Bol. Soc. Paran. Mat. (3s.) **v. 29** 2 (2011): 51–60.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v29i2.12847

On $\lambda-$ summable entire sequences of fuzzy numbers

N. Subramanian, K. Chandrasekhara Rao and K. Balasubramanian

ABSTRACT: In this paper the concept of strongly $(\lambda)_p -$ Cesáro summability of a sequence of fuzzy numbers and strongly $\lambda-$ statistically convergent sequences of fuzzy numbers are introduced.

Key Words: Fuzzy numbers, statistical convergence, entire sequence.

Contents

1 Introduction **51**

2 Definitions and Preliminaries **53**

3 Main Results **54**

1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh[18] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka[10] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Matloka[10] also has shown that every convergent sequence of fuzzy numbers is bounded. Later on sequences of fuzzy numbers have been discussed by Nanda[12], Nuray [14], Kwon[9], Savas[15], Wu and Wang[17], Bilgin[3] Basarir and Mursaleen [2,11],Aytar[1], Fang and Huang[5], and many others. The notion of statistical convergence was introduced by Fast[6] and Schoenberg[16] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic thoery, number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Fridy[7], Kwon[9], Nuray[14], Savas[15] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of Stone-Čech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion

2000 *Mathematics Subject Classification:* 40A05,40C05,40D05.

depends on the density of subsets of the subset N of natural numbers. The natural density of a set A of positive integers is defined by

$$\delta(A) = \lim_n \frac{1}{n} |\{k \leq n : k \in A\}|,$$

where $|\{k \leq n : k \in A\}|$ denotes the number of elements of $A \subseteq N$ not exceeding n [13]. It is clear that any finite subset of N have zero natural density and $\delta(A^c) = 1 - \delta(A)$. If a property $P(k)$ holds for all $k \in A$ with $\delta(A) = 1$, we say that P holds for almost all k, we abbreviate this by "$a.a.k$". A sequence (x_k) is said to be statistically convergent to L if for every $\epsilon > 0, \delta(\{k \in N : |x_k - L| \geq \epsilon\}) = 0$. In this case we write $S - lim x_k = L$. The existing literature on statistical convergence appears to have been restricted to real or complex sequences, but in [9] Kwon, Nuray [14] and Savas[15] extended the idea to apply to sequences of fuzzy numbers.

Let $C(R^n) = \{A \subset R^n : A\text{compact and convex}\}$. The space $C(R^n)$ has linear structure induced by the operations $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \in C(R^n)$ and $\lambda \in R$. The Hausdorff distance between A and B of $C(R^n)$ is defined as

$$\delta_\infty(A, B) = \max\left\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\right\}.$$

It is well known that $(C(R^n), \delta_\infty)$ is a complete metric space. The fuzzy number is a function X from R^n to [0,1] which is normal, fuzzy convex, upper semi-continuous and the closure of $\{x \in R^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the $\alpha-$level set $[X]^2 = \{x \in R^n : X(x) \geq \alpha\}$ is a nonempty compact convex subset of R^n, with support $X^0 = \{x \in R^n : X(x) > 0\}$. Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces the addition $X + Y$ and scalar multiplication $\lambda X, \lambda \in R$, in terms of $\alpha-$level sets, by $|X + Y|^\alpha = |X|^\alpha + |Y|^\alpha, |\lambda X|^\alpha = \lambda |X|^\alpha$ for each $0 \leq \alpha \leq 1$. Define, for each $1 \leq q < \infty$,

$$d_q(X, Y) = \left(\int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q \, d\alpha\right)^{1/q}, and \qquad d_\infty = sup_{0\leq\alpha\leq1}\delta_\infty(X^\alpha, Y^\alpha),$$

where δ_∞ is the Hausdorff metric. Clearly $d_\infty(X, Y) = lim_{q\to\infty} d_q(X, Y)$ with $d_q \leq d_r$, if $q \leq r$ [4].

Throughout the paper, d will denote d_q with $1 \leq q \leq \infty$. Let w be set of all sequences of fuzzy numbers. The generalized de la Vallée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k\in I_n} x_k,$$

where $\lambda = (\lambda_n)$ is a nondecreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1 = 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. A sequence $x(x_k)$ is said to be $(V, \lambda)-$ summable to a number L [8] if $t_n(x) \to L$ as $n \to \infty$. $(V, \lambda)-$

summability reduces to $(C, 1)$ summability when $\lambda_n = n$ for all n. A complex sequence, whose k^{th} term is x_k, is denoted by $\{x_k\}$ or simply x. Let ϕ be the set of all finite sequences. Let ℓ_∞, c, c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. In respect of ℓ_∞, c, c_0 we have $\|x\| = \overset{sup}{k} |x_k|$, where $x = (x_k) \in c_0 \subset c \subset \ell_\infty$. A sequence $x = \{x_k\}$ is said to be analytic if $sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called entire sequence if $lim_{k\to\infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ. Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$, $\delta^{(n)} = (0, 0, \ldots, 1, 0, 0, \ldots)$, 1 in the n^{th} place and zero's elsewhere.

2. Definitions and Preliminaries

Let w denote the set of all fuzzy complex sequences $x = (x_k)_{k=1}^{\infty}$. Consider
$\Gamma = \left\{x \in w : \lim_{k\to\infty} \left(|x_k|^{1/k}\right) = 0\right\}$ and
$\Lambda = \left\{x \in w : \sup_k \left(|x_k|^{1/k}\right) < \infty\right\}$.
Γ and Λ are metric spaces with the metric

$$d(x, y) = \inf\left\{\sup_k \left(|x_k - y_k|^{1/k}\right) \le 1\right\} \tag{1}$$

for all $x = \{x_k\}$ and $y = \{y_k\}$ in Γ.

In the present paper we introduce and examine the concepts of $\lambda-$ statistical convergence and strongly $(\lambda)_p -$ Cesáro convergence of sequences of fuzzy numbers. Firstly in section 2, we give the definition of $\lambda-$ statistical convergence and strongly $(\lambda)_p -$ Cesáro convergence of sequence of fuzzy numbers. In section 3, we establish some inclusion relation between the sequences $s(\lambda)$ and $(\lambda)_p$. We now give the following new definitions which will be needed in the sequel.

Definition 2.1 *Let $X = (X_k)$ be a sequence of fuzzy numbers. A sequence $X = (X_k)$ of fuzzy numbers is said to converge to fuzzy number X_0 if for every $\epsilon > 0$ there is a positive integer N_0 such that $\left(d\left(|X_k|^{1/k}\right), X_0\right) < \epsilon$ for $k \ge N_0$. $X = (X_k)$ is said to be Cauchy sequence if for every $\epsilon > 0$ there is a positive integer N_0 such that $\left(d\left(|X_k|^{1/k}\right), X_\ell\right) < \epsilon$ for $k, \ell \ge N_0$.*

Definition 2.2 *A sequence $X = (X_k)$ of fuzzy numbers is said to be analytic if the set $\left\{|X_k|^{1/k} : k \in N\right\}$ of fuzzy numbers is analytic.*

Definition 2.3 *A sequence $X = (X_k)$ of fuzzy numbers is said to be $\lambda-$ statistically convergent to a fuzzy number X_0 if for every $\epsilon > 0$, we have*

$$\tfrac{1}{n}\left|\left\{k \in I_n : \left(d\left(|X_k|^{1/k}\right), X_0\right) \ge \epsilon\right\}\right| \to 0\, as\, n \to \infty.$$

In this case we shall write $S_\lambda - lim_{k\to\infty}\left(|X_k|^{1/k}\right) = X_0$.
It can be shown that if a sequence $X = (X_k)$ of fuzzy numbers is convergent to a fuzzy number X_0, then it is statistically convergent to the fuzzy number X_0. But the converse does not hold. For example, we define $X = (X_k)$ such that

$$\left(|X_k|^{1/k}\right) = \begin{cases} A & \text{if } k = n^2, n = 1, 2, 3, \cdots \\ 0 & \text{otherwise} \end{cases}$$

where A is a fixed fuzzy number. Then $X = (X_k)$ is statistically convergent but is not convergent.

Definition 2.4 *A sequence $X = (X_k)$ of fuzzy numbers is said to be strongly $\lambda-$summable if there is a fuzzy number X_0 such that $\frac{1}{\lambda_n}\sum_{k\in I_n}\left(d\left(|X_k|^{1/k}\right), X_0\right) \to 0\, as\, n \to \infty$.*

Definition 2.5 *A sequence $X = (X_k)$ of fuzzy numbers is said to be strongly λ_p Cesáro summable if there is a fuzzy number X_0 such that $\frac{1}{\lambda_n}\sum_{k\in I_n}\left(d\left(|X_k|^{1/k}\right), X_0\right)^p \to 0\, as\, n \to \infty$. The set of all strongly λ_p- Cesáro summable sequences of fuzzy numbers is denoted by λ_p.*

Definition 2.6 *A sequence $X = (X_k)$ of fuzzy numbers is said to be $\lambda-$ statistically convergent or S_λ to a fuzzy number X_0 if for every $\epsilon > 0$, we have*

$$\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left(d\left(|X_k|^{1/k}\right), X_0\right) \geq \epsilon\right\}\right| \to 0\, as\, n \to \infty.$$

In this case we shall write $S_\lambda - lim\left(|X_k|^{1/k}\right) = X_0$. In the special case $(\lambda)_n = n$ for all $n \in N$, then $\lambda-$ statistically convergent is same as statistically convergent.

3. Main Results

Theorem 3.1 *(i)If a sequence $X = (X_k)$ is strongly λ_p- Cesáro summable to X_0, then it is $\lambda-$ statistically convergent to X_0.*
(ii)If $X = (X_k)$ is a sequence $\lambda-$ analytic and $\lambda-$ statistically convergent to X_0, then it is strongly $(\lambda)_p-$ Cesáro summable to X_0, and hence X is strongly $\lambda-$ Cesáro summable to X_0.

Proof: (i) Let $\epsilon > 0$ and $X \in (\lambda)_p$. We have
$\sum_{k\in I_n}\left(d\left(|X_k|^{1/k}\right), X_0\right) \geq \sum_{k\in I_n\,, d(X_k, X_0)\geq\epsilon}\left(d\left(|X_k|^{1/k}\right), X_0\right)^p$
$\geq \left|\left\{k \in I_n : \left(d\left(|X_k|^{1/k}\right), X_0\right) \geq \epsilon\right\}\right|\epsilon^p.$

Therefore (X_k) is $\lambda-$ statistically convergent to X_0.

(ii)Suppose that $X = (X_k)$ is analytic and $\lambda-$ statistically convergent to X_0.

Since $X \in \Lambda$, there exists a constant $M > 0$ such that $\left(d\left(|X_k|^{1/k}\right), X_0\right) \leq M$ for all k. Let $\epsilon > 0$ be given and choose N_ϵ such that

$\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left(d\left(|X_k|^{1/k}\right), X_0\right) \geq \left(\frac{\epsilon}{2}\right)^{1/p}\right\}\right| \leq \frac{\epsilon}{2M^p}$ for all $n > N_\epsilon$, and

set $L_n = \left|\left\{k \in I_n : \left(d\left(|X_k|^{1/k}\right), X_0\right) \geq \left(\frac{\epsilon}{2}\right)^{1/p}\right\}\right|$. Now for all $n > N_\epsilon$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(|X_k|^{1/k}\right), X_0\right]^p = \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(|X_k|^{1/k}\right), X_0\right]^p +$$
$$\frac{1}{\lambda_n}\sum_{k\notin I_n}\left[d\left(|X_k|^{1/k}\right), X_0\right]^p \leq \frac{1}{\lambda_n}\left(\frac{\lambda_n\epsilon}{2M^p}\right)M^p + \frac{1}{\lambda_n}\left(\frac{\lambda_n\epsilon}{2}\right) = \epsilon.$$

Hence $\left(|X_k|^{1/k}\right) \to X_0\,(\lambda)_p$. Further we have,

$$\frac{1}{n}\sum_{k=1}^{n}\left[d\left(|X_k|^{1/k}\right), X_0\right] = \frac{1}{n}\sum_{k=1}^{n-\lambda_n}\left[d\left(|X_k|^{1/k}\right), X_0\right] +$$
$$\frac{1}{n}\sum_{k\in I_n}\left[d\left(|X_k|^{1/k}\right), X_0\right]$$
$$\leq \frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n}\left[d\left(|X_k|^{1/k}\right), X_0\right] + \leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(|X_k|^{1/k}\right), X_0\right]$$
$$\leq \frac{2}{\lambda_n}\sum_{k\in I_n}\left[d\left(|X_k|^{1/k}\right), X_0\right].$$

Hence X is strongly Cesáro summable to X_0, since X is strongly λ Cesáro summable to X_0. This completes the proof. $\square$

Theorem 3.2 *Let (X_k) and (Y_k) be sequence of fuzzy numbers.*
(i)If $S_\lambda - lim\left(|X_k|^{1/k}\right) = X_0$ and $c \in R$, then $S_\lambda - lim\left(c\,|X_k|^{1/k}\right) = cX_0$; (ii)If $S_\lambda - lim\left(|X_k|^{1/k}\right) = X_0$ and $S_\lambda - lim\left(|Y_k|^{1/k}\right) = Y_0$, then $S_\lambda - lim\left(|X_k|^{1/k} + \left(|Y_k|^{1/k}\right)\right) = X_0 + Y_0$.

Proof: (i) Let $\alpha \in [0,1]$ and $c \in R$. Let $\left(|X_k^\alpha|^{1/k}\right), \left(|Y_k^\alpha|^{1/k}\right), X_0^\alpha$ and Y_0^α be $\alpha-$ level sets of $\left(|X_k|^{1/k}\right), \left(|Y_k|^{1/k}\right)$, X_0 and Y_0 respectively. Since $\delta_\infty\left(c\,|X_k|^{1/k}, X_0^\alpha\right) = |c|\,\delta_\infty\left(|X_k^\alpha|^{1/k}, X_0^\alpha\right)$, we have $d\left(c\,|X_k|^{1/k}, cX_0\right) = |c|\,d\left(|X_k|^{1/k}, X_0\right)$. For given $\epsilon > 0$ we have $\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left(d\left(c\,|X_k|^{1/k}, X_0\right)\right) \geq \epsilon\right\}\right|$
$\leq \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left(d\left(|X_k|^{1/k}, X_0\right)\right) \geq \frac{\epsilon}{|c|}\right\}\right|$. Hence $S_\lambda - lim\left(|X_k|^{1/k}\right) = cX_0$.
(ii) Suppose that $S_\lambda - lim\left(|X_k|^{1/k}\right) = X_0$ and $S_\lambda - lim\left(|Y_k|^{1/k}\right) = Y_0$. Firstly we have, $\delta_\infty\left(|X_k^\alpha|^{1/k} + |Y_k^\alpha|^{1/k}, X_0^\alpha + Y_0^\alpha\right) \leq$
$\delta_\infty\left(|X_k^\alpha|^{1/k} + |Y_k^\alpha|^{1/k}, \left(|Y_k^\alpha|^{1/k}\right) + X_0^\alpha\right) + \delta_\infty\left(|Y_k^\alpha|^{1/k} + X_0^\alpha, X_0^\alpha + Y_0^\alpha\right) =$
$\delta_\infty\left(|X_k^\alpha|^{1/k}, X_0^\alpha\right) + \delta_\infty\left(|Y_k^\alpha|^{1/k}, Y_0^\alpha\right)$.
By Minkowski's inequality we get
$d\left(|X_k^\alpha|^{1/k} + |Y_k|^{1/k}, X_0 + Y_0\right) \leq d\left(|X_k^\alpha|^{1/k}, X_0\right) + d\left(|Y_k^\alpha|^{1/k}, Y_0\right)$. Therefore

given $\epsilon > 0$ we have
$\frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(|X_k|^{1/k} + |Y_k|^{1/k}, X_0 + Y_0\right) \geq \epsilon\right\}\right| \leq$
$\frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(|X_k|^{1/k}, X_0\right) + d\left(|Y_k|^{1/k}, Y_0\right) \geq \epsilon\right\}\right| \leq$
$\frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(|X_k|^{1/k}, X_0\right) \geq \frac{\epsilon}{2}\right\}\right| +$
$\frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(|Y_k|^{1/k}, Y_0\right) \geq \frac{\epsilon}{2}\right\}\right|$.
Hence $S_\lambda - lim\left(|X_k|^{1/k} + |Y_k|^{1/k}\right) = X_0 + Y_0$. This completes the proof. □

Theorem 3.3 *If a sequence $X = (X_k)$ is statistically convergent to X_0 and $lim\, inf_{(n)}\left(\frac{\lambda_n}{n}\right) > 0$, then it is $\lambda-$ statistically convergent to X_0.*

Proof: For given $\epsilon > 0$, we have $\left|\left\{k \in n : d\left(|X_k|^{1/k}, X_0\right) \geq \epsilon\right\}\right|$
$\supset \left|\left\{k \in I_n : d\left(|X_k|^{1/k}, X_0\right) \geq \epsilon\right\}\right|$. Therefore
$\frac{1}{n}\left|\left\{k \leq n : d\left(|X_k|^{1/k}, X_0\right) \geq \epsilon\right\}\right| \geq \frac{1}{n}\left|\left\{k \in I_n : d\left(|X_k|^{1/k}, X_0\right) \geq \epsilon\right\}\right|$
$\geq \frac{\lambda_n}{n}\frac{1}{(\lambda)_n}\left|\left\{k \in I_n : d\left(|X_k|^{1/k}, X_0\right) \geq \epsilon\right\}\right|$.
Taking lim as $n \to \infty$ and using $lim\ \inf_{(n)}\left(\frac{\lambda_n}{n}\right) > 0$, we get $X = (X_k)$ is $\lambda-$ statistically convergent to X_0. This completes the proof. □

Definition 3.1 *Let $p = (p_k)$ be any sequence of positive real numbers. Then we define $(\lambda)_p = \left\{X = \{X_k\} : \frac{1}{\lambda_n}\sum_{k \in I_n}\left[d\left(|X_k|^{1/k}, X_0\right)\right]^{p_k} \to 0\, as\, n \to \infty\right\}$. Suppose that p_k is a constant for all k, then $(\lambda)_p = \lambda$.*

Theorem 3.4 *Let $0 \leq p_k \leq q_k$ and let $\left\{\frac{q_k}{p_k}\right\}$ be bounded. Then $(\lambda)_p \subset (\lambda)_q$.*

Proof: Let
$$X \in (\lambda)_q \tag{2}$$
$\frac{1}{\lambda_n}\sum_{k \in I_n}\left[d\left(|X_k|^{1/k}, X_0\right)\right]^{q_k} \to 0\, as\, n \to \infty.$

Let $t_k = \frac{1}{\lambda_n}\sum_{k \in I_n}\left[d\left(|X_k|^{1/k}, X_0\right)\right]^{p_k}$ and $\lambda_k = \frac{p_k}{q_k}$ Since $p_k \leq q_k$, we have $0 \leq \lambda_k \leq 1$.
Take $0 < \lambda < \lambda_k$. Define $u_k = t_k\,(t_k \geq 1)\,; u_k = 0\,(t_k < 1)$ and $v_k = 0\,(t_k \geq 1)\,;$ $v_k = t_k\,(t_k < 1)\,. t_k = u_k + v_k.$ (i.e)$t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that
$$u_k^{\lambda_k} \leq u_k \leq t_k \quad and \quad v_k^{\lambda_k} \leq v_k^{\lambda}. \tag{3}$$

Since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, $t_k^{\lambda_k} \leq t_k + v_k^{\lambda}$.

$\frac{1}{\lambda_n}\sum_{k \in I_n}\left[d\left(|X_k|^{1/k}, X_0\right)^{q_k}\right]^{\lambda_k} \leq \frac{1}{\lambda_n}\sum_{k \in I_n}\left[d\left(|X_k|^{1/k}, X_0\right)\right]^{q_k}$

$\Rightarrow \frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right)^{q_k} \right]^{p_k/q_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{q_k}$
$\Rightarrow \frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{p_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{q_k}$.
But $\frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{q_k} \to 0 \, as \, n \to \infty$.
Hence $\frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{p_k} \to 0 \, as \, n \to \infty$. Hence

$$X \in (\lambda)_p \tag{4}$$

From (2) and (4) we get $(\lambda)_q \subset (\lambda)_p$. This completes the proof. □

Theorem 3.5 *(a)Let* $0 < inf p_k \leq p_k \leq 1$. *Then* $(\lambda)_p \subset \lambda$ *(b) Let* $1 \leq p_k \leq \sup p_k < \infty$. *Then* $\lambda \subset (\lambda)_p$.

Proof: (a)Let

$$X \in (\lambda)_p \tag{5}$$

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{p_k} \to 0 \, as \, n \to \infty. \tag{6}$$

Since $0 < inf \, p_k \leq p_k \leq 1$

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right] \leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{p_k} \tag{7}$$

From (6) and (7) it follows that

$$X \in \lambda \tag{8}$$

Thus

$$(\lambda)_p \subset \lambda. \tag{9}$$

This completes the proof.
Proof:(b)Let $p_k \geq 1$ for each k and $sup \, p_k < \infty$. Let $X \in \lambda$

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right] \to 0 \, as \, n \to \infty \tag{10}$$

Since $1 \leq p_k \leq sup \, p_k < \infty$ we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{p_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right] \tag{11}$$

$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{p_k} \to 0 \, as \, n \to \infty$ (by using 10)

Therefore $X \in (\lambda)_p$. This completes the proof. □

Theorem 3.6 *Let $0 < p_k \leq q_k < \infty$ for each k. Then $(\lambda)_p \subseteq (\lambda)_q$.*

Proof: Let

$$X \in (\lambda)_p . \tag{12}$$

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{p_k} \to 0 \, as \, n \to \infty. \tag{13}$$

This implies that $\frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right] \leq 1$, for sufficiently large n, we get

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{q_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{p_k} . \tag{14}$$

$\Rightarrow \frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(|X_k|^{1/k}, X_0 \right) \right]^{q_k} \to 0 \, as \, n \to \infty$ (by using 13). Hence

$$X \in (\lambda)_q . \tag{15}$$

From (12) and (15) we get $(\lambda)_p \subseteq (\lambda)_q$. This completes the proof. □

References

1. S.Aytar, Statistical limit points of sequences of fuzzy numbers, *Inform. Sci.*, **165**(2004), 129-138.

2. M.Basarir and M.Mursaleen, Some sequence spaces of fuzzy numbers generated by infinite matrices, *J. Fuzzy Math.*, **11(3)**(2003), 757-764.

3. T.Bilgin, $\Delta-$statistical and strong $\Delta-$Cesáro convergence of sequences of fuzzy numbers,*Math. Commun.*, **8**(2003), 95-100.

4. P.Diamond and P.Kloeden, Metric spaces of fuzzy sets, *Fuzzy sets Syst.*, **35**(1990), 241-249.

5. Jin-xuan Fang and Huan Huang, On the level convergence of a sequence of fuzzy numbers, *Fuzzy Sets Syst.*, **147**(2004), 417-435.

6. H.Fast, Surla convergence statistique, *Colloq. Math.*, **1951**, 241-244.

7. J.Fridy, On the statistical convergence, *Analysis*, **5**(1985), 301-313.

8. L.Leindler, *Ü*ber die Vallee-Pousinsche Summierbarkeit Allgemeiner Orthogonalreihen, *Acta Math Acad. Sci. Hungar.*, **16**(1965), 375-387.

9. J.S.Kwon, On Statistical and P-Cesàro convergence of fuzzy numbers, *Korean J. Compu. Appl. math.* , **7(1)**(2000), 195-203.

10. M.Matloka, Sequences of fuzzy numbers, *Busefal*, **28**(1986), 28-37.

11. M.Mursaleen and M.Basarir, On some new sequence spaces of fuzzy numbers, *Indian J. Pure Appl. Math.*, **34(9)** (2003), 1351-1357.

12. S.Nanda, On sequences of fuzzy numbers, *Fuzzy Sets Syst.* , **33**(1989), 123-126.

13. I.Niven and H.S.Zuckerman, An Introduction to the Theory of Numbers, *fourth ed.,John Wiley and Sons, New York*, **1980**.

14. F.Nuray, Lacunary statistical convergence of sequences of fuzzy numbers, *Fuzzy Sets Syst.*, **99**(1998), 353-356.

15. E.Savas, On strongly $\lambda-$ summable sequences of fuzzy numbers, *Inform. Sci.*, **125**(2000), 181-186.

16. I.J.Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly*, **66**(1959), 361-375.

17. Congxin Wu and Guixiang Wang, Convergence of sequences of fuzzy numbers and fixed point theorems for increasing fuzzy mappings and application, *Fuzzy Sets Syst.*, **130**(2002), 383-390.

18. L.A.Zadeh, Fuzzy sets, *Inform Control*, **8**(1965), 338-353.

19. H.Kizmaz, On certain sequence spaces, *Canad Math. Bull.* , **24(2)**(1981), 169-176.

20. M.Et and R.Colak, On some generalized difference sequence spaces, *Soochow J. Math.*, **21(4)**(1995), 377-386.

21. M.Et, On some topological properties of generalized difference sequence spaces, *Int. J. Math. Math. Sci.*, **24(11)**(2000), 785-791.

22. M.Et and F.Nuray, Δ^m- Statistical Convergence, *Indian J. Pure Appl. Math.*, **32(6)** (2001), 961-969.

23. R.Colak, M.Et and E.Malkowsky, Some topics of sequence spaces, *Lecture Notes in Mathematics, Firat University Press, Elazig, Turkey*, **2004**.

24. M.Isik, On statistical convergence of generalized difference sequences, *Soochow J. Math.*, **30(2)**(2004), 197-205.

25. Y.Altin and M.Et, Generalized difference sequence spaces defined by a modulus function in a locally convex space, *Soochow J. Math.* , **31(2)**(2005), 233-243.

26. W.Orlicz, *Ü*ber Raume $\left(L^M\right)$ *Bull. Int. Acad. Polon. Sci. A*, **(**1936), 93-107.

27. J.Lindenstrauss and L.Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.

28. S.D.Parashar and B.Choudhary, Sequence spaces defined by Orlicz functions, *Indian J. Pure Appl. Math.* , **25(4)**(1994), 419-428.

29. M.Mursaleen,M.A.Khan and Qamaruddin, Difference sequence spaces defined by Orlicz functions, *Demonstratio Math.* , **Vol. XXXII** (1999), 145-150.

30. C.Bektas and Y.Altin, The sequence space $\ell_M\,(p,q,s)$ on seminormed spaces, *Indian J. Pure Appl. Math.*, **34(4)** (2003), 529-534.

31. B.C.Tripathy,M.Et and Y.Altin, Generalized difference sequence spaces defined by Orlicz function in a locally convex space, *J. Analysis and Applications*, **1(3)**(2003), 175-192.

32. K.Chandrasekhara Rao and N.Subramanian, The Orlicz space of entire sequences, *Int. J. Math. Math. Sci.*, **68**(2004), 3755-3764.

33. M.A.Krasnoselskii and Y.B.Rutickii, Convex functions and Orlicz spaces, *Gorningen, Netherlands*, **1961**.

34. Nakano, Concave modulars, *J. Math. Soc. Japan*, **5**(1953), 29-49.

35. W.H.Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.*, **25**(1973), 973-978.

36. I.J.Maddox, Sequence spaces defined by a modulus, *Math. Proc. Cambridge Philos. Soc*, **100(1)** (1986), 161-166.

37. H.I.Brown, The summability field of a perfect $\ell-\ell$ method of summation, *J. Anal. Math.*, **20**(1967), 281-287.

38. A.Wilansky, Summability through Functional Analysis, *North-Holland Mathematical Studies, North-Holland Publishing, Amsterdam*, **Vol.85**(1984).

39. P.K.Kamthan and M.Gupta, Sequence spaces and Series, Lecture Notes in Pure and Applied Mathematics, *Marcel Dekker Inc. New York*, **65**(1981).

40. P.K.Kamthan, Bases in a certain class of Frechet space, *Tamkang J. Math.*, **1976**, 41-49.

41. C.Goffman and G.Pedrick, First Course in Functional Analysis, *Prentice-Hall India, New Delhi*, **1974**.

N.Subramanian
Department of Mathematics, SASTRA University,
Thanjavur-613 401, India.

and

K. Chandrasekhara Rao
Department of Mathematics,Srinivasa Ramanujan Centre, SASTRA University
Kumbakonon, 612-001, India

and

K.Balasubramanian,
Department of Mathematics,
SASTRA University, Thanjavur-613 401.
India.
E-mail address: nsmaths@yahoo.com
E-mail address: kchandrasekhara@rediffmail.com
E-mail address: k_bala27@yahoo.co.in