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Morita context and generalized $(\alpha, \beta)-$derivations

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ABSTRACT: Let R and S be rings of a semi-projective Morita context, and α, β be automorphisms of R. An additive mapping F: $R \to R$ is called a generalized (α, β)-derivation on R if there exists an (α, β)-derivation d: $R \to R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. For any $x, y \in R$, set $[x, y]_{\alpha,\beta} = x\alpha(y) - \beta(y)x$ and $(x \circ y)_{\alpha,\beta} = x\alpha(y) + \beta(y)x$. In the present paper, we shall show that if the ring S is reduced then it is a commutative, in a compatible way with the ring R. Also, we obtain some results on bi-algebras via Cauchy modules.

Key Words: Prime rings, $(\alpha, \beta)-$Derivations and Generalized $(\alpha, \beta)-$ Derivations, algebras, coalgebras, Cauchy modules, Morita context

Contents

1. Introduction

A classical problem in ring theory is to study and generalized conditions under which a ring becomes commutative. So far the best tools found for this purpose are the derivations on rings and also on their modules. We can also achieve this goal by comparing two rings and impose conditions on them. If one of the rings is commutative, in compatible way, the other ring will also become commutative. In order to explore these ideas Morita theory is found to be a suitable tool.

Unless otherwise stated the term rings with center $Z(R)$ is used here for associative rings. We assume throughout that the datum $K(R, S) = \{R, S, M, N, \mu_R, \tau_S\}$ is said to be *Morita context* (or MC) in which R and S are rings, M and N are (S, R) and (R, S) bimodules, respectively, $\mu_R : N \otimes_S M \to R$ and $\tau_S : M \otimes_R N \to S$ are bimodule homomorphisms with associative condition

$$m_1\mu_R(n \otimes m) = \tau_S(m_1 \otimes n)m$$

and

$$\mu_R(n \otimes m)n_1 = n\tau_S(m \otimes n_1)$$

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where μ_R and τ_S are called *Morita maps* (or MC maps). The images $\mu_R := I$ and $\tau_R := J$ are two-sided ideals of R and S, respectively, and are called the *trace ideals* of the MC. If both MC maps are epimorphism, i.e. $I = R$ and $J = S$, then $K(R, S)$ is said to be a *projective Morita context* (or PMC). If one of the MC maps is an epimorphism, then $K(R, S)$ is said to be *semi-projective Morita context* or semi-PMC. If $K(R, S)$ is PMC rings, then the rings R and S are said to be *Morita equivalent*.

An *algebra* over R (or an $R-$algebra) is an $(R, R)-$bimodule M together with module morphisms (we will also call them *linear maps*):

$$\mu : M \otimes_R M \longrightarrow M, \text{ and } \eta : R \longrightarrow M,$$

such that

$$M \otimes_R M \otimes_R M \underset{1_M \otimes \mu}{\overset{\mu \otimes 1_M}{\rightrightarrows}} M \otimes_R M \xrightarrow{\mu} M, \ (associativity)$$

with $\mu \circ (\mu \otimes 1_M) = \mu \circ (1_M \otimes \mu)$ and

$$R \underset{1_M \otimes \eta}{\overset{\eta \otimes 1_M}{\rightrightarrows}} M \otimes_R M \xrightarrow{\mu} M, \ (unit)$$

with $\mu \circ (\eta \otimes 1_M) = 1_M = \mu \circ (1_M \otimes \eta)$.
Let R be a commutative ring. An $R-$coalgebra is an $(R, R)-$bimodule C, with $R-$linear maps:

$$\Delta : C \longrightarrow C \otimes_R C \text{ and } \varepsilon : C \longrightarrow R,$$

such that

$$C \xrightarrow{\Delta} C \otimes_R C \underset{\Delta \otimes 1_C}{\overset{1_C \otimes \Delta}{\rightrightarrows}} C \otimes_R C \otimes_R C, \ (coassociativity)$$

with $(1_C \otimes \Delta) \circ \Delta = (\Delta \otimes 1_C) \circ \Delta$ and

$$C \xrightarrow{\Delta} C \otimes_R C \underset{\varepsilon \otimes 1_C}{\overset{1_C \otimes \varepsilon}{\rightrightarrows}} R, \ (counit)$$

with $(1_C \otimes \varepsilon) \circ \Delta = 1_C = (\varepsilon \otimes 1_C) \circ \Delta$.
Now we combine the notions of $R-$algebras and $R-$coalgebras. An $R-$*bialgebra* B is an $R-$module together with algebra and coalgebra structures.

Let R and S be rings and M an $(R, S)-$bimodule. Then the dual of M which is denoted by $M^* = Hom_R(M, R)$ is an $(S, R)-$bimodule, and for every left R-module L there is a canonical module morphism

$$\varphi_L^M : M^* \otimes_R L \longrightarrow Hom_R(M, L)$$

defined by

$$\varphi_L^M(m^* \otimes l)(m) = m^*(m)l \in L \text{ for all } m \in M, m^* \in M^*, l \in L.$$

If φ_L^M is an isomorphism for each left $R-$module L, then ${}_RM_S$ is called a *Cauchy module*. (see [1] and [5]).

For each $x, y \in R$, denote the *commutator* $xy - yx$ by $[x, y]$ and the *anti-commutator* $xy + yx$ by $x \circ y$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. By a *derivation* on R we mean the most natural derivation $d : R \longrightarrow R$ which is additive as well as satisfying the relation $d(xy) = d(x)y+xd(y)$ for all $x, y \in R$. In particular, for a fixed $a \in R$, the mapping $I_a : R \longrightarrow R$ given by $I_a(x) = [x, a]$ is a derivation called an *inner derivation* of R.

To understand our results it is best to review some generalizations of the notion of derivation in rings. Let α and β be the endomorphisms of R. An additive map $d : R \longrightarrow R$ is called an *(α, β)-derivation* if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. A $(1, 1)$-derivation is called simply a derivation, where 1 is the identity map on R. An example of an (α, β)-derivation, when R has a nontrivial central idempotent e is to let $d(x) = ex$, $\alpha(x) = (1 - e)x$ and $\beta = 1$ (or d). Here, d is not a derivation because $d(ee) = eee \neq 2eee = (ee)e + e(ee) = d(e)e + ed(e)$. In any ring with endomorphism β, if we set $d = 1 - \beta$, then d is a $(\beta, 1)$-derivation, but not a derivation when R is semiprime, unless $\beta = 1$. For a fixed a, the map $d_a : R \longrightarrow R$ given by $d_a(x) = [a, x]_{\alpha,\beta}$ for all $x \in R$ is an (α, β)-derivation which is said to be an (α, β)-inner derivation. An additive mapping $F : R \longrightarrow R$ is called a *generalized (α, β)-inner derivation* if $F(x) = a\alpha(x) + \beta(x)b$, for some fixed $a, b \in R$ and for all $x \in R$. A simple computation yields that if F is a generalized (α, β)-inner derivation, then for all $x, y \in R$, we have

$$F(xy) = F(x)\alpha(y) + \beta(x)d_{-b}(y),$$

where d_{-b} is an (α, β)-inner derivation. With this viewpoint, an additive map $F : R \longrightarrow R$ is called a *generalized (α, β)-derivation* associated with an (α, β)-derivation $d : R \longrightarrow R$ such that

$$F(xy) = F(x)\alpha(y) + \beta(x)d(y) \text{ holds for all } x, y \in R.$$

Clearly this notion includes those of (α, β)-derivation when $F = d$, of derivation when $F = d$ and $\alpha = \beta = 1$, and of generalized derivation, when is the case $\alpha = \beta = 1$.

An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. Clearly, every ideal is a Lie ideal but the converse need not be true in general. A Lie ideal L is said to be square closed if $a^2 \in L$ for all $a \in L$. In Section 2 we have established one lemma by involving square closed Lie ideals and have used them to obtain the main results in Section 3.

In the present paper our aim is to prove that if R and S are rings of a semi-PMC, in which R admits generalized $(\alpha, \beta)-$derivations F and G satisfying certain differential identities in rings such that S is reduced, then S is commutative. Some results related to division rings, bi-algebras and Cauchy module are stated and proved.

2. Preliminaries Results

Following are some useful identities which hold for every $x, y, z \in R$. We will use them in the proof of our theorems.

- $[xy, z]_{\alpha,\beta} = x[y, z]_{\alpha,\beta} + [x, \beta(z)]y = x[y, \alpha(z)] + [x, z]_{\alpha,\beta}y$;
- $[x, yz]_{\alpha,\beta} = \beta(y)[x, z]_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z)$;
- $(x \circ (yz))_{\alpha,\beta} = (x \circ y)_{\alpha,\beta}\alpha(z) - \beta(y)[x, z]_{\alpha,\beta} = \beta(y)(x \circ z)_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z)$;
- $((xy) \circ z)_{\alpha,\beta} = x(y \circ z)_{\alpha,\beta} - [x, \beta(z)]y = (x \circ z)_{\alpha,\beta}y + x[y, \alpha(z)]$.

Let us consider three important remarks.

Remark 2.1. Let R be a prime ring and H an additive subgroups of R. Let $f : H \to R$ and $g : H \to R$ be additive functions such that $f(s)Rg(s) = \{0\}$ for all $s \in H$. Then either $f(s) = 0$ for all $s \in H$, or $g(s) = 0$ for all $s \in H$.

Remark 2.2. Let R and S be rings of an MC $K(R, S) = \{R, S, M, N, \mu_R, \tau_S\}$ such that R is commutative and $R \cong S$, then $M \otimes_R N \cong N \otimes_R M$ and the datum $\{R, M, N, \mu_R\}$ is MC where the map $\mu_R : M \otimes_R N \longrightarrow R$ satisfies the associative condition

$$\mu_R(m \otimes n)m_1) = m\mu_R(n \otimes m_1).$$

Remark 2.3. Let R be any ring and I be a nonzero ideal of R. If $\alpha : R \to R$ is an automorphism of R such that $0 \neq z \in Z(R)$, then $\alpha(z) \in Z(R)$.

The proof of Remark 2.1 is rather elementary and is based on the fact that a group cannot be written as the set-theoretic union of its two proper subgroups. Also the proof of Remark 2.2 is clear by using elementary properties of bimodules and the definition of MC. Similarly, Remark 2.3 can also be verified easily.

We begin our discussion with the following results. For the sake of interest, Lemma 2.5 in this section is stated in more general setting, that is, in terms of Lie ideals. Their application is restricted to ideals in the last section.

Lemma 2.4 ([12], Lemma 2.1)**.** *Let R be a prime ring and I be a nonzero ideal of R. If $[x, y] \in Z(R)$ for all $x, y \in I$ or if $(x \circ y) \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Lemma 2.5 ([8], Lemma 2.4)**.** *Let R be a prime ring with char$(R) \neq 2$, and I be a nonzero square closed Lie ideal of R. Let α, β be automorphisms of R. If $[x, y]_{\alpha,\beta} = 0$, for all $x, y \in I$, then $I \subseteq Z(R)$.*

In view of Lemma 2.5 we get the following corollary:

Corollary 2.6. *Let R be a prime ring with $char(R) \neq 2$, and I be a nonzero ideal of R. Let α, β be automorphisms of R. If $[x, y]_{\alpha,\beta} = 0$, for all $x, y \in I$, then R is commutative.*

Lemma 2.7 ([9], Theorem 2.1)**.** *Let R and S be rings of semi-PMC $K(R, S)$ in which τ_S epic. If R is commutative and S is reduced, then S is also commutative.*

Lemma 2.8 ([9], Corollary 2.4)**.** Let $K(R, S)$ be a PMC of rings in which R is commutative. Then

(a) If S is a reduced ring, then R is also reduced and $R \cong S$.

(b) If S is a domain, then both R and S become isomorphic integral domains.

(c) If S is a division ring, then both R and S are isomorphic field.

Lemma 2.9 ([1], Theorem 3.7)**.** *Let R be a commutative ring, M and N Cauchy $R-$modules. Then the datum $\{R, M, N, \mu_R\}$ is MC if and only if $M \otimes_R N$ is a $R-$bialgebra.*

Lemma 2.10. *Let R be a prime ring of characteristic different from two and I be a nonzero ideal of R. Let α, β be automorphisms of R. If $[x, y]_{\alpha,\beta} \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof: For any $x, y \in I$, we have

$$[x, y]_{\alpha,\beta} \in Z(R). \tag{2.1}$$

Replacing x by $x\alpha(y)$ in (2.1), we get $[x, y]_{\alpha,\beta}\alpha(y) \in Z(R)$, this implies that $[[x, y]_{\alpha,\beta}\alpha(y), r] = 0$ for all $x, y \in I$, $r \in R$. Thus, as an application of (2.1), we find that $[x, y]_{\alpha,\beta}[\alpha(y), r] = 0$. Again replacing r by $r\alpha(m)$ and using the last expression, we get $[x, y]_{\alpha,\beta}R[\alpha(y), \alpha(m)] = \{0\}$, for all $x, y, m \in I$. Thus, by Remark 2.1, either $\alpha([y, m]) = 0$ for all $y, m \in I$, or $[x, y]_{\alpha,\beta} = 0$ for all $x, y \in I$. In the first case R is commutative by Lemma 2.4. In the second one, R is commutative by Corollary 2.6. □

Lemma 2.11. *Let R be a prime ring of characteristic different from two and I be a nonzero ideal of R. Let α, β be automorphisms of R. If $(x \circ y)_{\alpha,\beta} \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof: For all $x, y \in I$, we have

$$(x \circ y)_{\alpha,\beta} \in Z(R). \tag{2.2}$$

Replacing x by $\beta(y)x$, we get $\beta(y)(x \circ y)_{\alpha,\beta} \in Z(R)$, which implies that $[\beta(y)(x \circ y)_{\alpha,\beta}, r] = 0$, for all $r \in R$. Hence, by (2.2), we get $[\beta(y), r](x \circ y)_{\alpha,\beta} =$

0, for all $x, y \in I$, and $r \in R$. Now replace r by $\beta(m)r$, to get $[y, m]R\beta^{-1}((x \circ y)_{\alpha,\beta}) = \{0\}$, for all $x, y, m \in I$. By Remark 2.1, we conclude that either $[y, m] = 0$ for all $y, m \in I$, or $\beta^{-1}((x \circ y)_{\alpha,\beta}) = 0$ for all $x, y \in I$. In the first case, R is commutative by Lemma 2.4. On the other hand if $\beta^{-1}((x \circ y)_{\alpha,\beta}) = 0$ for all $x, y \in I$, then $(x \circ y)_{\alpha,\beta} = 0$. Replacing y by ym, and using the last expression, we get $\beta(y)[x, m]_{\alpha,\beta} = 0$. Again replace y by yr for all $r \in R$, to get $IR\beta^{-1}([x, m]_{\alpha,\beta}) = 0$. Since I is nonzero ideal and R is prime which yields that $[x, m]_{\alpha,\beta} = 0$ for all $x, m \in I$, and hence R is commutative by Lemma 2.10. □

3. Centralizing in Generalized (α, β)-Derivations via Morita Context

Theorem 3.1. *Let $K(R, S)$ be a semi-PMC in which the trace ideal I is nonzero and τ_S is epic. Suppose that α, β are automorphisms of R, and R admits a generalized (α, β)-derivation F with associated (α, β)-derivation d with $\{0\} \neq d(Z(R)) \subseteq Z(R)$, such that either*

(i) $[F(x), x]_{\alpha,\beta} \in Z(R)$ for all $x \in I$, or

(ii) $(F(x) \circ x)_{\alpha,\beta} \in Z(R)$ for all $x \in I$.

Further, if R is a prime ring of characteristic different from two and S is reduced, then S is commutative.

Proof: (i) For all $x \in I$, we have

$$[F(x), x]_{\alpha,\beta} \in Z(R). \tag{3.1}$$

Linearizing (3.1), we get

$$[F(x), y]_{\alpha,\beta} + [F(y), x]_{\alpha,\beta} \in Z(R) \text{ for all } x, y \in I. \tag{3.2}$$

For any $z \in Z(R)$, replacing y by yz in (3.2), using (3.2), and Remark 2.3, we get

$$\beta(y)[F(x), z]_{\alpha,\beta} + [\beta(y), x]_{\alpha,\beta} d(z) \in Z(R) \text{ for all } x, y \in I.$$

Again, replacing y by my with $m \in I$ and using the above expression, we get

$$\beta(m)\beta(y)[F(x), z]_{\alpha,\beta} + \beta(m)[\beta(y), x]_{\alpha,\beta} d(z) + \beta([m, x])\beta(y)d(z) \in Z(R).$$

Thus, in particular, we have

$$[\beta(m)\{\beta(y)[F(x), z]_{\alpha,\beta} + [\beta(y), x]_{\alpha,\beta} d(z)\} + \beta([m, x])\beta(y)d(z), \beta(m)] = 0.$$

This gives

$$[\beta([m, x])\beta(y)d(z), \beta(m)] = 0 \text{ for all } x, y, m \in I. \tag{3.3}$$

Since R is prime and $\{0\} \neq d(Z(R)) \subseteq Z(R)$, we find that $\beta([[m, x]y, m]) = 0$ for all $x, y, m \in I$, that is $[m, x][y, m] + [[m, x], m]y = 0$. Again, replacing y by yx and using the above expression, we get $[m,x]y[m,x]=0$, for all $x,y,m\in I$. That is $[m,x]I[m,x]=0$ for all $m,x\in I$. Thus, primeness of R forces that $[m,x]=0$, and hence R is commutative by Lemma 2.4. Since S is reduced, we get the required result by Lemma 2.7.

(ii) For all $x\in I$, we have

$$(F(x)\circ x)_{\alpha,\beta}\in Z(R). \tag{3.4}$$

Linearizing (3.4), we get

$$(F(x)\circ y)_{\alpha,\beta}+(F(y)\circ x)_{\alpha,\beta}\in Z(R) \text{ for all } x,y\in I.$$

For any nonzero $z\in Z(R)$, replacing y by yz in the last expression and using Remark 2.3, we get $-\beta(y)[x,z]_{\alpha,\beta}+(\beta(y)\circ x)_{\alpha,\beta}d(z)+\beta(y)[d(z),\alpha(x)]\in Z(R)$. Since $\{0\}\neq d(Z(R))\subseteq Z(R)$, then

$$-\beta(y)[x,z]_{\alpha,\beta}+(\beta(y)\circ x)_{\alpha,\beta}d(z)\in Z(R).$$

Again replacing y by my, we get

$$\beta(m)\{-\beta(y)[x,z]_{\alpha,\beta}+(\beta(y)\circ x)_{\alpha,\beta}d(z)\}-[\beta(m),\beta(x)]\beta(y)d(z)\in Z(R) \text{ for all } x,y,m\in I.$$

Thus, in particular, we have

$$[\beta(m)\{-\beta(y)[x,z]_{\alpha,\beta}+(\beta(y)\circ x)_{\alpha,\beta}d(z)\}-[\beta(m),\beta(x)]\beta(y)d(z),\beta(m)]=0.$$

This gives

$$[[\beta(m),\beta(x)]\beta(y)d(z),\beta(m)]=0 \quad \text{for all } x,y,m\in I.$$

Now using similar arguments as used in the proof of (i) after equation (3.3), we get the required result. □

Theorem 3.2. *Let $K(R,S)$ be a semi-PMC in which the trace ideal I is nonzero and τ_S is epic. Suppose that α,β are automorphisms of R, and R admits a generalized (α,β)-derivation F with associated (α,β)-derivation d with $\{0\}\neq d(Z(R))\subseteq Z(R)$, such that $F=0$ or $d\neq 0$ and R satisfies any one of the following conditions*

(i) $F([x,y])-[x,y]_{\alpha,\beta}\in Z(R)$ *for all* $x,y\in I$,

(ii) $F(x\circ y)-(x\circ y)_{\alpha,\beta}\in Z(R)$ *for all* $x,y\in I$,

(iii) $F([x,y])-(x\circ y)_{\alpha,\beta}\in Z(R)$ *for all* $x,y\in I$,

(iv) $F(x\circ y)-[x,y]_{\alpha,\beta}\in Z(R)$ *for all* $x,y\in I$,

(v) $(F(x)\circ F(y))-[x,y]_{\alpha,\beta}\in Z(R)$ *for all* $x,y\in I$,

(vi) $[F(x),d(y)]-[x,y]_{\alpha,\beta}\in Z(R)$ *for all* $x,y\in I$.

Further, if R is a prime ring of characteristic different from two and S is reduced, then S is commutative.

Proof: (i) For all $x, y \in I$, we have

$$F([x,y]) - [x,y]_{\alpha,\beta} \in Z(R). \tag{3.5}$$

If $F = 0$, then $[x,y]_{\alpha,\beta} \in Z(R)$ for all $x, y \in I$, thus R is commutative by Lemma 2.10. Since S is reduced then by Lemma 2.7, S is commutative.

Therefore, we shall assume that $d \neq 0$. For any nonzero $z \in Z(R)$, since $\alpha(z) \in Z(R)$ by Remark 2.3, replacing y by yz in (3.5) and using (3.5), we get

$$\beta([x,y])d(z) - \beta(y)[x,z]_{\alpha,\beta} \in Z(R). \tag{3.6}$$

Again, replacing y by my in (3.6), we find that

$$\beta(m)\{\beta([x,y])d(z) - \beta(y)[x,z]_{\alpha,\beta}\} + \beta([x,m])\beta(y)d(z) \in Z(R) \text{ for all } x, y, m \in I.$$

Thus, in particular

$$[\beta(m)\{\beta([x,y])d(z) - \beta(y)[x,z]_{\alpha,\beta}\} + \beta([x,m])\beta(y)d(z), \beta(m)] = 0.$$

This implies that $[\beta([x,m])\beta(y)d(z), \beta(m)] = 0$ for all $x, y, m \in I$. Notice that the arguments given in the last paragraph of the proof of Theorem 3.1 (i) are still valid in the present situation, and hence repeating the same process, we get the required result.

(ii) It is given that F is a generalized (α, β)- generalized derivation on R. If $F = 0$, then $(x \circ y)_{\alpha,\beta} \in Z(R)$, thus R is commutative by Lemma 2.11. Since S is reduced so we get the required result by Lemma 2.7.

Therefore, we shall assume that $d \neq 0$. Now for all $x, y \in I$, we have

$$F(x \circ y) - (x \circ y)_{\alpha,\beta} \in Z(R). \tag{3.7}$$

For any nonzero $z \in Z(R)$, replacing y by yz in (3.7) and using (3.7), we get

$$\beta(x \circ y)d(z) - \beta(y)[x,z]_{\alpha,\beta} \in Z(R). \tag{3.8}$$

Again replacing y by my in (3.8), we get

$$\beta(m)\{\beta(x \circ y)d(z) - \beta(y)[x,z]_{\alpha,\beta}\} + \beta(([x,m])\beta(y)d(z) \in Z(R) \text{ for all } x, y, m \in I.$$

Thus, in particular

$$[\beta(m)\{\beta(x \circ y)d(z) - \beta(y)[x,z]_{\alpha,\beta}\} + \beta(([x,m])\beta(y)d(z), \beta(m)] = 0.$$

Hence, we obtain $[\beta([x,m])\beta(y), \beta(m)]d(z) = 0$ for all $x, y, m \in I$. Again since $0 \neq d(z) \in Z(R)$ and R prime, we get $\beta([[x,m]y,m]) = 0$ and hence using the similar arguments as used in the last paragraph of the proof of Theorem 3.1 (i) to get the required result.

(iii) It is given that F is a generalized (α, β)-derivation on R. If $F = 0$, then $(x \circ y)_{\alpha,\beta} \in Z(R)$, thus by Lemma 2.11 R is commutative. Since S is reduced so S is commutative by Lemma 2.7.

Hence we shall assume that $d \neq 0$. Now for all $x, y \in I$, we have

$$F([x,y]) - (x \circ y)_{\alpha,\beta} \in Z(R). \tag{3.9}$$

Replacing y by yz for any $z \in Z(R)$ in (3.9), we get

$$\beta[x,y]d(z) + \beta(y)[x,z]_{\alpha,\beta} \in Z(R), \text{ for all } x, y \in I.$$

Now, applying similar technique as used after (3.6) in the proof of (i) yields the required result.

(iv) It is given that F is a generalized (α, β)-derivation. If $F = 0$, then $[x,y]_{\alpha,\beta} \in Z(R)$, for all $x, y \in I$, and hence R is commutative by Lemma 2.10. Since S is reduced so S is commutative by Lemma 2.7.

Therefore, we shall assume that $d \neq 0$. For all $x, y \in I$, we have

$$F(x \circ y) - [x,y]_{\alpha,\beta} \in Z(R). \tag{3.10}$$

Replacing y by yz for any $z \in Z(R)$ in (3.10), we get

$$\beta(x \circ y)d(z) - \beta(y)[x,z]_{\alpha,\beta} \in Z(R), \text{ for all } x, y \in I.$$

The last expression is the same as the equation (3.8) and hence the result follows.

(v) It is given that F is a generalized (α, β)-derivation. If $F = 0$, then $[x,y]_{\alpha,\beta} \in Z(R)$, for all $x, y \in I$, and hence R is commutative by Lemma 2.10. Since S is reduce so we get the required result by Lemma 2.7.

Therefore, we shall assume that $d \neq 0$. Now for all $x, y \in I$, we have

$$(F(x) \circ F(y)) - [x,y]_{\alpha,\beta} \in Z(R). \tag{3.11}$$

Replace y by yz for any $z \in Z(R)$, in (3.11), to get

$$(F(x) \circ \beta(y))d(z) - \beta(y)[x,z]_{\alpha,\beta} \in Z(R).$$

Again replacing y by my with $m \in I$ in the last expression, we get

$$\beta(m)\{(F(x) \circ \beta(y))d(z) - \beta(y)[x,z]_{\alpha,\beta}\} + [F(x), \beta(m)]\beta(y)d(z) \in Z(R).$$

Thus, in particular

$$[\beta(m)\{(F(x)\circ\beta(y))d(z)-\beta(y)[x,z]_{\alpha,\beta}\}+[F(x),\beta(m)]\beta(y)d(z),\beta(m)]=0,$$

and hence $[[F(x),\beta(m)]\beta(y)d(z),\beta(m)]=0$ for all $x,y,m\in I$. This can be rewritten as $[[F(x),\beta(m)]\beta(y),\beta(m)]d(z)=0$. Since $0\neq d(Z(R))\subseteq Z(R)$ and R is prime we find that $[[F(x),\beta(m)],\beta(m)]\beta(y)+[F(x),\beta(m)][\beta(y),\beta(m)]=0$. Again replace y by yt with $t\in I$ in the last expression, we obtain $[F(x),\beta(m)]\beta(y)\beta([t,m])=0$ and hence $\beta^{-1}([F(x),\beta(m)])y[t,m]=0$. Thus, by Remark 2.1 either $\beta^{-1}([F(x),\beta(m)])=0$ for all $x,m\in I$ or $[t,m]=0$ for all $t,m\in I$. If $[t,m]=0$, then R is commutative by Lemma 2.4. Now, since S is reduced so we get the required result by Lemma 2.7. On the other hand if $\beta^{-1}([F(x),\beta(m)])=0$, then $[F(x),\beta(m)]=0$ for all $x,m\in I$. Again replacing x by xz for any nonzero $z\in Z(R)$, and using Remark 2.3 we get $\beta([x,m])d(z)=0$, for all $x,m\in I$. Since $0\neq d(z)\in Z(R)$ and R is prime then, we get $\beta([x,m])=0$ for all $x,m\in I$, and hence R is commutative by Lemma 2.4. Since S is reduced, we get the required result by Lemma 2.7.

(vi) It is given that F is a generalized (α,β)-derivation. If $F=0$, then $[x,y]_{\alpha,\beta}\in Z(R)$, for all $x,y\in I$, and thus R is commutative by Lemma 2.10. Since S is reduced, we get the required result by Lemma 2.7.

Therefore, we shall assume that $d\neq 0$. For all $x,y\in I$, we have

$$[F(x),d(y)]-[x,y]_{\alpha,\beta}\in Z(R). \tag{3.12}$$

Replacing y by yz for any nonzero $z\in Z(R)$ in (3.12), we get

$$[F(x),\beta(y)]d(z)-\beta(y)[x,z]_{\alpha,\beta}\in Z(R),\ \text{for all } x,y\in I.$$

Replacing y by my with $m\in I$ in the last expression we find that

$$\beta(m)\{[F(x),\beta(y)]d(z)-\beta(y)[x,z]_{\alpha,\beta}\}+[F(x),\beta(m)]\beta(y)d(z)\in Z(R).$$

Hence, in particular

$$[\beta(m)\{[F(x),\beta(y)]d(z)-\beta(y)[x,z]_{\alpha,\beta}\}+[F(x),\beta(m)]\beta(y)d(z),\beta(m)]=0.$$

This implies that $[[F(x),\beta(m)]\beta(y)d(z),\beta(m)]=0$ for all $x,y,m\in I$. Now using the same arguments as used in the last paragraph of (v), we get the required result. □

Theorem 3.3. *Let $K(R,S)$ be a semi-PMC in which the trace ideal I is nonzero and τ_S is epic. Suppose that α,β are automorphisms of R, and R admits a generalized (α,β)-derivation F with associated (α,β)-derivation d with $\{0\}\neq d(Z(R))\subseteq Z(R)$, such that*

(i) *$[F(x),F(y)]\in Z(R)$ for all $x,y\in I$,* *(ii)* $F([x, y]) - [F(x), y]_{\alpha,\beta} \in Z(R)$, *for all* $x, y \in I$,

(iii) $F(x \circ y) - (F(x) \circ y)_{\alpha,\beta} \in Z(R)$ *for all* $x, y \in I$.

Further, if R *is a prime ring of characteristic different from two and* S *is reduced, then* S *is commutative.*

Proof: (i) For all $x, y \in I$, we have

$$[F(x), F(y)] \in Z(R). \tag{3.13}$$

Replacing y by yz for any nonzero $z \in Z(R)$ in (3.13) using (3.13), we get

$$[F(x), \beta(y)]d(z) \in Z(R).$$

Since $\{0\} \neq d(Z(R)) \subseteq Z(R)$ and R is prime then we have

$$[F(x), \beta(y)] \in Z(R), \text{ for all } x, y \in I.$$

For any nonzero $z \in Z(R)$, replacing x by xz in the above expression and using Remark 2.3, we find that $\beta([x, y])d(z) \in Z(R)$ for all $x, y \in I$. Again, since $\{0\} \neq d(Z(R)) \subseteq Z(R)$ and R is prime, we obtain $\beta([x, y]) \in Z(R)$ that is, $[x, y] \in Z(R)$ for all $x, y \in I$, and hence R is commutative by Lemma 2.4. Since S is reduced so by Lemma 2.7 we get the required result.

(ii) For all $x, y \in I$, we have

$$F[x, y] - [F(x), y]_{\alpha,\beta} \in Z(R). \tag{3.14}$$

Replacing y by yz for any nonzero $z \in Z(R)$ in (3.14)and using (3.14), we get

$$\beta([x, y])d(z) - \beta(y)[F(x), z]_{\alpha,\beta} \in Z(R), \text{ for all } x, y \in I.$$

Notice that the arguments given in the proof of Theorem 3.2 (i) after equation (3.6) are still valid in the present situation, and hence repeating the same process, we get the required result.

(iii) For all $x, y \in I$, we have

$$F(x \circ y) - (F(x) \circ y)_{\alpha,\beta} \in Z(R). \tag{3.15}$$

Replacing y by yz for any $z \in Z(R)$ using (3.15), to get

$$\beta(x \circ y)d(z) - \beta(y)[F(x), z]_{\alpha,\beta} \in Z(R).$$

Now using the similar arguments as used in the proof of Theorem 3.2 (ii) after equation 3.8, we get the required result. □

Theorem 3.4. *Let $K(R,S)$ be a semi-PMC in which the trace ideal I is nonzero and τ_S is epic. Suppose that α,β are automorphisms of R, and R admits a generalized (α,β)-derivations F and G with associated (α,β)-derivations d and g, respectively, with $\{0\} \neq g(Z(R)) \subseteq Z(R)$, such that $F = 0$ (or $G = 0$) or $d \neq 0$ (or $g \neq 0$) and R satisfy the condition $[F(x),G(y)] - [x,y]_{\alpha,\beta} \in Z(R)$ for all $x,y \in I$. Further, if R is a prime ring of characteristic different from two and S is reduced, then S is commutative.*

Proof: It is given that F and G are generalized $(\alpha,\beta)-$derivations on R. If $F = 0$ (or $G = 0$) then $[x,y]_{\alpha,\beta} \in Z(R)$, for all $x,y \in I$, and hence by Lemma 2.10 R is commutative. Since S is reduced so we get the required result by Lemma 2.7.

Now we shall assume that $g \neq 0$. Then for all $x,y \in I$, we have

$$[F(x),G(y)] - [x,y]_{\alpha,\beta} \in Z(R). \tag{3.16}$$

Replacing y by yz for any nonzero $z \in Z(R)$, in (3.16), using (3.16) and Remark 2.3, we get

$$[F(x),\beta(y)]g(z) - \beta(y)[x,z]_{\alpha,\beta} \in Z(R). \tag{3.17}$$

Again, replacing y by my with $m \in I$ in (3.17), we get

$$\beta(m)\{[F(x),\beta(y)]g(z) - \beta(y)[x,z]_{\alpha,\beta}\} + [F(x),\beta(m)]\beta(y)g(z) \in Z(R).$$

Thus, in particular

$$[\beta(m)\{[F(x),\beta(y)]g(z) - \beta(y)[x,z]_{\alpha,\beta}\} + [F(x),\beta(m)]\beta(y)g(z),\beta(m)] = 0$$

and hence $[[F(x),\beta(m)]\beta(y)g(z),\beta(m)] = 0$ for all $x,y,m \in I$. Using the same arguments as used in the last paragraph of Theorem 3.2 (v), yields the required result.

□

In view of these results, we get the following corollaries:

Corollary 3.5. *In each of the above, from Theorem 3.1 to Theorem 3.4, if $K(R,S)$ be a PMC in which M and N are Cauchy modules, then $M \otimes_R N$ is an $R-$bialgebra if and only if the datum $K(R,S)$ is MC.*

Proof: Suppose that $M \otimes_R N$ is an $R-$bialgebra, since R is commutative and $R \cong S$ from the above theorems, then by Remark 2.2 and Lemma 2.9, respectively, the datum $\{R,M,N,\mu_R\}$ is MC. On the other hand, if the datum $K(R,S)$ is MC, R is commutative and $R \cong S$ by the above theorems, then by Remark 2.2 the datum $\{R,M,N,\mu_R\}$ is MC, thus by Lemma 2.9 $M \otimes_R N$ is an $R-$bialgebra. □

Corollary 3.6. *By the same argument as Corollary 3.5, in the cases, from Theorem 3.1 to Theorem 3.4, if R and S are Morita equivalent rings, then by Lemma 2.8 (a), R is also reduced and $Z(R) \cong Z(S)$. Since R and S are commutative, $R = Z(R)$ and $S = Z(S)$ and hence $R \cong S$. If S is division ring then S is a field. Since S is commutative division ring, by Lemma 2.8 (c), R and S are becomes isomorphic fileds.*

Corollary 3.7. *Let $K(R, S)$ be a PMC in which rings R and S are equipped with multiplicative identity 1. Then $Z(R) \cong Z(S)$. If the conditions of either Theorem 3.1, or of Theorem 3.2, or of Theorem 3.3, or ofTheorem 3.4 are satisfied, then $S \cong Z(R)$. Hence R can be treated as an $S-$Algebra. Moreover in this case S becomes prime, as being prime is a Morita invariant property.*

Corollary 3.8. *Let $K(R, S)$ be a semi-PMC in which τ_S is epic. Then the generalized matrix ring $T = \begin{bmatrix} R & M \\ N & S \end{bmatrix}$ and S are Morita equivalent [10, Theorem 2.1]. Hence, trivially, in this case, if the conditions of either of the Theorems 3.1, 3.2, 3.3, or 3.4, are satisfied, then $Z(T) \cong S$.*

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