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Asymptotically Lacunary Statistical Equivalent Sequences of Fuzzy Numbers

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abstract: In this article we present the following definition which is natural combination of the definition for asymptotically equivalent and lacunary statistical convergence of fuzzy numbers. Let $\theta = (k_r)$ be a lacunary sequence. The two sequnces $X = (X_k)$ and $Y = (Y_k)$ of fuzzy numbers are said to be asymptotically lacunary statistical equivalent to multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \bar{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon\right\}\right| = 0.$$

Key Words: Asymptotically equivalent, lacunary sequence, fuzzy numbers.

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1. Introduction

In 1993, Marouf [2] presented definitions for asymptotically equivalent sequences of real numbers. In 2003, Patterson [4] extended these definitions by presenting on asymptotically statistical equivalent analog of these definitions. For sequences of fuzzy numbers Savaş [5] introduced and studied asymptotically $\lambda-$statistical equivalent sequences. Later Esi and Esi [1] studied $\Delta-$asymptotically equivalent sequences of fuzzy numbers. The goal of this paper is to extended the idea to apply to asymptotically equivalent and lacunary statistical convergence of fuzzy numbers.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, 3, ...$ where $k_0 = 0$, we shall mean an increasing sequence of nonnegative integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

Let D denote the set of all closed and bounded intervals on R , the real line. For $X, Y \in D$ we define

$$d(X, Y) = max(|a_1 - b_1|, |a_2 - b_2|)$$

where $X = [a_1, a_2]$ and $Y = [b_1, b_2]$. It is known that (D, d) is a complete metric space. A fuzzy real number X is a fuzzy set on R , i.e. a mapping $X : R \to I$ $(= [0, 1])$ associating each real number t with its grade of membership $X(t)$.

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The set of all upper - semi continous, normal,convex fuzzy real numbers is denoted by $R\ (I)$. Throughout the paper, by a fuzzy real number X ,we mean that $X \in R(I)$.

The $\alpha-$ cut or $\alpha-$ level set $[X]^{\alpha}$ of the fuzzy real number X , for $0 < \alpha \leq 1$ defined by $[X]^{\alpha} = \{t \in R : X(t) \geq \alpha\}$; for $\alpha = 0$, it is the closure of the strong $0-$cut , i.e. closure of the set $\{t \in R : X(t) > 0\}$. The linear structure of $R(I)$ induces addition $X+Y$ and scalar multipliction μX , $\mu \in R$, in terms of $\alpha-$ level set , by $[X+Y]^{\alpha} = [X]^{\alpha} + [Y]^{\alpha}$, $[\mu X]^{\alpha} = \mu [X]^{\alpha}$ for each $\alpha \in [0,1]$.

Let

$$\overline{d} : R(I) \times R(I) \to R$$

be defined by

$$\overline{d}(X,Y) = \sup_{\alpha \in (0,1]} d([X]^{\alpha}, [Y]^{\alpha}).$$

Then $\overline{d}$ defines a metric on $R\ (I)$. It is well known that $R\ (I)$ is complete with respect to $\overline{d}$.

A sequence (X_k) of fuzzy real numbers is said to be convergent to the fuzzy real numbers X_0,if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(X_k, X_0) < \varepsilon$ for all $k \geq n_0$.Nuray and Savaş [3] defined the notion of statistical convergence for sequences of fuzzy numbers as follows :A sequence of fuzzy numbers is said to be statistically convergent to the fuzzy real number X_0, if for evey $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \leq n : \overline{d}(X_k, X_0) \geq \varepsilon\right\}\right| = 0.$$

2. Definitions and Notations

Definition 2.1 *Let $\theta = (k_r)$ be lacunary sequence. The two sequences $X = (X_k)$ and $Y = (Y_k)$ of fuzzy numbers are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon \succ 0$,*

$$\lim_r \frac{1}{h_r} \left|\left\{k \in I_r : \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon\right\}\right| = 0.$$

$$\left(\text{denoted by } X \overset{S_\theta^L(F)}{\sim} Y\right)$$

and simply asymptotically lacunary statistical equivalent if $L = \overline{1}$ (where $\overline{1}$ is unity element for addition in $R\ (I)$). Furthermore, let $S_\theta^L(F)$ denotes the set of $X = (X_k)$ and $Y = (Y_k)$ of fuzzy numbers such that $X \overset{S_\theta^L(F)}{\sim} Y$.

Definition 2.2 *Let $\theta = (k_r)$ be lacunary sequence. The two sequences $X = (X_k)$ and $Y = (Y_k)$ of fuzzy numbers are strong asymptotically lacunary statistical equivalent of multiple L provided that*

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \overline{d}\left(\frac{X_k}{Y_k}, L\right) = 0,$$

$\left(\text{denoted by } X \overset{N_\theta^L(F)}{\sim} Y\right)$ and simply strong asymptotically lacunary statistical equivalent if $L = \overline{1}$. In addition, let $N_\theta^L(F)$ denotes the set of $X = (X_k)$ and $Y = (Y_k)$ of fuzzy numbers such that $X \overset{N_\theta^L(F)}{\sim} Y$.

3. Main Results

Theorem 3.1 *Let $\theta = (k_r)$ be lacunary sequence. Then*

(a) *If $X \overset{N_\theta^L(F)}{\sim} Y$ then $X \overset{S_\theta^L(F)}{\sim} Y$,*

(b) *if $X \in \ell_\infty(F)$ and $X \overset{S_\theta^L(F)}{\sim} Y$ then $X \overset{N_\theta^L(F)}{\sim} Y$,*

(c) *$S_\theta^L(F) \cap \ell_\infty(F) = N_\theta^L(F) \cap \ell_\infty(F)$*

where $\ell_\infty(F)$ the set of all bounded sequences of fuzzy numbers.

Proof: **(a)** If $\varepsilon > 0$ and $X \overset{N_\theta^L(F)}{\sim} Y$, then

$$\sum_{k \in I_r} \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \sum_{k \in I_r \;\&\; \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon} \overline{d}\left(\frac{X_k}{Y_k}, L\right)$$
$$\geq \varepsilon . \left|\left\{k \in I_r : \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon\right\}\right| .$$

Therefore $X \overset{S_\theta^L(F)}{\sim} Y$.

(b) Suppose that $X = (X_k)$ and $Y = (Y_k) \in \ell_\infty(F)$ and $X \overset{S_\theta^L(F)}{\sim} Y$.Then we can assume that $\overline{d}\left(\frac{X_k}{Y_k}, L\right) \leq T$ for all k.Given $\varepsilon > 0$

$$\frac{1}{h_r} \sum_{k \in I_r} \overline{d}\left(\frac{X_k}{Y_k}, L\right) = \frac{1}{h_r} \sum_{k \in I_r \;\&\; \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon} \overline{d}\left(\frac{X_k}{Y_k}, L\right) + \frac{1}{h_r} \sum_{k \in I_r \;\&\; \overline{d}\left(\frac{X_k}{Y_k}, L\right) < \varepsilon} \overline{d}\left(\frac{X_k}{Y_k}, L\right)$$
$$\leq \frac{T}{h_r} \left|\left\{k \in I_r : \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon\right\}\right| + \varepsilon$$

Therefore $X \overset{N_\theta^L(F)}{\sim} Y$.

(c) It follows from (a) and (b). □

Theorem 3.2 *Let $\theta = (k_r)$ be lacunary sequence with $\liminf_r q_r > 1$, then $X \overset{S_L(F)}{\sim} Y$ implies $X \overset{S_\theta^L(F)}{\sim} Y$,*

where $S^L(F)$ (see [5]) the set of $X = (X_k)$ and $Y = (Y_k)$ of fuzzy numbers such that

$$\lim_n \frac{1}{n} \left|\left\{k \leq n : \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon\right\}\right| = 0.$$

Proof: Suppose that $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1+\delta$ for sufficiently large r, which implies

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$$

if $X \overset{S^L(F)}{\sim} Y$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\frac{1}{k_r}\left|\left\{k \leq k_r : \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon\right\}\right| \geq \frac{1}{k_r}\left|\left\{k \in I_r : \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon\right\}\right|$$
$$\geq \frac{\delta}{1+\delta}.\frac{1}{h_r}\left|\left\{k \in I_r : \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon\right\}\right|.$$

This complete the proof. □

Theorem 3.3 *Let* $\theta = (k_r)$ *be lacunary sequence with* $\limsup_r q_r < \infty$, *then* $X \overset{S^L_\theta(F)}{\sim} Y$ *implies* $X \overset{S_L(F)}{\sim} Y$.

Proof: If $\limsup_r q_r < \infty$,then there exists $B > 0$ such that $q_r < C$ for all $r \geq 1$.Let $X \overset{S^L_\theta(F)}{\sim} Y$ and $\varepsilon > 0$.There exists $B > 0$ such that for every $j \geq B$

$$A_J = \frac{1}{h_j}\left|\left\{k \in I_j : \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon\right\}\right| < \varepsilon.$$

we can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, 3, ...$ Now let n be any integer with $k_{r-1} < n < k_r$,where $r \geq B$. Then

$$\frac{1}{n}\left|\left\{k \leq n : \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon\right\}\right| \leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : \overline{d}\left(\frac{X_k}{Y_k}, L\right) \geq \varepsilon\right\}\right|$$

$$= \frac{1}{k_{r-1}} \left| \left\{ k \in I_1 : \overline{d}\left(\frac{X_k}{Y_k}, L \right) \geq \varepsilon \right\} \right| + \frac{1}{k_{r-1}} \left| \left\{ k \in I_2 : \overline{d}\left(\frac{X_k}{Y_k}, L \right) \geq \varepsilon \right\} \right|$$
$$+ ... + \frac{1}{k_{r-1}} \left| \left\{ k \in I_r : \overline{d}\left(\frac{X_k}{Y_k}, L \right) \geq \varepsilon \right\} \right|$$
$$= \frac{k_1}{k_{r-1}k_1} \left| \left\{ k \in I_1 : \overline{d}\left(\frac{X_k}{Y_k}, L \right) \geq \varepsilon \right\} \right| + \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)} \left| \left\{ k \in I_2 : \overline{d}\left(\frac{X_k}{Y_k}, L \right) \geq \varepsilon \right\} \right|$$
$$+ ... + \frac{k_B - k_{B-1}}{k_{r-1}(k_B - k_{B-1})} \left| \left\{ k \in I_B : d\left(\frac{X_k}{Y_k}, L \right) \geq \varepsilon \right\} \right|$$
$$+ ... + \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})} \left| \left\{ k \in I_r : \overline{d}\left(\frac{X_k}{Y_k}, L \right) \geq \varepsilon \right\} \right|$$
$$= \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + ... + \frac{k_B - k_{B-1}}{k_{r-1}} A_B + ... + \frac{k_r - k_{r-1}}{k_{r-1}} A_r$$
$$\leq \left\{ \sup_{j \geq 1} A_j \right\} \frac{k_B}{k_{r-1}} + \left\{ \sup_{j \succeq B} A_j \right\} \frac{k_r - k_B}{k_{r-1}}$$
$$\leq K. \frac{k_B}{k_{r-1}} + \varepsilon.C.$$

This complete the proof. □

Theorem 3.4 *Let $\theta = (k_r)$ be lacunary sequence $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, then $X \overset{S_L(F)}{\sim} Y \Leftrightarrow X \overset{S_\theta^L(F)}{\sim} Y$.*

Proof: The result follow from Theorem 3.2 and Theorem 3.3. □

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