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Solidity and some double sequence spaces

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ABSTRACT: In this paper we investigate the solidity (normality) of the sequence spaces $c^2_A, \ell^2_A, \Lambda^2_A$ and Γ^2_A.

Key Words: entire sequence, analytic sequence, double sequence.

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1. Introduction

Throughout w, Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in N$ the set of positive integers. Then w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich[2]. Later on it was investigated by Hardy[3], Moricz[4], Moricz and Rhoades[5], Basarir and Solankan[1], Tripathy[6], Colak and Turkmenoglu[7], Turkmenoglu[8], and many others.
We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$is called convergent if and only if the double sequence.$(s_{mn})$is called convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...)$ (see[9]). A sequence $x = (x_{mn})$is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by Γ^2. Let $\phi = \{all finite sequences\}$. Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$

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section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m, n \in N$,

$$\delta_{mn} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0, & 0, & ...1, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero other wise. An FK-space(or a metric space)X is said to have AK property if (δ_{mn}) is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$. An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in N)$ are also continuous. If X is a sequence space, we give the following definitions:
(i)$X^{'}$= the continuous dual of X;
(ii)$X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \, for \, each \, x \in X\right\}$
(iii)$X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \, is \, convegent, \, foreach \, x \in X\right\}$
(iv)$X^{\gamma} = \left\{a = (a_{mn}) : \overset{sup}{m,n} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, foreachx \in X\right\}$;
(v)$let \, X \, beanFK - space \supset \phi; \, then \, X^{f} = \left\{f(\delta_{mn}) : f \in X^{'}\right\}$;
(vi)$X^{\Lambda} = \left\{a = (a_{mn}) : \overset{sup}{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \, foreach \, x \in X\right\}$;
$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha - (or Köthe - Toeplitz)$dual of $X, \beta - (or \, generalized - Köthe - Toeplitz) \, dual \, of X, \gamma - \, dual \, of \, X, \, \Lambda - dual \, of X \, respectively.$

2. Definitions and Preliminaries

Let w^2 denote the set of all complex double sequences. A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all prime sense double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called prime sense double entire sequence if $|x_{mn}|^{1/m+n} \to 0 \, as \, m, n \to \infty$. The double entire sequences will be denoted by Γ^2. The space $\Lambda^2 \, and \, \Gamma^2$ is a metric space with the metric

$$d(x, y) = sup_{mn} \left\{|x_{mn} - y_{mn}|^{1/m+n} : m, n : 1, 2, 3, ...\right\} \tag{2}$$

$forall \, x = \{x_{mn}\} \, and \, y = \{y_{mn}\} in \, \Gamma^2$.

c^2= the space of all double convergent sequences.

ℓ^2=the space of all sequences $x = \{x_{mn}\}$ such that $\sum_{m,n=1}^{\infty} |x_{mn}|$ converges.

Let $A = \left(a_{mn}^{jk}\right) (m, n, j, k = 1, 2, 3, \cdots)$ be an infinite matrix. Given a sequence $x = \{x_{mn}\}$ we write formally

$$y_{mn} = A_{mn}(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}^{jk} x_{mn} \, (j, k = 1, 2, \cdots)$$

The sequence $\{y_{mn}\} = \{A_{mn}(x)\}$ will be denoted by Ax or y. Let X be a sequence space and let X_A be the set of all those sequences $x = \{x_{mn}\}$ for which $Ax \in X$.

The set of all matrices transforming X into X will be denoted by (X, X). We recall the following :
A sequence space X is called solid (or normal) if and only if $\Lambda^2 X \subset X$.
Any matrix in (c^2, c^2) is called a conservative matrix. A conservative matrix which preserves the limit is said to be a Toeplitz matrix.

3. Main Results

Proposition 3.1 *If A is a conservative matrix, which fails to sum an analytic sequence, then c_A^2 is not solid.*

Proof: The constant sequence

$$e = \begin{pmatrix} 1, & 1, & ...1, & 1, & 0, ... \\ 1, & 1, & ...1, & 1, & 0, ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 1, & 1, & ...1, & 1, & 0, ... \\ 0, & 0, & ...0, & 0, & 0, ... \end{pmatrix}$$

is in c_A^2
with 1 in the upto $(m, n)^{th}$ position and zero other wise.
By our hypothesis, there exists a analytic sequence b such that $b \notin c_A^2$. That is $b \cdot e \notin c_A^2$. Therefore, $\Lambda^2 \cdot c_A^2 \not\subset c_A^2$. Showing that c_A^2 is not solid. This completes the proof. □

Corollary 3.2 *If A is a Toeplitz matrix, then c_A^2 is not solid.*

Proposition 3.3 *If $A \in (\ell^2, \ell^2)$, then ℓ_A^2 is in general not solid.*

Proof: Let

$$A = \begin{pmatrix} 1, & 1, & 0, & 0, & 0, ...0, & 0, & 0, ... \\ 0, & 0, & 1, & 1, & 0, ...0, & 0, & 0, ... \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0, & 0, & ...1, & 1, & 0, & 0, & 0, ... \\ 0, & 0, & ...0, & 0, & 0, & 0, & 0, ... \end{pmatrix}$$

with 1 in the $(m, n)^{th}$ position and 1 in the $(m+1, n+1)^{th}$ position and zero other wise.

That

$a_{m,2n-1}a_{m,2n} - a_{3m-1,n}a_{2m,2n} = 1, (m, n = 1, 2, 3, \cdots)$

$a_{m,2n}a_{m,2n} - a_{3m,n}a_{2m,2n} = 1, (m, n = 1, 2, 3, \cdots)$

$a^{jk}_{mn} = 0$, Otherwise.

Then $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|a^{jk}_{mn}\right| = 1$ for each fixed j, k

Showing that $A \in \left(\ell^2, \ell^2\right)$ We note that

$x \in \ell^2_A$ if and only if $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|x_{m,2n-1}x_{m,2n} - x_{3m-1,n}x_{2m,2n}\right|$ converges. Take

$$x = \begin{pmatrix} 1, & -1, & ...1, & -1, & 0, ... \\ 1, & -1, & ...1, & -1, & 0, ... \\ . \\ . \\ . \\ 1, & -1, & ...1, & -1, & 0, ... \\ 0, & 0, & ...0, & 0, & 0, ... \end{pmatrix}$$

so that $x \in \ell^2_A$.
with 1 in the $(m, n)^{th}$ position and -1 in the upto $(m+1, n+1)^{th}$, zero other wise. Take

$$b = x = \begin{pmatrix} 1, & -1, & ...1, & -1, & 0, ... \\ 1, & -1, & ...1, & -1, & 0, ... \\ . \\ . \\ . \\ 1, & -1, & ...1, & -1, & 0, ... \\ 0, & 0, & ...0, & 0, & 0, ... \end{pmatrix}.$$

Then b is in Λ^2.
with 1 and -1 alternatively up to $(m, n)^{th}$ position and zero other wise. Now

$$y = bx \begin{pmatrix} 1, & 1, & ...1, & 1, & 0, ... \\ 1, & 1, & ...1, & 1, & 0, ... \\ . \\ . \\ . \\ 1, & 1, & ...1, & 1, & 0, ... \\ 0, & 0, & ...0, & 0, & 0, ... \end{pmatrix} = e$$

with 1 upto $(m, n)^{th}$ position and zero other wise.

For e, we have $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|y_{m,2n-1}y_{m,2n}+y_{3m-1,n}y_{2m,2n}|=2+2+\cdots$

which is a divergent series. Thus $\Lambda^2\cdot\ell_A^2\subset\ell_A^2$. Hence ℓ_A^2 is not solid. This completes the proof. □

Proposition 3.4 *If* $A\in\left(\Lambda^2,\Lambda^2\right)$, *then* Λ_A^2 *is in general, not solid.*

Proof: Let

$$A=\begin{pmatrix}-1, & 1, & 0, & 0, & 0,...0, & 0, & 0,...\\ 0, & 0, & -1, & 1, & 0,...0, & 0, & 0,...\\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0, & 0, & ...-1, & 1, & 0, & 0, & 0,...\\ 0, & 0, & ...0, & 0, & 0, & 0, & 0,...\end{pmatrix}$$

with -1 in the $(m,n)^{th}$ position and 1 in the $(m+1,n+1)^{th}$ position and zero other wise.
Inother words, $A=\left(a_{mn}^{jk}\right)$ is defined by

$a_{m,2n-1}a_{m,2n}-a_{3m-1,n}a_{2m,2n}=1,(m,n=1,2,3,\cdots)$

$a_{m,2n}a_{m,2n}-a_{3m,n}a_{2m,2n}=1,(m,n=1,2,3,\cdots)$

$a_{mn}^{jk}=0$, Otherwise.

Then $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|a_{mn}^{jk}\right|^{1/m+n}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|(a_{m,2n-1}a_{m,2n}-a_{3m-1,n}a_{2m,2n})|+$

$|(a_{m,2n}a_{m,2n}-a_{3m,n}a_{2m,2n})|=2$ for each fixed j,k.

Consequently, $A\in\left(\Lambda^2,\Lambda^2\right)$
Note that $x\in\Lambda_A^2$ if and only if

$$Ax=\begin{pmatrix}-a_{11}+a_{12}, & -a_{13}+a_{14}, & ...\\ -a_{21}+a_{22}, & -a_{23}+a_{24}, & ...\\ . & & \\ . & & \\ . & & \end{pmatrix}\in\Lambda^2$$

we take

$$A = \begin{pmatrix} 1, & 2, & 3, & 4, & ... \\ 1, & 2, & 3, & 4, & ... \\ . \\ . \\ . \end{pmatrix} sothat$$

$$Ax = \begin{pmatrix} 1, & 1, & ...1, & 1, & 0, ... \\ 1, & 1, & ...1, & 1, & 0, ... \\ . \\ . \\ . \\ 1, & 1, & ...1, & 1, & 0, ... \\ 0, & 0, & ...0, & 0, & 0, ... \end{pmatrix} \text{ and } x \in \Lambda^2_A.$$

$$b = \begin{pmatrix} -1, & 1, & ... -1, & 1, & 0, ... \\ -1, & 1, & ... -1, & 1, & 0, ... \\ . \\ . \\ . \\ -1, & 1, & ... -1, & 1, & 0, ... \\ 0, & 0, & ...0, & 0, & 0, ... \end{pmatrix}$$

Then

$$bx = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . \\ . \\ . \\ 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix}$$

and

$$A(bx) = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . \\ . \\ . \\ 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix} \notin \Lambda^2$$

Thus $\Lambda^2 \cdot \Lambda^2_A \not\subset \Lambda^2_A$. Hence Λ^2_A is not a solid space. This completes the proof.□

Proposition 3.5 *If* $A \in \left(\Gamma^2, \Gamma^2\right)$, *the* Γ^2_A *is not necessarily solid.*

Proof: Let

$$A=\begin{pmatrix} -1, & 1, & 0, & 0, & 0,...0, & 0, & 0,... \\ 0, & 0, & -1, & 1, & 0,...0, & 0, & 0,... \\ . \\ . \\ . \\ 0, & 0, & ...-1, & 1, & 0, & 0, & 0,... \\ 0, & 0, & ...0, & 0, & 0, & 0, & 0,... \end{pmatrix}$$

writing $\{t_{mn}\}$ for the transform of $\{x_{mn}\}$, so that

$$t_{mn}=-\left(x_{m,2n-1}x_{m,2n}-x_{3m-1,n}x_{2m,2n}\right)+\left(x_{m,2n}x_{m,2n}-x_{3m,n}x_{2m,2n}\right),(m,n=1,2,3,\cdots)$$

We can verify directly that

$$|x_{mn}|^{1/m+n}\to 0\Rightarrow |t_{mn}|^{1/m+n}\to 0\,(m,n\to\infty)$$

For if $\eta<1$, then $|a+b|^{\eta}<|a|^{\eta}+|b|^{\eta}$ so that

$$|t_{mn}|^{1/m+n}<|x_{m,2n-1}x_{m,2n}-x_{3m-1,n}x_{2m,2n}|^{1/m+n}+|x_{m,2n}x_{m,2n}-x_{3m,n}x_{2m,2n}|^{1/m+n} \tag{3}$$

Since $|x_{mn}|^{1/m+n}\to 0\,(m,n\to\infty)$, we have $|x_{mn}|<1$ for sufficiently large m,n. Supposing that m,n is large enough for

$$|x_{m,2n-1}x_{m,2n}-x_{3m-1,n}x_{2m,2n}|<1, |x_{m,2n}x_{m,2n}-x_{3m,n}x_{2m,2n}|<1.$$

Hence if $|x_{mn}|^{1/m+n}\to 0$, then $|t_{mn}|^{1/m+n}\to 0\,(m,n\to\infty)$.

It is now trivial that

$$\begin{pmatrix} 1, & 1, & ...1, & 1, & 0,... \\ 1, & 1, & ...1, & 1, & 0,... \\ . \\ . \\ . \\ 1, & 1, & ...1, & 1, & 0,... \\ 0, & 0, & ...0, & 0, & 0,... \end{pmatrix}$$

belongs to Γ^2_A but

$$\begin{pmatrix} -1, & 1, & ...-1, & 1, & 0,... \\ -1, & 1, & ...-1, & 1, & 0,... \\ . & & & & \\ . & & & & \\ . & & & & \\ -1, & 1, & ...-1, & 1, & 0,... \\ 0, & 0, & ...0, & 0, & 0,... \end{pmatrix}$$

does not.

Here we have

$$b = \begin{pmatrix} -1, & 1, & ...-1, & 1, & 0,... \\ -1, & 1, & ...-1, & 1, & 0,... \\ . & & & & \\ . & & & & \\ . & & & & \\ -1, & 1, & ...-1, & 1, & 0,... \\ 0, & 0, & ...0, & 0, & 0,... \end{pmatrix}$$

in Λ^2

So Γ^2_A is not solid. This completes the proof. □

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