Bol. Soc. Paran. Mat. (3s.) **v. 29** 2 (2011): 61–68.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v29i2.12848

Preopen sets in ideal bitopological spaces

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abstract: The aim of this paper is to introduce and characterize the concepts of preopen sets and their related notions in ideal bitopological spaces.

Key Words: Ideal bitopological spaces, (i, j)-pre-$\mathcal{I}$-open sets, (i, j)-pre-$\mathcal{I}$-closed sets.

Contents

1. Introduction and Preliminaries

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [4] and Vaidyanathasamy [5]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^*$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [5] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^*(.)$ for a topology $\tau^*(\tau, \mathcal{I})$ called the $*$-topology, finer than τ is defined by $\mathrm{Cl}^*(A) = A \cup A^*(\tau, \mathcal{I})$ when there is no chance of confusion, $A^*(\mathcal{I})$ is denoted by A^* and τ_i-$\mathrm{Int}^*(A)$ denotes the interior of A in $\tau_i^*(\mathcal{I})$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau_1, \tau_2, \mathcal{I})$ is called an ideal bitopological space. A subset A of an ideal bitopological space is said to be (i, j)-$\mathcal{I}$-open [2] if $A \subset \tau_i$-$\mathrm{Int}(A_j^*)$, where $A_j^* = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau_j(x)\}$. Let A be a subset of a bitopological space (X, τ_1, τ_2). We denote the closure of A and the interior of A with respect to τ_i by τ_i-$\mathrm{Cl}(A)$ and τ_i-$\mathrm{Int}(A)$, respectively. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-preopen [3] if $A \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A))$, where $i, j = 1, 2$ and $i \neq j$. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be pre-$\mathcal{I}$-open [1] if $S \subset \mathrm{Int}(\mathrm{Cl}^*(S))$. The family of all pre-$\mathcal{I}$-open sets of $(X, \tau, \mathcal{I})$ is denoted by $P\mathcal{I}O(X, \tau)$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-precontinuous [3] if the inverse image of every σ_j-open set in (Y, σ_1, σ_2) is (i, j)-preopen in $(X, \tau_1, \tau_2, \mathcal{I})$, where $i \neq j$, i, j=1, 2.

2000 *Mathematics Subject Classification:* 54D10

2. (i,j)-pre-$\mathcal{I}$-open sets

Definition 2.1 *A subset A of an ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ is said to be (i,j)-pre-$\mathcal{I}$-open if and only if $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(A))$, where $i,j = 1,2$ and $i \neq j$. The family of all (i,j)-pre-$\mathcal{I}$-open sets of $(X,\tau_1,\tau_2,\mathcal{I})$ is denoted by $P\mathcal{I}O(X,\tau_1,\tau_2)$ or (i,j)-$P\mathcal{I}O(X)$. Also, The family of all (i,j)-pre-$\mathcal{I}$-open sets of $(X,\tau_1,\tau_2,\mathcal{I})$ containing x is denoted by (i,j)-$P\mathcal{I}O(X,x)$.*

Remark 2.2 *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on (X,τ_1,τ_2). If $\mathcal{I} \subset \mathcal{J}$, then $P\mathcal{J}O(X,\tau_1,\tau_2)$ $\subset P\mathcal{I}O(X,\tau_1,\tau_2)$.*

Remark 2.3 *It is clear that every τ_i-open sets is (i,j)-pre-$\mathcal{I}$-open but the converse is not true in general as it can be seen from the following example.*

Example 2.4 *Let $X = \{a,b,c\}$, $\tau_1 = \{\varnothing, \{a\}, X\}$, $\tau_2 = \{\varnothing, \{b\}, X\}$ and $\mathcal{I}$ $= \{\varnothing, \{a\}\}$. Then $\{b\}$ is $(1,2)$-pre-$\mathcal{I}$-open but not τ_1-open and therefore not τ_1-preopen.*

Remark 2.5 *It is clear that $P\mathcal{I}O(X,\tau_1,\tau_2) \neq P\mathcal{I}O(X,\tau_1) \cup P\mathcal{I}O(X,\tau_2)$.*

Example 2.6 *Let $(X,\tau_1,\tau_2,\mathcal{I})$ be as in Example 2.4. Then $P\mathcal{I}O(X,\tau_1) = \{\varnothing,$ $\{a\}, \{a,b\}, \{a,c\}, X\}$, $P\mathcal{I}O(X,\tau_2) = \{\varnothing, \{b\}, \{a,b\}, \{b,c\}, X\}$. But $P\mathcal{I}O(X,\tau_1,\tau_2)$ $= \{\varnothing, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X\}$.*

Proposition 2.7 *Every (i,j)-$\mathcal{I}$-open set is (i,j)-pre-$\mathcal{I}$-open.*

Proof: Let A be an (i,j)-$\mathcal{I}$-open set. Then $A \subset \tau_i\text{-Int}(A_j^*) \subset \tau_i\text{-Int}(A \cup A_j^*) = \tau_i$-Int$(\tau_j\text{-Cl}^*(A))$. Therefore, $A \in (i,j)$-$P\mathcal{I}O(X)$. □

Example 2.8 *Let $X = \{a,b,c\}$, $\tau_1 = \{\varnothing, \{a\}, X\}$, $\tau_2 = \{\varnothing, \{b\}, X\}$ and $\mathcal{I} =$ $\{\varnothing, \{a\}\}$. Then $\{a\}$ is $(1,2)$-pre-$\mathcal{I}$-open but not $(1,2)$-$\mathcal{I}$-open.*

Proposition 2.9 *Every (i,j)-pre-$\mathcal{I}$-open set is (i,j)-preopen.*

Proof: Let $(X,\tau_1,\tau_2,\mathcal{I})$ be an ideal bitopological space and let $A \in (i,j)$-P$\mathcal{I}O(X)$. Then $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(A)) = \tau_i\text{-Int}(A \cup A_j^*) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}(A) \cup A) = \tau_i\text{-Int}(\tau_j$-Cl$(A))$. This shows that $A \in$(i, j)$-PO(X)$. □

The following example shows that the converse of Proposition 2.9 is not true in general.

Example 2.10 *Let $X = \{a,b,c\}$, $\tau_1 = \{\varnothing, \{b\}, X\}$, $\tau_2 = \{\varnothing, \{a\}, \{b,c\}, X\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$. Then $\{a\}$ is $(1,2)$-preopen but not $(1,2)$-pre-$\mathcal{I}$-open.*

Proposition 2.11 *For an ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ and $A \subset X$, we have:*

(i) If $\mathcal{I} = \emptyset$, then A is (i,j)-pre-$\mathcal{I}$-open if and only if A is (i,j)-preopen.

(ii) If $\mathcal{I} = \mathcal{P}(X)$, then A is (i,j)-pre-$\mathcal{I}$-open if and only if A is τ_i-open.

Proof: The proof follows from the fact that

(i) if $\mathcal{I} = \emptyset$, then $A^* = \mathrm{Cl}(A)$.

(ii) if $\mathcal{I} = P(X)$, then $A^* = \emptyset$ for every subset A of X.

□

Proposition 2.12 *Let A be a subset of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ and A be an (i,j)-pre-$\mathcal{I}$-open set. Then we have the following:*

1. τ_j-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(A))) = \tau_j$-$\mathrm{Cl}(A)$.
2. τ_j-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(A))) = \tau_j$-$\mathrm{Cl}^*(A)$.

Proof: The proof is obvious. □

Remark 2.13 *The intersection of two (i,j)-pre-$\mathcal{I}$-open sets need not be (i,j)-pre-$\mathcal{I}$-open as it can be seen from the following example.*

Example 2.14 *Let $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, \{a, c\}, X\}$, $\tau_2 = \{\varnothing, \{b, c\}, X\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$. Then the sets $\{a, b\}$ and $\{a, c\}$ are $(1,2)$-pre-$\mathcal{I}$-open sets of $(X, \tau_1, \tau_2, \mathcal{I})$ but their intersection $\{a\}$ is not an $(1,2)$-pre-$\mathcal{I}$-open set of $(X, \tau_1, \tau_2, \mathcal{I})$.*

Theorem 2.15 *If $\{A_\alpha\}_{\alpha\in\Omega}$ is a family of (i,j)-pre-$\mathcal{I}$-open sets in $(X, \tau_1, \tau_2, \mathcal{I})$, then $\underset{\alpha\in\Omega}{\cup} A_\alpha$ is (i,j)-pre-$\mathcal{I}$-open in $(X, \tau_1, \tau_2, \mathcal{I})$.*

Proof: Since $\{A_\alpha : \alpha \in \Omega\} \subset (i,j)$-$\mathrm{P}\mathcal{I}O(X)$, then $A_\alpha \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(A_\alpha))$ for every $\alpha \in \Omega$. Thus, $\underset{\alpha\in\Omega}{\cup} A_\alpha \subset \underset{\alpha\in\Omega}{\cup} \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(A_\alpha)) \subset \tau_i$-$\mathrm{Int}(\underset{\alpha\in\Omega}{\cup} \tau_j$-$\mathrm{Cl}^*(A_\alpha)) = \tau_i$-$\mathrm{Int}(\underset{\alpha\in\Omega}{\cup} (A_\alpha)^*_j \cup A_\alpha) = \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\underset{\alpha\in\Omega}{\cup} A_\alpha))$. Therefore, we obtain $\underset{\alpha\in\Omega}{\cup} A_\alpha \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\underset{\alpha\in\Omega}{\cup} A_\alpha))$. Hence any union of (i,j)-pre-$\mathcal{I}$-open sets is (i,j)-pre-$\mathcal{I}$-open. □

Theorem 2.16 *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space. If $U \in \tau_1 \cap \tau_2$ and $V \in P\mathcal{I}O(X, \tau_1, \tau_2,)$, then $U \cap V \in PO(X, \tau_1, \tau_2, \mathcal{I})$.*

Proof: By definition, we have $U \cap V \subset U \cap \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(V)) \subset U \cap (\tau_i$-$\mathrm{Int}(V \cup V^*_j)) = \tau_i$-$\mathrm{Int}((U \cap V) \cup (U \cap V^*_j)) \subset \tau_i$-$\mathrm{Int}(U \cap V) \cup (U \cap V)^*_j) = \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(U \cap V))$. Therefore, $U \cap W \in (i,j) - P\mathcal{I}O(X)$. □

Lemma 2.17 *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space with $A \subset B \subset X$, then $A^*_i(\mathcal{I}_{|B}, \tau_{|B}) = A^*_i(\mathcal{I}, \tau_i) \cap B$ for $i = 1, 2$.*

Theorem 2.18 *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space. If $U \in \tau_1 \cap \tau_2$ and $W \in (i, j)$-$PO(X)$, then $U \cap W \in (i, j)$-$PO(U, \tau_{1|U}, \tau_{2|U}, \mathcal{I}_{|U})$.*

Proof: Since $U \in \tau_1 \cap \tau_2$, we have τ_i-$\mathrm{Int}_U(A) = \tau_i$-$\mathrm{Int}(A)$ where $i = 1, 2$ for any subset A of U. By using this fact and Lemma 2.17, the result follows immediately. □

Definition 2.19 *In an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$, $A \subset X$ is said to be (i, j)-pre-$\mathcal{I}$-closed if $X \backslash A$ is (i, j)-pre-$\mathcal{I}$-open in X, $i, j = 1, 2$ and $i \neq j$.*

Theorem 2.20 *If A is an (i, j)-pre-$\mathcal{I}$-closed set in an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ if and only if τ_i-$\mathrm{Cl}(\tau_j$-$\mathrm{Int}^*(A)) \subset A$.*

Proof: The proof follows from the definitions. □

Theorem 2.21 *If a subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is (i, j)-pre-$\mathcal{I}$-closed, then τ_i-$\mathrm{Cl}^*(\tau_j$-$\mathrm{Int}(A)) \subset A$.*

Proof: Since A is (i, j)-pre-$\mathcal{I}$-closed, $X \backslash A$ is (i, j)-pre-$\mathcal{I}$-open in $(X, \tau_1, \tau_2, \mathcal{I})$. Then $X \backslash A \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(X \backslash A)) \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(X \backslash A)) = X \backslash (\tau_i$-$\mathrm{Cl}(\tau_j$-$\mathrm{Int}(A)) \subset X \backslash (\tau_i$-$\mathrm{Cl}^*(\tau_j$-$\mathrm{Int}(A)))$. Therefore, we obtain τ_i-$\mathrm{Cl}^*(\tau_j$-$\mathrm{Int}(A)) \subset A$. □

Theorem 2.22 *Arbitrary intersection of (i, j)-pre-$\mathcal{I}$-closed sets is always (i, j)-pre-$\mathcal{I}$-closed.*

Proof: Follows from Theorems 2.15 and 2.21. □

Definition 2.23 *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space, S a subset of X and x be a point of X. Then*

- *(i) x is called an (i, j)-pre-$\mathcal{I}$-interior point of S if there exists $V \in (i, j)$-$P\mathcal{I}O(X)$ such that $x \in V \subset S$.*
- *ii) the set of all (i, j)-pre-$\mathcal{I}$-interior points of S is called (i, j)-pre-$\mathcal{I}$-interior of S and is denoted by (i, j)-$p\mathcal{I}\,\mathrm{Int}(S)$.*

Theorem 2.24 *Let A and B be subsets of $(X, \tau_1, \tau_2, \mathcal{I})$. Then the following properties hold:*

- *(i) (i, j)-$p\mathcal{I}\,\mathrm{Int}(A) = \cup\{T : T \subset A$ and $T \in (i, j)$-$P\mathcal{I}O(X)\}$.*
- *(ii) (i, j)-$p\mathcal{I}\,\mathrm{Int}(A)$ is the largest (i, j)-pre-$\mathcal{I}$-open subset of X contained in A.*
- *(iii) A is (i, j)-pre-$\mathcal{I}$-open if and only if $A = (i, j)$-$p\mathcal{I}\,\mathrm{Int}(A)$.*
- *(iv) (i, j)-$p\mathcal{I}\,\mathrm{Int}((i, j)$-$p\mathcal{I}\,\mathrm{Int}(A)) = (i, j)$-$p\mathcal{I}\,\mathrm{Int}(A)$.*

(v) *If* $A \subset B$*, then* (i,j)*-*$p\mathcal{I}\,\mathrm{Int}(A) \subset (i,j)$*-*$p\mathcal{I}\,\mathrm{Int}(B)$*.*

(vi) (i,j)-$p\mathcal{I}\,\mathrm{Int}(A) \cup (i,j)$-$p\mathcal{I}\,\mathrm{Int}(B) \subset (i,j)$-$p\mathcal{I}\,\mathrm{Int}(A \cup B)$.

(vii) (i,j)-$p\mathcal{I}\,\mathrm{Int}(A \cap B) \subset (i,j)$-$p\mathcal{I}\,\mathrm{Int}(A) \cap (i,j)$-$p\mathcal{I}\,\mathrm{Int}(B)$.

Proof: (i). Let $x \in \cup\{T : T \subset A$ and $T \in (i,j)$-$P\mathcal{I}O(X)\}$. Then, there exists $T \in (i,j)$-$P\mathcal{I}O(X,x)$ such that $x \in T \subset A$ and hence $x \in (i,j)$-$p\mathcal{I}\,\mathrm{Int}(A)$. This shows that $\cup\{T : T \subset A$ and $T \in (i,j)$-$P\mathcal{I}O(X)\} \subset (i,j)$-$p\mathcal{I}\,\mathrm{Int}(A)$. For the reverse inclusion, let $x \in (i,j)$-$p\mathcal{I}\,\mathrm{Int}(A)$. Then there exists $T \in (i,j)$-$P\mathcal{I}O(X,x)$ such that $x \in T \subset A$. we obtain $x \in \cup\{T : T \subset A$ and $T \in (i,j)$-$P\mathcal{I}O(X)\}$. This shows that (i,j)-$p\mathcal{I}\,\mathrm{Int}(A) \subset \cup\{T : T \subset A$ and $T \in (i,j)$-$P\mathcal{I}O(X)\}$. Therefore, we obtain (i,j)-$p\mathcal{I}\,\mathrm{Int}(A) = \cup\{T : T \subset A$ and $T \in (i,j)$-$P\mathcal{I}O(X)\}$.
The proof of (ii)-(v) are obvious and the proofs of (vi) and (vii) are obvious from (v). □

Definition 2.25 *Let* $(X, \tau_1, \tau_2, \mathcal{I})$ *be an ideal bitopological space,* S *a subset of* X *and* x *be a point of* X*. Then*

(i) *x is called an (i,j)-pre-$\mathcal{I}$-cluster point of S if $V \cap S \neq \emptyset$ for every $V \in (i,j)$-$P\mathcal{I}O(X,x)$.*

(ii) *the set of all (i,j)-pre-$\mathcal{I}$-cluster points of S is called (i,j)-pre-$\mathcal{I}$-closure of S and is denoted by (i,j)-$p\mathcal{I}\,\mathrm{Cl}(S)$.*

Theorem 2.26 *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space and $A \subset X$. A point $x \in (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in (i,j)$-$P\mathcal{I}O(X,x)$.*

Proof: The proof follows from Definition 2.25. □

Theorem 2.27 *Let A and B be subsets of $(X, \tau_1, \tau_2, \mathcal{I})$. Then the following properties hold:*

(i) (i,j)-$p\mathcal{I}\,\mathrm{Cl}(A) = \cap\{F : A \subset F$ *and* $F \in (i,j)$-$P\mathcal{I}C(X)\}$.

(ii) *(i,j)-$p\mathcal{I}\,\mathrm{Cl}(A)$ is the smallest (i,j)-pre-$\mathcal{I}$-closed subset of X containing A.*

(iii) *A is (i,j)-pre-$\mathcal{I}$-closed if and only if $A = (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A)$.*

(iv) (i,j)-$p\mathcal{I}\,\mathrm{Cl}((i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A) = (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A)$.

(v) *If $A \subset B$, then (i,j)-$p\mathcal{I}\,\mathrm{Cl}(A) \subset (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(B)$.*

(vi) (i,j)-$p\mathcal{I}\,\mathrm{Cl}(A) \cup (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(B) \subset (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A \cup B)$.

(vii) (i,j)-$p\mathcal{I}\,\mathrm{Cl}(A \cap B) \subset (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A) \cap (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(B)$.

Proof: (i). Suppose that $x \notin (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A)$. Then there exists $V \in (i,j)$-$P\mathcal{I}O(X,x)$ such that $V \cap A = \emptyset$. Since $X\backslash V$ is (i,j)-pre-$\mathcal{I}$-closed set containing A and $x \notin X\backslash V$, we obtain $x \notin \cap\{F : A \subset F$ and $F \in (i,j)$-$P\mathcal{I}C(X)\}$. Conversely, suppose there exists $F \in (i,j)$-$P\mathcal{I}C(X)$ such that $A \subset F$ and $x \notin F$. Since $X\backslash F$ is (i,j)-pre-$\mathcal{I}$-open set containing x, we obtain $(X\backslash F) \cap A = \emptyset$. This shows that $x \notin (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A)$. Therefore, we obtain (i,j)-$p\mathcal{I}\,\mathrm{Cl}(A) = \cap\{F : A \subset F$ and $F \in (i,j)$-$P\mathcal{I}C(X)\}$.
The other proofs are obvious. □

Theorem 2.28 *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space and $A \subset X$. Then the following propeties hold:*

(i) (i,j)-$p\mathcal{I}\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A)$;

(i) (i,j)-$p\mathcal{I}\,\mathrm{Cl}(X\backslash A) = X\backslash(i,j)$-$p\mathcal{I}\,\mathrm{Int}(A)$.

Proof: (i). Let $x \notin (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A)$. There exists $V \in (i,j)$-$P\mathcal{I}O(X,x)$ such that $V \cap A = \emptyset$; hence we obtain $x \in (i,j)$-$p\mathcal{I}\,\mathrm{Int}(X\backslash A)$. This shows that $X\backslash(i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A) \subset (i,j)$-$p\mathcal{I}\,\mathrm{Int}(X\backslash A)$. Let $x \in (i,j)$-$p\mathcal{I}\,\mathrm{Int}(X\backslash A)$. Since (i,j)-$p\mathcal{I}\,\mathrm{Int}(X\backslash A) \cap A = \emptyset$, we obtain $x \notin (i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A)$; hence $x \in X\backslash(i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A)$. Therefore, we obtain (i,j)-$p\mathcal{I}\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)$-$p\mathcal{I}\,\mathrm{Cl}(A)$.
(ii). Follows from (i). □

Definition 2.29 *A subset B_x of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be an (i,j)-pre-$\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists an (i,j)-pre-$\mathcal{I}$-open set U such that $x \in U \subset B_x$.*

Theorem 2.30 *A subset of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is (i,j)-pre-$\mathcal{I}$-open if and only if it is an (i,j)-pre-$\mathcal{I}$-neighbourhood of each of its points.*

Proof: Let G be an (i,j)-pre-$\mathcal{I}$-open set of X. Then by definition, it is clear that G is an (i,j)-pre-$\mathcal{I}$-neighbourhood of each of its points, since for every $x \in G$, $x \in G \subset G$ and G is (i,j)-pre-$\mathcal{I}$-open. Conversely, suppose G is an (i,j)-pre-$\mathcal{I}$-neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in (i,j)$-$P\mathcal{I}O(X)$ such that $S_x \subset G$. Then $G = \bigcup\{S_x : x \in G\}$. Since each S_x is (i,j)-pre-$\mathcal{I}$-open and an arbitrary union of (i,j)-pre-$\mathcal{I}$-open sets is (i,j)-pre-$\mathcal{I}$-open, G is (i,j)-pre-$\mathcal{I}$-open in $(X, \tau_1, \tau_2, \mathcal{I})$. □

3. pairwise pre-$\mathcal{I}$-continuous functions

Definition 3.1 *A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-pre-$\mathcal{I}$-continuous (resp. (i,j)-$\mathcal{I}$-continuous [2]) if the inverse image of every σ_i-open set in (Y, σ_1, σ_2) is (i,j)-pre-$\mathcal{I}$-open (resp. (i,j)-$\mathcal{I}$-open)in $(X, \tau_1, \tau_2, \mathcal{I})$, where $i \neq j$, i,j=1, 2.*

Proposition 3.2 *(i) Every (i,j)-$\mathcal{I}$-continuous function is (i,j)-pre-$\mathcal{I}$-continuous.*

(ii) Every (i, j)-pre-$\mathcal{I}$-continuous function is (i, j)-precontinuous

Proof: The proof follows from Propositions 2.7 and 2.9. □

However, the converse may be false.

Example 3.3 *Let $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, \{a\}, X\}$, $\tau_2 = \{\varnothing, \{a\}, X\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is $(1, 2)$-pre-$\mathcal{I}$-continuous. but not $(1, 2)$-$\mathcal{I}$-continuous.*

Example 3.4 *Let $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, \{b\}, X\}$, $\tau_2 = \{\varnothing, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is $(1, 2)$-precontinuous but not $(1, 2)$-pre-$\mathcal{I}$-continuous.*

Theorem 3.5 *For a function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:*

(i) f is pairwise pre-$\mathcal{I}$-continuous;

(ii) For each point x in X and each σ_i-open set F in Y such that $f(x) \in F$, there is a (i, j)-pre-$\mathcal{I}$-open set A in X such that $x \in A$, $f(A) \subset F$;

(iii) The inverse image of each σ_i-closed set in Y is (i, j)-pre-$\mathcal{I}$-closed in X;

(iv) For each subset A of X, $f((i, j)\text{-}p\mathcal{I}\,\mathrm{Cl}(A)) \subset \sigma_i\text{-}\mathrm{Cl}(f(A))$;

(v) For each subset B of Y, $(i, j)\text{-}p\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(B))$;

(vi) For each subset C of Y, $f^{-1}(\sigma_i\text{-}\mathrm{Int}(C)) \subset (i, j)\text{-}p\mathcal{I}\,\mathrm{Int}(f^{-1}(C))$.

Proof: (i)⇒(ii): Let $x \in X$ and F be a σ_i-open set of Y containing $f(x)$. By (i), $f^{-1}(F)$ is (i, j)-pre-$\mathcal{I}$-open in X. Let $A = f^{-1}(F)$. Then $x \in A$ and $f(A) \subset F$.
(ii)⇒(i): Let F be σ_i-open in Y and let $x \in f^{-1}(F)$. Then $f(x) \in F$. By (ii), there is an (i, j)-pre-$\mathcal{I}$-open set U_x in X such that $x \in U_x$ and $f(U_x) \subset F$. Then $x \in U_x \subset f^{-1}(F)$. Hence $f^{-1}(F)$ is (i, j)-pre-$\mathcal{I}$-open in X.
(i)⇔(iii): This follows due to the fact that for any subset B of Y, $f^{-1}(Y \backslash B) = X \backslash f^{-1}(B)$.
(iii)⇒(iv): Let A be a subset of X. Since $A \subset f^{-1}(f(A))$ we have $A \subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(f(A)))$. Now, $\sigma_i\text{-}\mathrm{Cl}(f(A))$ is σ_i-closed in Y and hence $(i, j)\text{-}p\mathcal{I}\,\mathrm{Cl}(A) \subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(f(A)))$, for $(i, j)\text{-}p\mathcal{I}\,\mathrm{Cl}(A)$ is the smallest (i, j)-pre-$\mathcal{I}$-closed set containing A. Then $f((i, j)\text{-}p\mathcal{I}\,\mathrm{Cl}(A)) \subset \sigma_i\text{-}\mathrm{Cl}(f(A))$.
(iv)⇒(v): Let B be any subset of Y. Now, $f((i, j)\text{-}p\mathcal{I}\,\mathrm{Cl}(f^{-1}(B))) \subset \sigma_i\text{-}\mathrm{Cl}(f(f^{-1}(B))) \subset \sigma_i\text{-}\mathrm{Cl}(B)$. Consequently, $(i, j)\text{-}p\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(B))$.
(v)⇒(iii): Let B be any σ_i-closed subset of Y. Then $(i, j)\text{-}p\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(B)) = f^{-1}(B)$; hence $f^{-1}(B)$ is (i, j)-pre-$\mathcal{I}$-closed in X.
(i)⇒(vi): Let B be a σ_i-open set in Y. Clearly, i$f^{-1}(\sigma_i\text{-}\mathrm{Int}(B))$ is (i, j)-pre-$\mathcal{I}$-open and we have $f^{-1}(\sigma_i\text{-}\mathrm{Int}(B)) \subset (i, j)\text{-p}\mathcal{I}\,\mathrm{Int}(f^{-1}(\sigma_i\text{-}\mathrm{Int}(B))) \subset (i, j)\text{-}p\mathcal{I}\,\mathrm{Int}(f^{-1}(B)$.

(vi)⇒(i): Let B be a σ_i-open set in Y. Then σ_i-Int$(B) = B$ and $f^{-1}(B) \subset f^{-1}(\sigma_i\text{-Int}(B)) \subset (i,j)\text{-}p\mathcal{I}\,\text{Int}(f^{-1}(B))$. Hence we have $f^{-1}(B) = (i,j)\text{-}p\mathcal{I}\,\text{Int}(f^{-1}(B))$. This shows that $f^{-1}(B)$ is (i,j)-pre-$\mathcal{I}$-open in X. □

Theorem 3.6 *If $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is pairwise pre-$\mathcal{I}$-continuous and $A \in \tau_1 \cap \tau_2$, then $f|A : (A, \tau_{1|A}, \tau_{2|A}, \mathcal{I}_{|A}) \to (Y, \sigma_1, \sigma_2)$ is pairwise pre-$\mathcal{I}_{|A}$-continuous.*

Proof: The proof follows from Theorem 2.18. □

Acknowledgement The authors thank the referee for his/her valuable comments and suggestions.

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