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# Strict monotonicity and unique continuation of the biharmonic operator

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abstract: In this paper, we will show that the strict monotonicity of the eigenvalues of the biharmonic operator holds if and only if some unique continuation property is satisfied by the corresponding eigenfunctions.

Key Words: variational methods, eigenvalues, biharmonic operator, unique continuation.

## Contents



## 1. Introduction

Consider the Navier boundary value problem involving the biharmonic operator

$$\begin{cases} \text{find } \ (\lambda, u) \in \mathbb{R} \times \big((H_0^1(\Omega) \cap H^2(\Omega))\backslash\{0\}\big) \ \text{ such that} \\ \Delta^2 u = \lambda m u \ \text{ on } \ \Omega, \\ u = \Delta u = 0 \ \text{ in } \ \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is an open bounded with a boundary $\partial\Omega$ and $\Delta^2$ denotes the biharmonic operator defined by $\Delta^2 u = \Delta(\Delta u)$. The weight function $m(x)$ is assumed to be in $L^\infty(\Omega)$ and meas $\{x \in \Omega; m(x) \neq 0\} > 0$.
It is very well known (cf. [7]), that the spectrum $\sigma(\Delta^2, m)$ of the problem (1) contains a sequence of positive eigenvalues

$$0 < \lambda_1(m) < \lambda_2(m) \leq \lambda_3(m) \leq \cdots \to +\infty.$$

Moreover, $\lambda_k(m)$ has the variational characterization :

$$\frac{1}{\lambda_k(m)} = \sup_{F_k} \inf_{u \in F_k} \left\{ \int_\Omega m u^2, \int_\Omega |\Delta u|^2 = 1 \right\}, \tag{2}$$

where $F_k$ varies over all k-dimensional subspaces of $H_0^1(\Omega) \cap H^2(\Omega)$ and $\lambda_k(m)$ is repeated with its order of multiplicity.
For convenience, we now give the following definitions :

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**Definition 1.1** *We say that solutions of the problem (1) satisfies the unique continuation property, in short (U.C.P.) if the only solution $u \in H^2_{Loc}(\Omega)$ which vanishes on a set of positive measure in $\Omega$ is $u \equiv 0$.*

There is an extensive literature on unique continuation. We refer to the works of Jerison-Kenig and Garofalo-Lin, among others,( cf. [4], [8]...) and the references therein. The unique continuation property as defined above differs from the more usual notions of unique continuation. See ( [3]), for more details.

**Definition 1.2** *We say that $\lambda_k(.)$ is strict monotone with respect to weight if $\lambda_k(m) > \lambda_k(\widehat{m})$, for all $m \underset{\not\equiv}{<} \widehat{m}$.*

Here we use the notation $\underset{\not\equiv}{<}$ to mean inequality a.e. together with strict inequality on a set of positive measure.
Over the last years, the notion of unique continuation has been receiving increasing attention from workers in partial differential equations since the pioneering work of Carleman [9] in 1939, see for instance [4,6,8] and the references therein.
The unique continuation property of the biharmonic operator was proved recently by Fabrizio Cuccu and Giovanni Porru [5].
In [3], Djairo G. de Figueiredo and Jean-Pierre Gossez proved that strict monotonicity holds if and only if some unique continuation property is satisfied by the corresponding eigenfunctions of an uniformly elliptic operator of second order. The aim of this work is to extend this last result to higher order, namely the biharmonic operator. As consequence, we will show the following :

**Corollary 1.3** *Let $m \in L^\infty(\Omega)$ and $k \in \mathbb{N}$. If $\lambda_k(1) \underset{\not\equiv}{<} m \underset{\not\equiv}{<} \lambda_{k+1}(1)$, then the only solution of the problem*

$$(P_m) \quad \begin{cases} \Delta^2 u = mu & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

*is $u \equiv 0$.*

This "unique continuation" result is very useful in the study of nonlinear problems [1].

## 2. Strict monotonicity and unique continuation

In this section, we will prove that the strict monotonicity of the eigenvalues of the biharmonic operator is equivalent to a unique continuation property.

**Theorem 2.1** *Let $m$ and $\widehat{m}$ be two weights with $m \underset{\not\equiv}{<} \widehat{m}$ and $k \in \mathbb{N}$. If the eigenfunctions $\varphi_k$ associated to $\lambda_k(m)$ enjoy the (U.C.P), then $\lambda_k(m) > \lambda_k(\widehat{m})$.*

**Theorem 2.2** *Let $m$ be a weight and $k \in \mathbb{N}$. If the eigenfunctions $\varphi_k$ associated to $\lambda_k(m)$ do not enjoy the (U.C.P), then there exists a weight $\widehat{m}$ with $m \lneqq \widehat{m}$, such that, for some $i \in \mathbb{N}$ with $\lambda_i(m) = \lambda_k(m)$, one has $\lambda_i(m) = \lambda_i(\widehat{m})$.*

**Proof of Theorem 2.1.**
Let k in $\mathbb{N}$, we put

$$F_k =< \varphi_1, \varphi_2, ......., \varphi_k >,$$

where $\varphi_i$ is the eigenfunction associated to $\lambda_i(m)$, with $\int_\Omega |\varphi_i|^2 = 1$ for $i = 1 \ldots k$.
It is clear

$$\frac{1}{\lambda_k(m)} = \min_{u \in F_k} \left\{ \int_\Omega mu^2, \int_\Omega |\Delta u|^2 = 1 \right\} = \int_\Omega m\varphi_k^2. \tag{3}$$

Let u $\in F_k$ with $\int_\Omega |\Delta u|^2 = 1$, either $u$ achieves the infimum in (3) or not. In the first case, $u$ is an eigenfunction associated to $\lambda_k(m)$.
So, by the (U.C.P) and since $u^2(x) > 0$ a.e. $x \in \Omega$ we have

$$\frac{1}{\lambda_k(m)} = \int_\Omega mu^2 < \int_\Omega \widehat{m}u^2.$$

In the second case

$$\frac{1}{\lambda_k(m)} < \int_\Omega mu^2 \leq \int_\Omega \widehat{m}u^2.$$

Thus, in any case,

$$\frac{1}{\lambda_k(m)} < \int_\Omega \widehat{m}u^2.$$

It then follows, by a simple compactness argument, that

$$\frac{1}{\lambda_k(m)} < inf \left\{ \int_\Omega \widehat{m}u^2, \int_\Omega |\Delta u|^2 = 1, u \in F_k \right\}.$$

This yields the desired inequality

$$\frac{1}{\lambda_k(m)} < \frac{1}{\lambda_k(\widehat{m})}.$$

$\square$

**Proof of Theorem 2.2.**
Denote by $u$ an eigenfunction associated to $\lambda_k(m)$ which vanishes on a set of positive measure. Take $i$ such that $\lambda_k(m) = \lambda_i(m) < \lambda_{i+1}(m)$. Define

$$\widehat{m}(x) = \begin{cases} m(x) & \text{if } u(x) \neq 0, \\ m(x) + \epsilon & \text{if } u(x) = 0, \end{cases}$$

where $\epsilon > 0$ is chosen so that

$$\lambda_i(m) < \lambda_{i+1}(\widehat{m}), \tag{4}$$

which is possible by the continuous dependence of the eigenvalues with respect to the weight. We have

$$\Delta^2 u = \lambda_i(m)mu = \lambda_i(m)\widehat{m}u,$$

which shows that $\lambda_i(m)$ is an eigenvalue for the weight $\widehat{m}$ i.e $\lambda_i(m) = \lambda_l(\widehat{m})$ for some $l \in \mathbb{N}$. Let us choose the largest $l$ such that this equality holds. It follows from (4) that $l < i + 1$. Moreover, by the monotone dependence, $\lambda_i(\widehat{m}) \leq \lambda_i(m)$ which implies $l \geq i$. Then we conclude that $l = i$, so $\lambda_i(\widehat{m}) = \lambda_i(m)$. □

**Proof of Corollary 1.3** Suppose by contradiction that $(P_m)$ has nontrivial solution
i.e

$$1 \in \sigma(\Delta^2, m).$$

On the other hand, from the inequality $\lambda_k(1) \underset{\not\equiv}{\leq} m \underset{\not\equiv}{\leq} \lambda_{k+1}(1)$ and the strict monotonicity, we deduce

$$\lambda_n(\lambda_n(1)) > \lambda_n(m) \quad \text{and} \quad \lambda_{n+1}(m) > \lambda_{n+1}(\lambda_{n+1}(1)),$$

since

$$\lambda_{n+1}(\lambda_{n+1})(1)) = \lambda_n(\lambda_n(1)) = 1$$

we deduce that

$$\lambda_n(m) < 1 < \lambda_{n+1}(m)$$

which is a contradiction, completing the proof. □

**Remark 2.3** *These results (theorem 2.1 and theorem 2.2) remain valid if one replaces Navier boundary condition by Dirichlet boundary conditions, namely,* $u = \frac{\partial u}{\partial \nu} = 0$ *on* $\partial\Omega$, *and the space* $H_0^1(\Omega) \cap H^2(\Omega)$ *by* $H_0^2(\Omega)$, *see [2].*

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