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The generalized difference gai sequences of fuzzy numbers defined by Orlicz functions

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ABSTRACT: In this paper we introduce the classes of gai sequences of fuzzy numbers using generalized difference operator Δ^m(m fixed positive integer) and the Orlicz functions. We study its different properties and also we obtain some inclusion results of these classes.

Key Words: Fuzzy numbers, difference sequence, Orlicz space, entire sequence, analytic sequence, gai sequence, complete

Contents

1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [18] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming.

In this paper we introduce and examine the concepts of Orlicz space of entire sequence of fuzzy numbers generated by infinite matrices.

Let $C\left(R^n\right)=\{A\subset R^n : A\,compact\,and\,convex\}$. The space $C\left(R^n\right)$ has linear structure induced by the operations $A+B=\{a+b : a\in A, b\in B\}$ and $\lambda A=\{\lambda a : a\in A\}$ for $A,B\in C\left(R^n\right)$ and $\lambda\in R$. The Hausdorff distance between A and B of $C\left(R^n\right)$ is defined as

$$\delta_\infty\left(A,B\right)=max\left\{sup_{a\in A}inf_{b\in B}\left\|a-b\right\|,sup_{b\in B}inf_{a\in A}\left\|a-b\right\|\right\}$$

It is well known that $\left(C\left(R^n\right),\delta_\infty\right)$ is a complete metric space.
The fuzzy number is a function X from R^n to [0,1] which is normal, fuzzy convex, upper semi-continuous and the closure of $\{x\in R^n : X(x)>0\}$ is compact. These properties imply that for each $0<\alpha\leq 1$, the $\alpha-$level set $[X]^\alpha=\{x\in R^n : X(x)\geq\alpha\}$ is a nonempty compact convex subset of R^n, with support $X^c=\{x\in R^n : X(x)>0\}$. Let $L\left(R^n\right)$ denote the set of all fuzzy numbers. The linear structure of $L\left(R^n\right)$ induces the addition $X+Y$ and scalar multiplication $\lambda X, \lambda\in R$, in terms of $\alpha-$ level

2000 *Mathematics Subject Classification:* 40A05,40C05,40D05 sets, by $[X+Y]^{\alpha} = [X]^{\alpha} + [Y]^{\alpha}, [\lambda X]^{\alpha} = \lambda [X]^{\alpha}$ for each $0 \leq \alpha \leq 1$. The absolute value $|X|$ of $X \in L(R^n)$ is defined by (see for instance Kaleva and Seikkala [42] $|X|(t) = \{ \begin{matrix} \max(X(t), X(-t)), & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{matrix}$ (1)Define, for each $1 \leq q < \infty$,

$$d_q(X,Y) = \left(\int_0^1 \delta_{\infty}(X^{\alpha}, Y^{\alpha})^q \, d\alpha \right)^{1/q}, \, and \, d_{\infty} = sup_{0 \leq \alpha \leq 1} \delta_{\infty}(X^{\alpha}, Y^{\alpha}),$$

where δ_{∞} is the Hausdorff metric. Clearly $d_{\infty}(X,Y) = lim_{q \to \infty} d_q(X,Y)$ with $d_q \leq d_r$, if $q \leq r$ [29].

The additive identity in $L(R^n)$ is denoted by $\bar{0}$. For simplicity in notation, we shall write throughout d instead of d_q with $1 \leq q \leq \infty$.

A metric on $L(R^n)$ is said to be translation invariant if $d(X+Z, Y+Z) = d(X,Y)$ for all $X, Y, Z \in L(R^n)$

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers into $L(R^n)$. The fuzzy number X_k denotes the value of the function at $k \in \mathbb{N}$. We denotes by $W(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers.

A complex sequence, whose k^{th} terms is x_k is denoted by $\{x_k\}$ or simply x. Let ϕ be the set of all finite sequences. Let ℓ_{∞}, c, c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. In respect of ℓ_{∞}, c, c_0 we have $\|x\| = \begin{matrix} sup \\ k \end{matrix} |x_k|$, where $x = (x_k) \in c_0 \subset c \subset \ell_{\infty}$. A sequence $x = \{x_k\}$ is said to be analytic if $sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called entire sequence if $lim_{k \to \infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ. A sequence x is called gai sequence if $lim_{k \to \infty} (k! |x_k|)^{1/k} = 0$. The vector space of all gai sequences will be denoted by χ. Orlicz [26] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [27] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (1 \leq p < \infty)$. Subsequently different classes of sequence spaces defined by Parashar and Choudhary [28], Mursaleen et al. [29], Bektas and Altin [30], Tripathy et al. [31], Rao and subramanian [32] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref. [33].
Recall([26],[33]) an Orlicz function is a function $M : [0, \infty) \to [o, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called modulus function, introduced by Nakano [34] and further discussed by Ruckle [35] and Maddox [36] and many others.
An Orlicz function M is said to satisfy Δ_2- condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \leq KM(u)(u \geq 0)$.Lindenstrauss and

Tzafriri [27] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \; for \; some \, \rho > 0 \right\}. \tag{1.1}$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\} \tag{1.2}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p, 1 \leq p < \infty$, the space ℓ_M coincide with the classical sequence space ℓ_p. Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, ..., x_n, 0, 0, ...\}$ $\delta^{(n)} = (0, 0, ..., 1, 0, 0, ...)$, 1 in the n^{th} place and zero's else where.

Remark 1.1 *An Orlicz function M satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$. Let $m \in \mathbb{N}$ be fixed, then the generalized difference operation*

$$\Delta^m : W(F) \rightarrow W(F)$$

is defined by

$$\Delta X_k = X_k - X_{k+1} \; and \; \Delta^m X_k = \Delta\left(\Delta^{m-1} X_k\right) (m \geq 2) \; for \; all \; k \in \mathbb{N}$$

2. Definitions and Prelimiaries

Let P_s denotes the class of subsets of $\mathbb{N}$, the natural numbers, those do not contain more than s elements. Throughout (ϕ_n) represents a non-decreasing sequence of real numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$.
The sequence $\chi(\phi)$ for real numbers is defined as follows:

$$\chi(\phi) = \left\{ (X_k) : \tfrac{1}{\phi_s} (k! \, |X_k|)^{1/k} \rightarrow 0 \, as \, k, s \rightarrow \infty \;\; for \;\; k \in \sigma \in P_s \right\}$$

The generalized sequence space $\chi(\Delta_n, \phi)$ of the sequence space $\chi(\phi)$ for real numbers is defined as follows

$$\chi(\Delta_n, \phi) = \left\{ (X_k) : \tfrac{1}{\phi_s} (k! \, |\Delta X_k|)^{1/k} \rightarrow 0 \, as \;\; k, s \rightarrow \infty \;\; for \;\; k \in \sigma \in P_s \right\}$$

where $\Delta_n X_k = X_k - X_{k+n}$ for $k \in \mathbb{N}$ and fixed $n \in \mathbb{N}$

In this article we introduce the following classes of sequences of fuzzy numbers:
Let M be an Orlicz function, then

$$\Lambda_M^F(\Delta^m) = \left\{ (X_k) \in W(F) : sup_k M\left(\frac{d\left(\left(|\Delta^m X_k|^{1/k}\right), \bar{0}\right)}{\rho}\right) < \infty \, for \, some \, \rho > 0 \right\}$$

$$\chi_M^F(\Delta^m) = \left\{ (X_k) \in W(F) : M\left(\frac{d\left((k! |\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \rightarrow 0 \, as \, k \rightarrow \infty, \right.$$

for some $\rho > 0\}$

$\Gamma_M^F(\Delta^m) = \left\{(X_k) \in W(F) : M\left(\frac{d\left(\left(|\Delta^m X_k|^{1/k}\right),\bar{0}\right)}{\rho}\right) \to 0\, as\, k \to \infty,\ for\, some\, \rho > 0\right\}$

$\chi_M^F(\Delta^m,\phi) = \left\{(X_k) \in W(F) : \frac{1}{\phi_s}M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho}\right) \to 0\, as\, k,s \to \infty,\right.$
for k$\in \sigma \in P_s\}$

$\Gamma_M^F(\Delta^m,\phi) = \left\{(X_k) \in W(F) : \frac{1}{\phi_s}M\left(\frac{d\left((|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho}\right) \to 0\, as\, k,s \to \infty,\right.$
for k$\in \sigma \in P_s\}$

3. Main Results

In this section we prove some results involving the classes of sequences of fuzzy numbers $\chi_M^F(\Delta^m,\phi), \chi_M^F(\Delta^m)$ and $\Lambda_M^F(\Delta^m)$.

Theorem 3.1 *If d is a translation invariant metric, then $\chi_M^F(\Delta^m,\phi)$ are closed under the operations of addition and scalar multiplication*

Proof: Since d is a translation invariant metric implies that

$$d\left(\left(k!\left(\Delta^m X_k + \Delta^m Y_k\right)\right)^{1/k},\bar{0}\right) \le d\left(\left(k!\left(\Delta^m X_k\right)\right)^{1/k},\bar{0}\right) + d\left(\left(k!\left(\Delta^m Y_k\right)\right)^{1/k},\bar{0}\right) \tag{3.1}$$

and

$$d\left(\left(k!\left(\Delta^m \lambda X_k\right)\right)^{1/k},\bar{0}\right) \le |\lambda|^{1/k} d\left(\left(k!\left(\Delta^m X_k\right)\right)^{1/k},\bar{0}\right) \tag{3.2}$$

where λ is a scalar and $|\lambda|^{1/k} > 1$. Let $X = (X_k)$ and $Y = (Y_k) \in \chi_M^F(\Delta^m,\phi)$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\frac{1}{\phi_s}M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho_1}\right) \to 0\, as\, k,s \to \infty,\ for\, k \in \sigma \in P_s$$
$$\frac{1}{\phi_s}M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho_2}\right) \to 0\, as\, k,s \to \infty,\ for\, k \in \sigma \in P_s$$

Let $\rho_3 = max(2\rho_1, 2\rho_2)$. By the equation (3.1) and since M is non-decreasing convex function, we have

$M\left(\frac{d\left((k!|\Delta^m X_k + \Delta^m Y_k|)^{1/k},\bar{0}\right)}{\rho}\right) \le M\left(\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho_3}\right) + \left(\frac{d\left((k!|\Delta^m Y_k|)^{1/k},\bar{0}\right)}{\rho_3}\right)\right) \le$
$\frac{1}{2}M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho_1}\right) + \frac{1}{2}M\left(\frac{d\left((k!|\Delta^m Y_k|)^{1/k},\bar{0}\right)}{\rho_2}\right)$

$\Rightarrow \frac{1}{\phi_s}M\left(\frac{d\left((k!|\Delta^m X_k + \Delta^m Y_k|)^{1/k},\bar{0}\right)}{\rho_3}\right) \le$
$\frac{1}{\phi_s}M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho_1}\right) + \frac{1}{\phi_s}M\left(\frac{d\left((k!|\Delta^m Y_k|)^{1/k},\bar{0}\right)}{\rho_2}\right)$ for $k \in \sigma \in P_s$

Hence $X + Y \in \chi_M^F(\Delta^m,\phi)$. Now, let $X = (X_k) \in \chi_M^F(\Delta^m,\phi)$ and $\lambda \in R\, with\, 0 < |\lambda|^{1/k} < 1$. By the condition (3.2) and Remark, we have

$M\left(\frac{d\left((k!|\Delta^m \lambda X_k|)^{1/k},\bar{0}\right)}{\rho}\right) \leq M\left(\frac{|\lambda|^{1/k} d\left((k!|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho}\right) \leq$
$|\lambda|^{1/k} M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho}\right)$

Therefore $\lambda X \in \chi_M^F(\Delta^m,\phi)$. This completes the proof. □

Theorem 3.2 *The space* $\chi_M^F(\Delta^m,\phi)$ *is a complete metric space with the metric by*

$g(X,Y) = d\left((k!\,|X_k - Y_k|)^{1/k}\right)$
$+\, inf\left\{\rho > 0 : sup_{k\in\sigma\in P_s}\frac{1}{\phi_s}\left(M\left(\frac{d\left(k!(|\Delta^m X_k - \Delta^m Y_k|)^{1/k}\right),\bar{0}}{\rho}\right)\right) \leq 1\right\}$

Proof: Let $\left(X^i\right)$ be a cauchy sequence in $\chi_M^F(\Delta^m,\phi)$. Then for each $\epsilon > 0$, there exists a positive integer n_0 such that $g\left(X^i,Y^j\right) < \epsilon$ for $i,j \geq n_0$, then

$\Rightarrow d\left(\left(k!\left|X_k^i - Y_k^j\right|\right)^{1/k}\right)$
$+\, inf\left\{\rho > 0 : sup_{k\in\sigma\in P_s}\frac{1}{\phi_s}\left(M\left(\frac{d\left(k!\left(\left|\Delta^m X_k^i - \Delta^m Y_k^j\right|\right)^{1/k},\bar{0}\right)}{\rho}\right)\right) \leq 1\right\} < \epsilon$, for all $i,j \geq n_0$

$$d\left(\left(k!\left|X_k^i - Y_k^j\right|\right)^{1/k},\bar{0}\right) < \epsilon\, for\, all\, i,j \geq n_0 \tag{3.3}$$

and

$$inf\left\{\rho > 0 : sup_{k\in\sigma\in P_s}\frac{1}{\phi_s}\left(M\left(\frac{d\left(k!\,(|\Delta^m X_k - \Delta^m Y_k|)^{1/k},\bar{0}\right)}{\rho}\right)\right) \leq 1\right\} < \epsilon, \tag{3.4}$$

for all i,j$\geq n_0$

By(3.3),d$\left(\left(k!\left|X_k^i - X_k^j\right|\right)^{1/k},\bar{0}\right) < \epsilon\, for\, all\, i,j \geq n_0$ and $k = 1,2,3,...,m$. It follows that $\left(X_k^i\right)$ is a cauchy sequence in $L(R)$ for $k = 1,2,3,...,m$. Since $L(R)$ is complete, then $\left(X_k^i\right)$ is convergent in $L(R)$. Let $lim_{i\to\infty}X_k^i = X_k$ for $k = 1,2,...,m$. Now (3.4) for a given $\epsilon > 0$, there exists some ρ_ϵ $(0 < \rho_\epsilon < \epsilon)$ such that

$$sup_{k\in\sigma\in P_s}\frac{1}{\phi_s}\left(M\left(\frac{d\left(k!\left(\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k},\bar{0}\right)}{\rho_\epsilon}\right)\right) \leq 1$$

Thus

$$\left(sup_{k\in\sigma\in P_s}\frac{1}{\phi_s}\left(M\left(\frac{d\left(k!\left(\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k},\bar{0}\right)}{\rho}\right)\right) \leq 1\right) \leq$$
$$\left(sup_{k\in\sigma\in P_s}\frac{1}{\phi_s}\left(M\left(\frac{d\left(k!\left(\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k},\bar{0}\right)}{\rho_\epsilon}\right)\right) \leq 1\right)$$

we have $d\left(\left(k!\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k}, \bar{0}\right) < \epsilon$ and the fact that

$$d\left(\left(k!\left|X_{k+m}^i - X_{k+m}^j\right|\right)^{1/k}, \bar{0}\right) \leq d\left(\left(k!\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k}, \bar{0}\right)$$

$$= \binom{m}{0} d\left(\left(k!\left|X_k^i - X_k^j\right|\right)^{1/k}, \bar{0}\right) + \binom{m}{1} d\left(\left(k!\left|X_{k+1}^i - X_{k+1}^j\right|\right)^{1/k}, \bar{0}\right) + \cdots + \binom{m}{m-1} d\left(\left(k!\left|X_{k+m-1}^i - X_{k+m-1}^j\right|\right)^{1/k}, \bar{0}\right)$$

So, we have $d\left(\left(k!\left|X_k^i - X_k^j\right|\right)^{1/k}, \bar{0}\right) < \epsilon$ for each $k \in \mathbb{N}$. Therefore $\left(X^i\right)$ is a cauchy sequence in $L(R)$. Since $L(R)$. is complete, then it is convergent in $L(R)$. Let $lim_{i\to\infty} X_k^i = X_k$ say, for each $k \in \mathbb{N}$. Since $\left(X^i\right)$ is a cauchy sequence, for each $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that $g\left(X^i, X^j\right) < \epsilon$ for all $i, j \geq n_0$. So we have $lim_{j\to\infty} d\left(\left(k!\left|X_k^i - X_k^j\right|\right)^{1/k}, \bar{0}\right) = d\left(\left(k!\left|X_k^i - X_k\right|\right)^{1/k}, \bar{0}\right) < \epsilon$ and $lim_{j\to\infty} d\left(\left(k!\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k}, \bar{0}\right) = d\left(\left(k!\left|\Delta^m X_k^i - \Delta^m X_k\right|\right)^{1/k}, \bar{0}\right) < \epsilon$ for all $i, j \geq n_0$. This implies that $g\left(X^i, X\right) < \epsilon$ for all $i \geq n_0$. That is $X^i \to X$ as $i \to \infty$, where $X = (X_k)$. Since $d\left(\left(k!\left|\Delta^m X_k - X_0\right|\right)^{1/k}, \bar{0}\right) \leq d\left(\left(k!\left|\Delta^m X_k^{n_0} - X_0\right|\right)^{1/k}, \bar{0}\right) + d\left(\left(k!\left|\Delta^m X_k^{n_0} - \Delta^m X_k\right|\right)^{1/k}, \bar{0}\right)$ we obtain $X = (X_k) \in \chi_M^F$. Therefore $\chi_M^F(\Delta^m, \phi)$ is complete metric space. This completes the proof. □

Proposition 3.3 *The space* $\Lambda_M^F(\Delta^m)$ *is a complete metric space with the metric by*

$$h(X,Y) = inf\left\{\rho > 0 : sup_k\left(M\left(\frac{d\left(\left(\left|\Delta^m X_k - \Delta^m Y_k\right|\right)^{1/k}\right), \bar{0}}{\rho}\right)\right) \leq 1\right\}$$

Theorem 3.4 *If* $\left(\frac{\phi_s}{\psi_s}\right) \to 0\, as\, s \to \infty$ *then* $\chi_M^F(\Delta^m, \phi) \subset \chi_M^F(\Delta^m, \psi)$

Proof: Let $\left(\frac{\phi_s}{\psi_s}\right) \to 0\, as\, s \to \infty$ and $X = (X_k) \in \chi_M^F(\Delta^m, \phi)$. Then, for some $\rho > 0$

$$\frac{1}{\phi_s} M\left(\frac{d\left(\left(k!\left|\Delta^m X_k\right|\right)^{1/k}, \bar{0}\right)}{\rho}\right) \to 0\, as\, k, s \to \infty,\, for\, k \in \sigma \in P_s$$

$$\Rightarrow \frac{1}{\psi_s} M\left(\frac{d\left(\left(k!\left|\Delta^m X_k\right|\right)^{1/k}, \bar{0}\right)}{\rho}\right) \leq \left(\frac{\phi_s}{\psi_s}\right)\left(\frac{1}{\phi_s} M\left(\frac{d\left(\left(k!\left|\Delta^m X_k\right|\right)^{1/k}, \bar{0}\right)}{\rho}\right)\right)$$

Therefore $X = (X_k) \in \chi_M^F(\Delta^m, \psi)$. Hence $\chi_M^F(\Delta^m, \phi) \subset \chi_M^F(\Delta^m, \psi)$. This completes the proof. □

Proposition 3.5 *If* $\left(\frac{\phi_s}{\psi_s}\right) \to 0$ *and* $\left(\frac{\psi_s}{\phi_s}\right) \to 0$ *as* $s \to \infty$ *then* $\chi_M^F(\Delta^m, \phi) = \chi_M^F(\Delta^m, \psi)$

Theorem 3.6 $\chi_M^F(\Delta^m) \subset \Gamma_M^F(\Delta^m, \phi)$

Proof: Let $X = (X_k) \in \chi_M^F(\Delta^m)$. Then we have

$$M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \to 0\, as\, k \to \infty,\ for\, some\, \rho > 0$$

Since (ϕ_n) is monotonic increasing, so we have
$\frac{1}{\phi_s} M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \leq \frac{1}{\phi_1} M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \leq \frac{1}{\phi_s} M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right)$
Therefore
$\frac{1}{\phi_s} M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \to 0\, as\, k, s \to \infty$ for $k \in \sigma \in P_s$. Hence
$\frac{1}{\phi_s} M\left(\frac{d\left((|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \to 0\, as\, k, s \to \infty$ for $k \in \sigma \in P_s$, and $(k!)^{1/k} \to 1$.
Thus $X = (X_k) \in \Gamma_M^F(\Delta^m, \phi)$. Therefore $\chi_M^F(\Delta^m) \subset \Gamma_M^F(\Delta^m, \phi)$. This completes the proof. □

Theorem 3.7 *Let* M_1 *and* M_2 *be Orlicz functions satisfying* Δ_2- *condition. Then* $\chi_{M_2}^F(\Delta^m, \phi) \subset \chi_{M_1 \circ M_2}^F(\Delta^m, \phi)$

Proof: Let $X = (X_k) \in \Gamma_{M_2}^F(\Delta^m, \phi)$. Then there exists $\rho > 0$ such that

$$\frac{1}{\phi_s} M_2\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \to 0\, as\, k, s \to \infty \text{ for } k \in \sigma \in P_s$$

Let $0 < \epsilon < 1$ and δ with $0 < \delta < 1$ such that $M_1(t) < \epsilon$ for $0 \leq 1 < \delta$. Let

$$y_k = M_2\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \text{ for all } k \in \mathbb{N}.$$

Now, let
$$M_1(y_k) = M_1(y_k) + M_1(y_k) \tag{3.5}$$
where the equation (3.5) RHS of the first term is over $y_k \leq \delta$ and the equation of (3.5) RHS of the second term is over $y_k > \delta$. By the Remark, we have

$$M_1(y_k) \leq M_1(1)\, y_k + M_1(2)\, y_k. \tag{3.6}$$

For $y_k > \delta$,

$$y_k < \tfrac{y_k}{\delta} \leq 1 + \tfrac{y_k}{\delta}.$$

Since M_1 is non-decreasing and convex, so

$$M_1(y_k) < M_1\left(1+\frac{y_k}{\delta}\right) < \frac{1}{2}M_1(2) + \frac{1}{2}M_1\left(\frac{2y_k}{\delta}\right).$$

Since M_1 satisfies Δ_2- condition, then there exists $K>1$ such that

$$M_1(y_k) < \frac{1}{2}KM_1(2)\frac{y_k}{\delta} + \frac{1}{2}KM_1(2)\frac{y_k}{\delta}.$$

Hence the equation (3.5) in RHS of second terms is

$$M_1(y_k) \leq max\left(1, K\delta^{-1}M_1(2)\right)y_k \tag{3.7}$$

By equation (3.6) and (3.7), we have $X=(X_k)\in\chi^F_{M_1\circ M_2}(\Delta^m,\phi)$.
Thus, $\chi^F_{M_2}(\Delta^m,\phi)\subset\chi^F_{M_1\circ M_2}(\Delta^m,\phi)$. This completes the proof. □

Proposition 3.8 *Let M be an Orlicz function which satisfies Δ_2- condition. Then $\chi^F(\Delta^m,\phi)\subset\chi^F_M(\Delta^m,\phi)$*

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