*(U \cap A)). \end{aligned}$$

Hence $U \cap A \in \widetilde{\Psi}_*(X, \mathcal{M})$.

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