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On Mannheim Curves in Riemann-Otsuki Space $R - O_3$

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ABSTRACT: In this paper, we studied the Mannheim partner curves and their new characterizations are also obtained in Riemann-Otsuki space $R - O_3$.

Key Words: Riemann-Otsuki spaces, Mannheim curve, Frenet formula.

Contents

1. Introduction

The basis of the theory of Otsuki spaces was introduced by T. Otsuki and A. Moor. The metric used to determine the observed space is Weyl-Otsuki's or Riemann-Otsuki's kind in [1,2]. Many authors studied on the theory by different aspects in [1,2,3]. Especially on curve theory, F. N. Djerdji obtained the Frenet formula of the Riemann-Otsuki space in terms of covariant and contravariant part of the connections in [1], and also observed auto-parallel curves of Riemann-Otsuki space in [3].

On the other hand for differential geometric point of view it is fundamental to study special curves and their characterizations. Mannheim curve, one of the special one, has many interesting features. The notion of Mannheim curve was introduced by A. Mannheim in 1878. These curves are characterized in Euclidean 3-space E^3 with respect to the curvature and torsion in the following way. A curve is a Mannheim curve if and only if its curvature k_1 and the torsion k_2 satisfies the equation $k_1 = \lambda\left(k_1^2 + k_2^2\right)$, where λ is a nonzero constant. R. Blum studied Mannheim curves by using the Riccati equation in [4]. Recently, H. Liu and F. Wang examined the Mannheim partner curves in Euclidean 3-space and Minkowski 3-space. They showed the necessary and sufficient conditions for the Mannheim partner curves in [5]. H. Öztekin and M. Ergüt focused on null Mannheim curves

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in Minkowski 3-space in [6]. The Mannheim curves have studied various scientists in different areas.

In the present paper, firstly we give a short view of the basis of Riemann-Otsuki space and secondly we study the Mannheim partner curves in $R-O_3$ and obtain some new characterizations for this curve.

2. Preliminaries

We obtain $R-O_n$ spaces from $W-O_n$ spaces assuming that the following relation $\nabla_k g_{ij} = \gamma_k g_{ij}$ holds $\gamma_k = 0$. Then $\nabla_k g_{ij} = 0$. Namely, $R-O_n$ space is an n-dimensional differentiable manifold with Riemannian metric tensor g_{ij} (det $\|g_{ij}\| \neq 0$) and connection of Otsuki. The basic elements of the $R-O_n$ spaces are metric g_{ij} and existence of a-priori P^i_j tensor (with respect to the local coordinates x^i of an n-dimensional differentiable manifold) which satisfies det $\left\|P^i_j\right\| \neq 0$, hence Q^i_j inverse tensor exists. The relation between tensor P^i_j and an inverse Q^i_j is given in the following way

$$a)\ P^i_j Q^j_r = \delta^i_r, \qquad b) P^i_j g_{ia} = P^i_a g_{ij}. \tag{2.1}$$

In Otsuki spaces the covariant differential of the tensor T^i_j is defined by

$$DT^i_j = P^i_a P^j_b \overline{D} T^a_b = P^i_a P^j_b \left(\partial_k T^a_b +' \Gamma^a_{rk} T^r_b -'' \Gamma^r_{bk} T^a_r \right) dx^k. \tag{2.2}$$

The Leibnitz formula does not hold for this differential. The differential $\overline{D}$ is the basic covariant differential. The different coefficients of the connection are the characteristic of the Otsuki spaces, and here are

$$\delta^i_j\,|_k =' \Gamma^i_{jk} -'' \Gamma^i_{jk} \neq 0. \tag{2.3}$$

Tensor P^i_j and the coefficient of the connection $'\Gamma^i_{jk}$ and $''\Gamma^i_{jk}$ satisfy the following Otsuki's relation

$$\partial_k P^i_j +'' \Gamma^i_{ak} P^a_j - P^{i'}_a \Gamma^a_{jk} = 0. \tag{2.4}$$

In Otsuki spaces it is possible to determine the covariant differentials D and $\overline{D}$ with respect only to the co-resp. contravariant part of the connection. So

$$'\overline{D} T^i_j =' \nabla_k T^i_j dx^k = \left(\partial_k T^i_j +' \Gamma^i_{rk} T^r_j -' \Gamma^r_{jk} T^i_r \right) dx^k, \tag{2.5}$$

holds. For this basic covariant differential the Leibnitz formula holds. The basic covariant differential $''\overline{D}$ can be defined in the same way.

It is characteristic that the basic covariant differential $'\overline{D}$ is identical in the case of contravariant indices with the basic covariant differential $\overline{D}$, and similarly in the case of covariant indices the basic covariant differential $''\overline{D}$ is identical with the basic covariant differential $\overline{D}$.

In the following we shall use the relations

$$'\overline{D}g_{ij} = dg_{ij} - \left({}'\Gamma^r_{ik}g_{rj} + {}'\Gamma^r_{jk}g_{ir}\right)dx^k, \tag{2.6}$$

$$''\overline{D}g_{ij} = dg_{ij} - \left({}''\Gamma^r_{ik}g_{rj} + {}''\Gamma^r_{jk}g_{ir}\right)dx^k = 0, \tag{2.7}$$

$$'\overline{D}g^{ra} = -g^{ia}g^{jr}\left({}'\overline{D}g_{ij}\right), \tag{2.8}$$

$$''\overline{D}g^{ra} = 0, \tag{2.9}$$

[1,2,3].

The following two sections are quoted from [1].

2.1. The Frenet Formula with Respect to the Contravariant Components of the Vectors

Let $C : x^i = x^i(s)$ be curve in $R-O_n$ at the point P such that s is the arc length parameter. In that point $v_0^i = \frac{dx^i}{ds}$ are the components of the unit tangent vector v_0.

Theorem 2.1. *If $C : x^i(s)$ is the curve of an $R-O_n$ space and v_l, $l = 0, ..., p-1$ $(p < n)$ are mutually orthogonal unit vectors which satisfy the relation*

$$\overline{D}v_l^j = -\kappa_l v_{l-1}^j + \kappa_{l+1}v_{l+1}^j + v_l^r\overline{D}\delta_r^j, \tag{2.10}$$

so that $\kappa_0 = 0$ and if $q = 1, ..., p-1$ then

$$\kappa_q = \left(g_{ij}\left(\overline{D}v_{q-1}^j + \kappa_{q-1}v_{q-2}^j\right)\right), \tag{2.11}$$

and v_{p+1} is the unit vector orthogonal to all before and $\kappa_0 = 0$, $\kappa_n = 0$ holds then the vector v_p satisfies the relation, (2.10) , too.

If we use Otsuki's covariant differential D, then from the connection $D_v^j = P_a^j\overline{D}v^a$ it follows that $\overline{D}v^a = Q_i^a Dv^i$. Applying this on (2.10) , we get

$$Dv_l^j = P_i^j(-\kappa_l v_{l-1}^i + \kappa_{l+1}v_{l+1}^i) + v_l^q Q_q^b D\delta_b^j, \tag{2.12}$$

with respect to $l = 0, ..., n-1$; $\kappa_0 = 0$; $\kappa_n = 0$.

Here, the equation (2.12) is the Frenet formula with respect to the contravariant components of the observed vectors.

Theorem 2.2. *If at the point M of the curve C in the $R-O_n$ space the mutually orthogonal unit vectors $v_0, v_1, ..., v_{n-1}$ satisfying relations (2.10) and (2.11) so that $\kappa_0 = 0$ and $\kappa_n = 0$ hold then (2.12) is the Frenet formula of the curve C of the $R-O_n$ space. This formula is applied with respect to the covariant differential D on the contravariant components of the observed vectors.*

Remark 2.3. *The relation (2.12) is the Frenet formula with respect to the covariant differential $'D$, too, applied on the contravariant components of the vectors.*

Now we can construct the first Frenet formula with respect to the basic covariant differential $''\overline{D}$ such that

$$''\overline{D}v_0^j = \kappa_1 \overline{v}_1^j,$$

where $\overline{v}_1 \perp v_0$. We shall apply the basic covariant differential $''\overline{D}$ to the tangent vectors v_0^i and v_0^j. Then, the belove theorem is given.

Theorem 2.4. *From the connection between the basic covariant differentials* $'\overline{D}$ *and* $''\overline{D}$ *it follows that* $v_1 = \overline{v}_1$ *and the value of* κ_1 *is equivalent to the value of* κ_1^*.

According to the characteristics of the covariant differential $''D$, we can state the following,

Theorem 2.5. *With respect to the basic covariant differential* $''\overline{D}$ *the Frenet formula of the curve* C *of the* $R-O_n$ *space is not different from the known formula of the Riemannian space . If* $v_0, v_1, ..., v_{n-1}$ *are in point* P *of the curve* C *in a suitable way constructed mutually orthogonal unit vectors, then*

$$''Dv_l^j = P_i^j(-\kappa_l^* v_{l-1}^i + \kappa_{l+1}^* v_{l+1}^i), \tag{2.13}$$

is the Frenet formula with respect to the covariant differential applied on the contravariant components of the observed vectors.

2.2. The Frenet Formula with Respect to the Covariant Components of the Vectors

According to the definition $v_{0i} = g_{ij}\frac{dx^j}{ds}$ holds and v_{0i} are the covariant components of the unit tangent vector v_0.

Now we can construct the mutually orthogonal unit vectors v_l $(l = 0, ..., n-1)$ as above, such that

$$'\overline{D}\overline{v}_l\ j = -\kappa_l^{**}\overline{v}_{l-1}^j + \kappa_{l+1}^{**}\overline{v}_{l+1}^j - \overline{v}_{lr}\overline{D}\delta_j^r, \tag{2.14}$$

holds with the remarks $\kappa_0^{**} = 0$, $\kappa_n^{**} = 0$ and if $l = 0, ..., n-2$ then

$$\kappa_{l+1}^{**} = \begin{array}{l} (g_{ij}\left('\overline{D}v_l\ j + \kappa_l^{**}\overline{v}_{l-1}j + \overline{v}_{l+1}j - \overline{v}_l r\overline{D}\delta_j^r\right) \\ \left('\overline{D}v_l\ i + \kappa_l^{**}\overline{v}_{l-1}i - \overline{v}_l q\overline{D}\delta_i^q\right))^{1/2}. \end{array} \tag{2.15}$$

We can now formulate the following

Theorem 2.6. *From the relation* $\overline{v}_l\ i = g_{ij}\overline{v}_l j$ *it follows that the value of* κ_l^{**} *is equal to the value of* κ_l *and* $\overline{v}_l \equiv v_l$ *holds.*

Theorem 2.7. *If in the point* M *of the curve* C *in the* $R-O_n$ *space the mutually orthogonal unit vectors* $v_0, ..., v_{n-1}$ *are constructed so that* v_p $(p = 1, ..., n-1)$ *satisfies*

$$v_p i = \frac{1}{\kappa_p^{**}}\left('\overline{D}v_{p-1}\ i + \kappa_{p-1}^{**}v_{p-2}^i + v_{p-1}r\overline{D}\delta_i^r\right),$$

with $\kappa_0^{**}=0$, $\kappa_n^{**}=0$ *then*

$$'Dv_p\ i = P_i^j\left(-\kappa_p^{**}v_{p-1}^i + \kappa_{p+1}^{**}v_{p+1}^i\right) - v_p r D\delta_i^a Q_a^r, \tag{2.16}$$

is the Frenet formula with respect to the covariant differential $'D$ *applied on the covariant components of the observed vectors.*

If we make above calculations with respect to the covariant differential $''\overline{D}$, according to equation (2.9) and the fact that in the case of covariant indices the basic covariant differentials $\overline{D}$ and $''\overline{D}$ are not different, it follows that this case is not different from the observation of Riemannian space. We can only say the following.

Remark 2.8. *The relation*

$$Dv_p\ i = P_i^j\left(-\kappa_p^{***}v_{p-1}\ i + \kappa_{p+1}^{***}v_{p+1}\ i\right), \tag{2.17}$$

is the Frenet formula with respect to Otsuki's covariant differential D *applied on the covariant components of the vectors.*

Here

$$\kappa_{p+1}^{***} = \left(g^{ij}\left(\overline{D}v_p\ i + \kappa_p^{***}v_{p-1}\ i\right)\left(\overline{D}v_p\ j + \kappa_p^{***}v_{p-1}\ j\right)\right)^{1/2}, \tag{2.18}$$

or using the $\overline{D}v\ i =' \overline{D}v\ i + v_r\overline{D}\delta_i^r$ *we get*

$$\begin{aligned}\kappa_{p+1}^{***} &= (g^{ij}\left('\overline{D}v_p\ i + \kappa_p^{***}v_{p-1}\ i + v_p\ r\overline{D}\delta_i^r\right)\\ &\quad \left('\overline{D}v_p\ j + \kappa_p^{***}v_{p-1}\ j + v_{pq}\overline{D}\delta_j^q\right))^{1/2},\end{aligned} \tag{2.19}$$

[1,2].

3. Mannheim Curves on Riemann-Otsuki Space $R-O_3$ with the Otsuki Covariant Differential

In this sections, we will define the Manheim curves in Riemann-Otsuki space $R-O_3$. We will get the necessary and sufficient conditions for the Mannheim curves in Riemann-Otsuki space $R-O_3$.

Definition 3.1. *Let* $X(s)$ *and* $X^*(s)$ *be regular curves in* $R-O_3$. $\left\{v_0^j\ (s), v_1^j\ (s), v_2^j\ (s)\right\}$ *and* $\left\{v_0^{*j}(s^*), v_1^{*j}(s^*), v_2^{*j}(s^*)\right\}$ *are Frenet frames, respectively on these curves.*

If there exist a corresponding relationship between the space curves $X(s)$ and $X^*(s^*)$ such that the binormal lines of $X(s)$ coincides with the principal normal lines of $X^*(s^*)$ at the corresponding points of the curves. Thus (X, X^*) is called Mannheim pair.

Theorem 3.2. *Let (X, X^*) be a Mannheim pair in $R - O_3$ and X, X^* are given (I, X) , (I, X^*) coordinate neighbourhood respectively. If $\lambda^{'} = Q_a^b D\delta_b^j$ then $d\left(X\left(s\right), X^*\left(s^*\right)\right) = const.$*

Proof: From Definition 3.1 we may write

$$X^*\left(s^*\right) = X\left(s\right) + \lambda\left(s\right) v_2^j\left(s\right). \tag{3.1}$$

Let us assume arclength parameters of X and X^* as s and s^*, respectively. Then we get

$$v_0^{*j}\left(s^*\right)\frac{ds^*}{ds} = X^{'}\left(s\right). + \lambda'\left(s\right) v_2^j\left(s\right) + \lambda\left(s\right)\left(v_2^j\left(s\right)\right)^{'}, \tag{3.2}$$

$$Dv_2^j\left(s\right) = P_i^j\left(-\kappa_2 v_1^i\right) + v_2^a Q_a^b D\delta_b^j, \tag{3.3}$$

$$v_0^{*j}\left(s^*\right)\frac{ds^*}{ds} = v_0^j\ \left(s\right) + \lambda^{'}\left(s\right) v_2^j\left(s\right) + \lambda\left(s\right)\left(P_i^j\left(-\kappa_2 v_1^i\right) + v_2^a Q_a^b D\delta_b^j\right), \tag{3.4}$$

$$v_0^{*j}\left(s^*\right)\frac{ds^*}{ds} = v_0^j\left(s\right) - \lambda\left(s\right) P_i^j \kappa_2 v_1^i\left(s\right) + \lambda^{'} v_2\ j\left(s\right) + \lambda\left(s\right) v_2^a Q_a^b D\delta_b^j. \tag{3.5}$$

Then using linear dependency of $v_1^{*j}\left(s^*\right)$ and $v_2^j\left(s\right)$, we get $\left\langle v_0^{*j}\left(s^*\right), v_2^j\left(s\right)\right\rangle = 0$ and from (3.5) we have

$$\begin{aligned} g_{ij} v_0^{*^j}\left(s^*\right) v_2^j\ \left(s\right)\frac{ds^*}{ds} &= g_{ij} v_0^j\ \left(s\right) v_2^j\left(s\right) - g_{ij}\lambda\left(s\right) P_i^j \kappa_2 v_1^i\left(s\right) v_2^j\left(s\right) \\ &+ \lambda^{'}\left(s\right) g_{ij} v_2^j\left(s\right) v_2^j\left(s\right) + \lambda\left(s\right) g_{ij} v_2^a v_2^j Q_a^b D\delta_b^j, \end{aligned} \tag{3.6}$$

then we write

$$\lambda^{'}\left(s\right) + \lambda\left(s\right) Q_a^b D\delta_b^j = 0, \tag{3.7}$$

$$\lambda\left(s\right) = -\frac{\lambda^{'}}{Q_a^b D\delta_b^j}.$$

From the hypothesis we recall that $\lambda^{'}\left(s\right) = Q_a^b D\delta_b^j$. Therefore

$$d\left(X\left(s\right), X^*\left(s^*\right)\right) = \left\|X^*\left(s^*\right) - X\left(s\right)\right\| = \left\|\lambda v_2^j\left(s\right)\right\| = \left\|\lambda\right\| = \left\|-\frac{\lambda^{'}}{Q_a^b D\delta_b^j}\right\| = const. \tag{3.8}$$

□

Theorem 3.3. *There exists a curve denoted by X such that (X, X^*) be a Mannheim pair in $R - O_3$.*

Proof: Since $v_1^{*j}\left(s^*\right)$ and $v_2^j\left(s\right)$ are linearly dependent, from (3.1) we have

$$X\left(s\right) = X^*\left(s^*\right) - \lambda v_1^{*j}\left(s^*\right). \tag{3.9}$$

Then, one can find a X curve for all values of λ, where λ is a nonzero constant.

□

Theorem 3.4. *Let (X, X^*) be a Mannheim pair in $R-O_3$. The torsion k_2 of X satisfies the following equation*

$$k_2 = \frac{k_1^*}{\lambda P_i^j \kappa_2^* g_j^i},$$

where we denote the curvature and torsion of X^* with k_1^* and k_2^*, respectively.

Proof: By considering λ is a nonzero constant in equation (3.5), we have

$$\begin{aligned} v_0^{*j}(s^*)\frac{ds^*}{ds} &= v_0^j(s) - \lambda P_i^j \kappa_2 v_1^i(s) + \lambda^{'} v_2\ j(s) + \lambda(s) v_2^a Q_a^b D\delta_b^j, \\ v_0^{*j}(s^*)\frac{ds^*}{ds} &= v_0^j(s) - \lambda P_i^j \kappa_2 g_j^i v_1^j(s) + \lambda^{'}\left(1-g_j^a\right) v_2^j(s). \end{aligned} \tag{3.10}$$

Note that $\lambda^{'}$ is a constant, so without loss of generality we may assume $\lambda^{'} = 0$, then recalling (2.12) we get

$$v_0^{*j}(s^*)\frac{ds^*}{ds} = v_0^j(s) - \lambda P_i^j \kappa_2 g_j^i v_1^j(s). \tag{3.11}$$

Then we can write above equation such that θ is the angle between v_0^j and v_0^{*j} at the corresponding points of X and X^*, respectively,

$$\begin{aligned} v_0^{*j}(s^*)\frac{ds^*}{ds} &= \cos\theta v_0^j(s) + \sin\theta v_1^j(s), \\ v_2^{*j}(s^*)\frac{ds^*}{ds} &= -\sin\theta v_0^j(s) + \cos\theta v_1^j(s). \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\cos\theta = \frac{ds}{ds^*}, \ \ \sin\theta = -\lambda P_i^j \kappa_2 g_j^i \frac{ds}{ds^*}. \tag{3.13}$$

If we differentiate of (3.9) with respect to s, we get

$$v_0^j(s) = \left(1 + \lambda P_i^{*j}\kappa_1^* g_j^i\right) v_0^{*j}(s)\frac{ds^*}{ds} - \lambda P_i^{*j}\kappa_2^* g_j^i v_2^{*j}(s)\frac{ds^*}{ds}. \tag{3.14}$$

From (3.12), we have

$$\begin{aligned} v_0^j(s) &= \cos\theta v_0^{*j}(s^*) - \sin\theta v_2^{*j}(s^*), \\ v_1^j(s) &= \sin\theta v_0^j(s) + \cos\theta v_2^{*j}(s^*). \end{aligned} \tag{3.15}$$

Taking into consideration equation (3.14) and (3.15), we get

$$\cos\theta = \left(1 + \lambda P_i^{*j}\kappa_1^* g_j^i\right)\frac{ds^*}{ds}, \ \ \sin\theta = \lambda P_i^{*j}\kappa_2^* g_j^i v_2^{*j}(s)\frac{ds^*}{ds}. \tag{3.16}$$

From (3.13) and (3.16), we can write

$$\cos^2\theta = 1 + \lambda P_i^{*j}\kappa_1^* g_j^i, \qquad \sin^2\theta = -\lambda g_j^{2i} P_i^j \kappa_2 P_i^{*j}\kappa_2^*. \tag{3.17}$$

Then, if we consider equation (3.17), one can obtain

$$k_2 = \frac{k_1^*}{\lambda P_i^j \kappa_2^* g_j^i}.$$

□

Corollary 3.5. *Let (X, X^*) be a Mannheim pair in $R - O_3$. Then product of the torsions κ_2 and κ_2^* at the corresponding points of the Mannheim curves are not constant where κ_2 and κ_2^* are the torsions of the curves X and X^*, respectively. Then Schell's theorem for Mannheim curves in $R - O_3$ is not valid.*

Theorem 3.6. *Let X and X^* are given (I, X) and (I, X^*) coordinate neighbourhoods respectively and (X, X^*) be a Mannheim pair in $R-O_3$. Then the relationship between the curvature and the torsion of the curve X^* is*

$$\mu k_2^* - \lambda k_1^* = 1, \tag{3.18}$$

where λ, μ are non-zero real numbers.

Proof: If we considering equation (3.16), we can write

$$\frac{\cos\theta}{1 + \lambda g_j^i P_i^{*j}\kappa_1^*} = \frac{\sin\theta}{\lambda g_j^i P_i^{*j}\kappa_2^*},$$

then, we can easily show that

$$\mu k_2^* - \lambda k_1^* = 1.$$

□

Corollary 3.7. *Let (X, X^*) be a Mannheim pair in $R - O_3$. Then , there is exist a linear relationship between κ_1^* and κ_2^* . Namely, Bertrand's theorem is valid for the Mannheim curves.*

Theorem 3.8. *Let (X, X^*) be a Mannheim pair in $R - O_3$. Then the following statements hold for the curvatures and the torsions of the curves X and X^*, respectively:*

i) $k_1 = -\dfrac{d\theta}{ds P_i^j g_j^i}$,

ii) $k_2 = -\dfrac{P_i^{*j} k_1^* \sin\theta \dfrac{ds^*}{ds} - P_i^{*j} k_2^* \cos\theta \dfrac{ds^*}{ds}}{P_i^j}$,

iii) $k_1^* = \dfrac{P_i^j k_2 \sin\theta}{P_i^{*j}} \dfrac{ds}{ds^*}$,

iv) $k_2^* = \dfrac{P_i^j k_2 \cos\theta}{P_i^{*j}} \dfrac{ds}{ds^*}$.

Proof: $i)$ If we derive the equation $\left\langle v_0^j, v_0^{*j} \right\rangle = \cos\theta$, we have

$$\left\langle P_i^j \kappa_1 v_1^i + v_0^a Q_a^b D\delta_b^j, v_0^{*j} \right\rangle + \left\langle v_0^j, P_i^{*j} \kappa_1^* v_1^{*i} + v_0^{*a} Q_a^{*b} D^* \delta_b^j \right\rangle = -\sin\theta \frac{d\theta}{ds},$$

by considering v_1^{*j} and v_2^j are linearly dependent, using (3.15) and (2.12) , we obtain

$$k_1 = -\frac{d\theta}{ds P_i^j g_j^i}.$$

By using the $\left\langle v_1^j, v_1^{*j} \right\rangle = 0$, $\left\langle v_2^j, v_0^{*j} \right\rangle = 0$ and $\left\langle v_2^j, v_2^{*j} \right\rangle = 0$, we can easily show that the proofs of (ii) , (iii) and (iv) holds. □

Corollary 3.9. *Let (X, X^*) be a Mannheim pair in $R-O_3$. Then the relationship between the curvatures and the torsions of the curves X and X^* is given by*

$$k_1^{*2} + k_2^{*2} = \left(\frac{P_i^j}{P_i^{*j}} \right)^2 k_2^2 \left(\frac{ds}{ds^*} \right)^2.$$

Theorem 3.10. *A curve is a Mannheim curve in $R-O_3$ if and only if*

$$k_1^* = -\lambda P_i^{*j} g_j^i \left(k_1^{*2} - k_2^{*2} \right),$$

where k_1^* and k_2^* is the curvatures and the torsions of the curve X^*, respectively, and λ is a nonzero constant.

Proof: By differentiating the equation (3.9) with respect to s^*, we have

$$v_0^j \frac{ds}{ds^*} = \left(1 + \lambda P_i^{*j} \kappa_1^* g_j^i \right) v_0^{*j} - \lambda P_i^{*j} \kappa_2^* g_j^i v_2^{*j}. \tag{3.19}$$

Then, if we differentiate (3.19) with respect to s^* and both sides of the equation (3.19) is multiply with v_2^j, then we get

$$k_1^* = -\lambda P_i^{*j} g_j^i \left(k_1^{*2} - k_2^{*2} \right),$$

□

Remark 3.11. *If Otsuki's covariant differential D applied on the covariant components of the vector. The above all calculating holds as the same of Riemannian case.*

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