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On Spacelike Biharmonic Slant Helices According to Bishop Frame in the Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

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abstract: In this paper, we study spacelike biharmonic slant helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. We characterize the spacelike biharmonic slant helices in terms of their curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Finally, we obtain parametric equations of spacelike biharmonic slant helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

Key Words: Biharmonic curve, Bishop frame, Rigid motions.

Contents

1. Introduction

Many important results in the theory of curves in E^3 were initiated by G. Monge; G. Darboux pioneered the moving frame idea. Thereafter, F. Frenet defined his moving frame and his special equations, which play an important role in mechanics and kinematics as well as in differential geometry. At the beginning of the twentieth century, Einstein's theory opened a door to new geometries such as Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold.

The notions of harmonic and biharmonic maps between Riemannian manifolds have been introduced by J. Eells and J.H. Sampson (see [4]).

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

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The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \operatorname{tr} R\left(\mathcal{T}(\phi), d\phi\right) d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study spacelike biharmonic slant helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. We characterize the spacelike biharmonic slant helices in terms of their curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Finally, we obtain parametric equations of spacelike biharmonic slant helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

2. Preliminaries

Let $\mathbb{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $\mathbb{E}(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$\mathbb{E}(1,1) \longrightarrow \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x, y, z)\,,$$

It's Lie algebra has a basis consisting of

$$\mathbf{X}_1 = \frac{\partial}{\partial x},\ \mathbf{X}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z},\ \mathbf{X}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3,\ \ [\mathbf{X}_2, \mathbf{X}_3] = 0,\ \ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2.$$

Put

$$x^1 = x,\ x^2 = \frac{1}{2}(y+z)\,,\ x^3 = \frac{1}{2}(y-z)\,.$$

Then, we get

$$\mathbf{X}_1 = \frac{\partial}{\partial x^1},\ \mathbf{X}_2 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} + e^{-x^1}\frac{\partial}{\partial x^3}\right),\ \mathbf{X}_3 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} - e^{-x^1}\frac{\partial}{\partial x^3}\right). \tag{2.1}$$

The bracket relations are

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3,\ \ [\mathbf{X}_2, \mathbf{X}_3] = 0,\ \ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2. \tag{2.2}$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. We consider left-invariant Lorentzian metric [10], given by

$$g = -\left(dx^1\right)^2 + \left(e^{-x^1}dx^2 + e^{x^1}dx^3\right)^2 + \left(e^{-x^1}dx^2 - e^{x^1}dx^3\right)^2, \tag{2.3}$$

where

$$g(\mathbf{X}_1, \mathbf{X}_1) = -1, \; g(\mathbf{X}_2, \mathbf{X}_2) = g(\mathbf{X}_3, \mathbf{X}_3) = 1. \tag{2.4}$$

Let coframe of our frame be defined by

$$\theta^1 = dx^1, \; \theta^2 = e^{-x^1}dx^2 + e^{x^1}dx^3, \; \theta^3 = e^{-x^1}dx^2 - e^{x^1}dx^3.$$

Proposition 2.1 *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ -\mathbf{X}_3 & 0 & -\mathbf{X}_1 \\ -\mathbf{X}_2 & -\mathbf{X}_1 & 0 \end{pmatrix}, \tag{2.5}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{X}_i}\mathbf{X}_j$ for our basis

$$\{\mathbf{X}_k, k = 1, 2, 3\} = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)W, Z).$$

Moreover we put

$$R_{ijk} = R(\mathbf{X}_i, \mathbf{X}_j)\mathbf{X}_k, \quad R_{ijkl} = R(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k, \mathbf{X}_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3.

$$R_{121} = \mathbf{X}_2, \quad R_{131} = \mathbf{X}_3, \quad R_{232} = \mathbf{X}_3$$

and

$$R_{1212} = -1, \quad R_{1313} = -1, \quad R_{2323} = -1. \tag{2.6}$$

3. Spacelike Biharmonic Slant Helices with Timelike Normal in the Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a non geodesic spacelike curve on the $\mathbb{E}(1,1)$ parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the $\mathbb{E}(1,1)$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion and

$$g(\mathbf{T},\mathbf{T}) = 1,\ g(\mathbf{N},\mathbf{N}) = -1,\ g(\mathbf{B},\mathbf{B}) = 1, \tag{3.2}$$

$$g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0.$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 - k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= k_2\mathbf{T}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} g(\mathbf{T},\mathbf{T}) = 1,\ g(\mathbf{M}_1,\mathbf{M}_1) = -1,\ g(\mathbf{M}_2,\mathbf{M}_2) = 1, \\ g(\mathbf{T},\mathbf{M}_1) = g(\mathbf{T},\mathbf{M}_2) = g(\mathbf{M}_1,\mathbf{M}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_1\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\tau(s) = \psi'(s)$, $\kappa(s) = \sqrt{|k_2^2 - k_1^2|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\sinh\psi(s), \\ k_2 &= \kappa(s)\cosh\psi(s). \end{aligned} \tag{3.5}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3. \end{aligned} \tag{3.6}$$

Theorem 3.1 $\gamma : I \longrightarrow \mathbb{E}(1,1)$*is a spacelike biharmonic curve with Bishop frame if and only if*

$$\begin{aligned} k_1^2 - k_2^2 &= constant = C \neq 0, \\ k_1'' + Ck_1 &= -k_1 \left[1 + 2\left(M_2^1\right)^2\right] + 2k_2 M_1^1 M_2^1, \quad (3.7) \\ k_2'' + Ck_2 &= -2k_1 M_1^1 M_2^1 - k_2 \left[-1 + 2\left(M_1^1\right)^2\right]. \end{aligned}$$

Proof: Using (3.1), we have

$$\begin{aligned} \tau_2(\gamma) &= \nabla_{\mathbf{T}}^3 \mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}} \mathbf{T})\mathbf{T} \\ &= (3k_1' k_1 - 3k_2' k_2)\mathbf{T} + (k_1'' + k_1^3 - k_1 k_2^2)\mathbf{M}_1 + (-k_2'' + k_2^3 - k_2 k_1^2)\mathbf{M}_2 \\ &\quad -k_1 R(\mathbf{T}, \mathbf{M}_1)\mathbf{T} - k_2 R(\mathbf{T}, \mathbf{M}_2)\mathbf{T}. \end{aligned}$$

Thus, the equality (5) can be written as follows

$$\begin{aligned} k_1' k_1 - k_2' k_2 &= 0, \\ k_1'' + k_1^3 - k_1 k_2^2 &= -k_1 R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_1) - k_2 R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_1), \quad (3.8) \\ -k_2'' + k_2^3 - k_2 k_1^2 &= k_1 R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_2) + k_2 R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_2). \end{aligned}$$

Making necessary calculations from (3.8), we have

$$\begin{aligned} k_1^2 - k_2^2 &= \text{constant} = C \neq 0, \\ k_1'' + Ck_1 &= -k_1 R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_1) - k_2 R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_1), \quad (3.9) \\ k_2'' + Ck_2 &= -k_1 R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_2) - k_2 R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_2). \end{aligned}$$

It is easy to prove that: right curvature equations of above system is equivalent to

$$\begin{aligned} R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_1) &= 1 + 2\left(M_2^1\right)^2, \\ R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_1) &= -2M_1^1 M_2^1, \quad (3.10) \\ R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_2) &= 2M_1^1 M_2^1, \\ R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_2) &= -1 + 2\left(M_1^1\right)^2. \end{aligned}$$

This shows (3.4), complete the proof of the theorem. □

Definition 3.2 *A regular spacelike curve* $\gamma : I \longrightarrow \mathbb{E}(1,1)$ *is called a slant helix provided the timelike unit vector* $\mathbf{M}_1$ *of the curve* $\gamma\theta$ *with some fixed timelike unit vector* u, *that is*

$$g\left(\mathbf{M}_1\left(s\right), u\right) = \cosh \wp \text{ for all } s \in I. \quad (3.11)$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

Theorem 3.3 *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a unit speed spacelike curve with non-zero natural curvatures. Then γ is a slant helix if and only if*

$$\frac{k_1}{k_2} = constant. \tag{3.12}$$

Proof: Differentiating (3.11) and by using the Bishop frame (3.3), we find

$$g\left(\nabla_{\mathbf{T}}\mathbf{M}_1, u\right) = g\left(k_1\mathbf{T}, u\right) = k_1 g\left(\mathbf{T}, u\right) = 0. \tag{3.13}$$

From (3.13) we get

$$g\left(\mathbf{T}, u\right) = 0.$$

Again differentiating from the last equality, we obtain

$$\begin{aligned} g\left(\nabla_{\mathbf{T}}\mathbf{T}, u\right) &= g\left(k_1\mathbf{M}_1 - k_2\mathbf{M}_2, u\right) \\ &= k_1 g\left(\mathbf{M}_1, u\right) - k_2 g\left(\mathbf{M}_2, u\right) \\ &= k_1 \cosh\wp - k_2 \sinh\wp = 0. \end{aligned}$$

Using above equation, we get

$$\frac{k_1}{k_2} = \tanh\wp = \text{constant}.$$

The converse statement is trivial. This completes the proof. □

Theorem 3.4 *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a unit speed spacelike biharmonic slant helix with non-zero Bishop curvatures. Then,*

$$k_1 = constant \text{ and } k_2 = constant. \tag{3.14}$$

Proof: Suppose that γ be a unit speed spacelike biharmonic slant helix according to Bishop frame. From (3.12) we have

$$k_1 = \tanh\wp k_2. \tag{3.15}$$

On the other hand, using first equation of (3.7), we obtain that k_2 is a constant. Similarly, k_1 is a constant. Hence, the proof is completed. □

Theorem 3.5 *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic slant helix with timelike normal in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the parametric equations of γ are*

$$x^1(s) = -\sinh\wp s + a_1,$$

$$x^2(s) = -\frac{\cosh\wp e^{-\sinh\wp s + a_1}}{2\left(D_1^2 + \sinh^2\wp\right)}\left[\left(\sinh\wp - D_1\right)\cos\left[D_1 s + D_2\right]\right. \tag{3.16}$$

$$+\left(\sinh\wp+D_1\right)\sin\left[D_1 s+D_2\right]\right]+a_2,$$

$$x^3\left(s\right)=-\frac{\cosh\wp e^{\sinh\wp s-a_1}}{2\left(D_1^2+\sinh^2\wp\right)}\left[\left(\sinh\wp-D_1\right)\cos\left[D_1 s+D_2\right]\right.$$

$$\left.+\left(\sinh\wp+D_1\right)\sin\left[D_1 s+D_2\right]\right]+a_3,$$

where D_1, D_2, a_1, a_2, a_3 *are constants of integration.*

Proof: Suppose that γ is a non geodesic spacelike biharmonic slant helix according to Bishop frame.

Since the curve γ is a spacelike slant helix, i.e. the vector $\mathbf{M}_1$ makes a constant angle with the constant timelike vector called the axis of the slant helix. So, without loss of generality, we take the axis of a slant helix as being parallel to the timelike vector $\mathbf{X}_1$. Then, using first equation of (3.9), we get

$$M_1^1=g\left(\mathbf{M}_1,\mathbf{X}_1\right)=\cosh\wp. \tag{3.17}$$

On other hand, the vector $\mathbf{M}_1$ is a unit timelike vector, so the following condition is satisfied:

$$\left(M_1^2\right)^2+\left(M_1^3\right)^2=-1+\cosh^2\wp. \tag{3.18}$$

The general solution of (3.18) can be written in the following form:

$$M_1^2=\sinh\wp\cos\phi\left(s\right), \tag{3.19}$$

$$M_1^3=\sinh\wp\sin\phi\left(s\right),$$

where ϕ is an arbitrary function of s.

So, substituting the components M_1^1, M_1^2 and M_1^3 in the first equation of (3.9), we have the following equation

$$\mathbf{M}_1=\cosh\wp\mathbf{X}_1+\sinh\wp\cos\phi\left(s\right)\mathbf{X}_2+\sinh\wp\sin\phi\left(s\right)\mathbf{X}_3. \tag{3.20}$$

If we substitute (3.20) in (3.3), we may choose

$$\phi'\left(s\right)\phi''\left(s\right)=0. \tag{3.21}$$

The general solution of (3.21) is

$$\phi\left(s\right)=D_1 s+D_2, \tag{3.22}$$

where D_1, D_2 are constants of integration.

Thus (3.20) and (3.22), imply

$$\mathbf{M}_1=\cosh\wp\mathbf{X}_1+\sinh\wp\cos\left[D_1 s+D_2\right]\mathbf{X}_2+\sinh\wp\sin\left[D_1 s+D_2\right]\mathbf{X}_3. \tag{3.23}$$

Using (2.1) in (3.23), we can choose

$$\mathbf{M}_2 = -\sin[D_1 s + D_2]\,\mathbf{X}_2 + \cos[D_1 s + D_2]\,\mathbf{X}_3.$$

From above equations we get

$$\mathbf{T} = -\sinh\wp \mathbf{X}_1 - \cosh\wp\cos[D_1 s + D_2]\,\mathbf{X}_2 - \cosh\wp\sin[D_1 s + D_2]\,\mathbf{X}_3. \quad (3.24)$$

By the formula of the (2.1), we have

$$\mathbf{T} = (-\sinh\wp, -\frac{1}{2}\cosh\wp e^{\sinh\wp\theta + a_1}(\cos[D_1 s + D_2] + \sin[D_1 s + D_2]),$$

$$-\frac{1}{2}\cosh\wp e^{-\sinh\wp\theta - a_1}(\cos[D_1 s + D_2] - \sin[D_1 s + D_2])), \quad (3.25)$$

where a_1 is constant of integration.

Integrating both sides of (3.25), we have (3.16) as desired. Thus, the proof is completed. □

We may use Mathematica in Theorem 3.5, yields

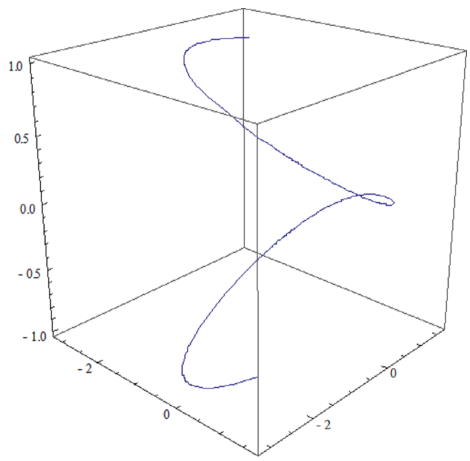

Figure 1.

In terms of Eqs. (3.15) and (3.16), we may give:

Corollary 3.6 *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic slant helix with timelike normal in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then,* *the parametric equations of γ in terms of Bishop curvatures are*

$$
\begin{aligned}
x^{1}(s) &= -\frac{k_1}{k_2}\cosh \wp s + a_1, \\
x^{2}(s) &= -\frac{\frac{k_2}{k_1}\sinh \wp e^{-\frac{k_1}{k_2}\cosh \wp s + a_1}}{2\left(D_1^2 + \sinh^2 \wp\right)}[\left(\frac{k_1}{k_2}\cosh \wp - D_1\right)\cos\left[D_1 s + D_2\right] \\
&\quad + \left(\frac{k_1}{k_2}\cosh \wp + D_1\right)\sin\left[D_1 s + D_2\right]] + a_2, \\
x^{3}(s) &= -\frac{\frac{k_2}{k_1}\sinh \wp e^{\frac{k_1}{k_2}\cosh \wp s - a_1}}{2\left(D_1^2 + \sinh^2 \wp\right)}[\left(\frac{k_1}{k_2}\cosh \wp - D_1\right)\cos\left[D_1 s + D_2\right] \\
&\quad + \left(\frac{k_1}{k_2}\cosh \wp + D_1\right)\sin\left[D_1 s + D_2\right]] + a_3,
\end{aligned}
$$

where D_1, D_2, a_1, a_2,a_3 are constants of integration.

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