Bol. Soc. Paran. Mat. (3s.) **v. 31** 1 (2013): 193–204.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v31i1.15264

Properties of Γ^2 defined by a modulus function

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ABSTRACT: In this article, we introduces the generalized difference paranormed double sequence spaces $\Gamma^2\left(\Delta^m_\gamma, f, p, q, s\right)$ and $\Lambda^2\left(\Delta^m_\gamma, f, p, q, s\right)$ defined over a semi-normed sequence space (X, q).

Key Words: entire sequence, analytic sequence, modulus function, semi norm, difference sequence, double sequence,duals.

Contents

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces are due to Bromwich [4]. Later on, the double sequence spaces were studied by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - |^{t_{mn}} = 1\, for\, some \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

2000 *Mathematics Subject Classification:* 40A05,40C05,40D05

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zeltser [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.
We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by Γ^2. By ϕ, we denote the set of all finite sequences.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is 1 in the $(i,j)^{th}$ place.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Orlicz [13] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\,(1 \le p < \infty)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [14], Mursaleen et al. [11], Bektas and Altin [3], Tripathy et al. [18], Rao and Subramanian [15], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].

Recalling [13] and [6], an Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [12] and further discussed by Ruckle [16] and Maddox [8], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \le KM(u)\,(u \ge 0)$. The Δ_2- condition is equivalent to $M(\ell u) \le K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\, some\, \rho > 0\right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \le p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.
If X is a sequence space, we give the following definitions:

(i)$X^{'}$ = the continuous dual of X;

(ii)$X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty,\ for\, each\, x \in X\right\}$;

(iii)$X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, is\, convegent,\; for\, each\, x \in X\right\}$;

(iv)$X^{\gamma} = \left\{a = (a_{mn}) : sup_{MN} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, for\, eachx \in X\right\}$;

(v)$let\, X\, beanFK - space \supset \phi;\, then\, X^{f} = \left\{f(\Im_{mn}) : f \in X^{'}\right\}$;

(vi)$X^{\delta} = \left\{a = (a_{mn}) : sup_{mn} \left|a_{mn}x_{mn}\right|^{1/m+n} < \infty,\; for\, each\, x \in X\right\}$;

$X^{\alpha}.X^{\beta}, X^{\gamma}$ and X^{δ} are called $\alpha - (orK\ddot{o}the - Toeplitz)$dual of $X, \beta - (orgeneralized - K\ddot{o}the - Toeplitz) dual\, of\, X, \gamma -\; dual\, of\, X,\; \delta \;-\; dual\, of\, X\, respectively. X^{\alpha}$ is defined by Gupta and Kamptan [20]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\alpha} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference spaces of single sequences was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_0 and ℓ_{∞} denote the classes of all, convergent,null and bounded sclar valued single sequences respectively. The above difference spaces are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \Gamma^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

2. Definitions and Preliminaries

Γ_M^2 and Λ_M^2 denote the Pringscheim's sense of double Orlicz space of entire sequences and Pringscheim's sense of double Orlicz space of bounded sequences respecctively.

The notion of a modulus function was introduced by Nakano [12]. We recall that a modulus f is a function from $[0,\infty) \to [0,\infty)$, such that
(1) $f(x) = 0$ if and only if $x = 0$
(2) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from condition (4) that f is continuous on $[0,\infty)$.
Let $p = (p_{mn})$ be a sequence of strictly positive real numbers and $s \geq 0$. Let X be semi normed space over the field $\mathbb{C}$ of complex numbers with the semi norm q. The symbol $w^2(X)$ denotes the space of all sequences defined over X. such that

$p_{mn} > 0$ for all m, n and $sup_{mn} p_{mn} = H < \infty, v = (v_{mn})$ be any fixed sequence of non-zero complex numbers and $m \in \mathbb{N}$ be fixed.
Define the sets :

$\Gamma^2_M = \left\{x \in w^2 : \left(M\left(\frac{((|x_{mn}|)^{1/m+n}}{\rho}\right)\right) \to 0\, as\, m, n \to \infty\, for\, some\, \rho > 0\right\}$ and

$$\Lambda^2_M = \left\{x \in w^2 : sup_{m,n\geq 1}\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)\right) < \infty\, for\, some\, \rho > 0\right\}.$$

The space Γ^2_M and Λ^2_M is a metric space with the metric

$$d(x, y) = inf\left\{\rho > 0 : sup_{m,n\geq 1}\left(M\left(\frac{|x_{mn} - y_{mn}|}{\rho}\right)\right)^{1/m+n} \leq 1\right\}$$

Now we define the following sequence spaces:

$\Gamma^2(\Delta^m_v, f, p, q, s) =$
$\left\{x \in w^2(X) : (mn)^{-s}\left(f\left(q\left(|\Delta^m_v x_{mn}|\right)^{1/m+n}\right)\right)^{p_{mn}} \to 0\,(m, n \to \infty), s \geq 0\right\}$

$\Lambda^2(\Delta^m_v, f, p, q, s) = \left\{x \in w^2(X) : sup_{mn}(mn)^{-s}\left(f\left(q\left(|\Delta^m_v x_{mn}|\right)^{1/m+n}\right)\right)^{p_{mn}} < \infty, s \geq 0\right\}$
where
$\Delta^0_v x_{mn} = (v_{mn} x_{mn}), \Delta_v x_{mn} = (v_{mn} x_{mn} - v_{mn+1} x_{mn+1} - v_{m+1n} x_{m+1n} + v_{m+1n+1} x_{m+1n+1})$

$\Delta^m_v x_{mn} = \Delta\Delta^{m-1}_v x_{mn} = \left(\Delta^{m-1}_v x_{mn} - \Delta^{m-1}_v x_{mn+1} - \Delta^{m-1}_v x_{m+1n} + \Delta^{m-1}_v x_{m+1n+1}\right)$
where f is a modulus function. The following inequality will be used through this article. Let $p = (p_{mn})$ be a sequence of positive real numbers with $0 < p_{mn} \leq sup_{mn} p_{mn} = H$, $D = max\left(1, 2^{H-1}\right)$. Then, for $a_{mn}, b_{mn} \in \mathbb{C}$, we have

$$|a_{mn} + b_{mn}|^{p_{mn}} \leq D\left\{|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}}\right\}$$

Some well-known spaces are obtained by specializing f, s, q, v, and m.

(1) If $f(x) = x$, $m = 0$, $v = (v_{mn}) = \begin{pmatrix} 1, & 1, & ...1, & 1, & 0, ... \\ 1, & 1, & ...1, & 1, & 0, ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 1, & 1, & ...1, & 1, & 0, ... \\ 0, & 0, & ...0, & 0, & 0, ... \end{pmatrix}$
with 1 in the upto $(m, n)^{th}$ position and zero other wise and $q(x) = |x|$, then $\Gamma^2(\Delta^m_v, f, p, q, s) = \Gamma^2(p, s)$ and $\Lambda^2(\Delta^m_v, f, p, q, s) = \Lambda^2(p, s)$.

(2) If $f(x) = x$, $m = 0$, $v = (v_{mn}) = \begin{pmatrix} 1, & 1, & ...1, & 1, & 0,... \\ 1, & 1, & ...1, & 1, & 0,... \\ . & & & & \\ . & & & & \\ . & & & & \\ 1, & 1, & ...1, & 1, & 0,... \\ 0, & 0, & ...0, & 0, & 0,... \end{pmatrix}$

with 1 in the upto $(m,n)^{th}$ position and zero other wise and $q(x) = |x|, s = 0$, then $\Gamma^2(\Delta_v^m, f, p, q, s) = \Gamma^2(p)$ and $\Lambda^2(\Delta_v^m, f, p, q, s) = \Lambda^2(p)$.

(3) If $m = 0, v = (v_{mn}) = \begin{pmatrix} 1, & 1, & ...1, & 1, & 0,... \\ 1, & 1, & ...1, & 1, & 0,... \\ . & & & & \\ . & & & & \\ . & & & & \\ 1, & 1, & ...1, & 1, & 0,... \\ 0, & 0, & ...0, & 0, & 0,... \end{pmatrix}$

with 1 in the upto $(m,n)^{th}$ position and zero other wise and $q(x) = |x|$, then $\Gamma^2(\Delta_v^m, f, p, q, s) = \Gamma^2(p, f, s)$ and $\Lambda^2(\Delta_v^m, f, p, q, s) = \Lambda^2(p, f, s)$.

3. Definitions

Definition 3.1. *Let p,q be semi norms on a vector space X. Then p is said to be stronger that q if (x_{mn}) is a sequence such that $p(x_{mn}) \to 0$, whenever $q(x_{mn}) \to 0$. If each is stronger than the other, then the p and q are said to be equivalent.*

Lemma 3.2. *Let p and q be semi norms on a linear space X. Then p is stronger than q if and only if there exists a constant M such that $q(x) \leq Mp(x)$ for all $x \in X$.*

Definition 3.3. *(1) A sequence space X is said to be solid or normal if $(\alpha_{mn}x_{mn}) \in X$ whenever $(x_{mn}) \in X$ and for all sequences of scalars (α_{mn}) with $|\alpha_{mn}| \leq 1$, for all $m, n \in \mathbb{N}$.*
(2) Symmetric if $(x_{mn}) \in X$ implies $\left(x_{\pi_{(mn)}}\right) \in X$, where $\pi(mn)$ is a permutation of $\mathbb{N} \times \mathbb{N}$;
(3) Sequence algebra if $x \cdot y \in X$ whenever $x, y \in X$.

Definition 3.4. *A sequence space X is said to be monotone if it contains the canonical pre-images of all its step spaces.*

Remark 3.5. *From Definition 3.3 and 3.4, it is clear that if a sequence space X is solid then X is monotone.*

Definition 3.6. *A sequence space X is said to be convergence free if $(y_{mn}) \in X$ whenever $(x_{mn}) \in X$ and $x_{mn} = 0$ implies that $y_{mn} = 0$.*

4. Main Results

Theorem 4.1. *Let $p = (p_{mn})$ be a analytic sequence. Then $\Gamma^2(\Delta_v^m, f, p, q, s)$ are linear spaces*

Proof: Let $x, y \in \Gamma^2(\Delta_v^m, f, p, q, s)$. For $\lambda, \mu \in \mathbb{C}$, there exists positive integers M_λ and N_μ, such that $|\lambda| \le M_\lambda$ and $|\mu| \le N_\mu$. Since f is subadditive, q is a seminorm, and Δ_v^m is linear, we have
$(mn)^{-s}\left[f\left(q\left(|\Delta_v^m(\lambda x_{mn} + \mu y_{mn})|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}}$
$\le D\left(max\left(1, |M_\lambda|^H\right)\right)(mn)^{-s}\left[f\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}} +$
$D\left(max\left(1, |N_\mu|^H\right)\right)(mn)^{-s}\left[f\left(q\left(|\Delta_v^m y_{mn}|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}} \to 0$ as $m, n \to \infty$. This means that $\lambda x + \mu y \in \Gamma^2(\Delta_v^m, f, p, q, s)$. Hence, $\Gamma^2(\Delta_v^m, f, p, q, s)$ is a linear space. □

Theorem 4.2. *The space $\Gamma^2(\Delta_v^m, f, p, q, s)$ is a paranormed space, paranormed by $g(x) = \sum_{i=1}^{\mu}\sum_{j=1}^{\eta} f(q(v_{ij}x_{ij})) + sup_{mn}(mn)^{-s}\left[f\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}/M}$ where $M = max(1, sup_{mn}p_{mn})$*

Proof: Clearly $g(x) = g(-x)$ for all $x \in \Gamma^2(\Delta_v^m, f, p, q, s)$. It is trivial that $(|\Delta_v^m(x_{mn})|)^{\frac{1}{m+n}} = \overline{\theta}$ for $x_{mn} = \overline{\theta}$,

where $\overline{\theta} = \begin{pmatrix} \theta, & \theta, & ...\theta, & \theta, ... \\ \theta, & \theta, & ...\theta, & \theta, ... \\ . & & & \\ . & & & \\ . & & & \\ \theta, & \theta, & ...\theta, & \theta, ... \end{pmatrix}$ and is the zero element of X. Since $q\left(\overline{\theta}\right) = 0$ and $f(0) = 0$, we get $g\left(\overline{\theta}\right) = 0$. Since $t_{mn} = p_{mn}/M \le 1$, if a_{mn} and b_{mn} are complex numbers, then we have

$$|a_{mn} + b_{mn}|^{t_{mn}} \le D\left\{|a_{mn}|^{t_{mn}} + |b_{mn}|^{t_{mn}}\right\}$$

Since $M \ge 1$, the above inequality implies that
$\sum_{i=1}^{\mu}\sum_{j=1}^{\eta} f(q(v_{ij}x_{ij} + v_{ij}y_{ij})) + sup_{mn}(mn)^{-s}\left[f\left(q\left(|\Delta_v^m(x_{mn} + y_{mn})|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}/M}$
$\le \sum_{i=1}^{\mu}\sum_{j=1}^{\eta} f(q(v_{ij}x_{ij})) + sup_{mn}(mn)^{-s}\left[f\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}/M} +$
$\sum_{i=1}^{\mu}\sum_{j=1}^{\eta} f(q(v_{ij}y_{ij})) + \sup_{mn}(mn)^{-s}\left[f\left(q\left(|\Delta_v^m y_{mn}|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}/M}$
Now, it follows that g is subadditive. Next, let λ be a non-zero scalar. The continuity of scalar multiplication follows from the inequality
$g(\lambda x) \le K_\lambda \sum_{i=1}^{\mu}\sum_{j=1}^{\eta} f(q(v_{ij}x_{ij})) +$
$sup_{mn}(mn)^{-s}\left(K_\lambda^{p_{mn}/M}\right)\left[f\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}/M} \le max\left(1, K_\lambda^{H/M}\right) g(x)$,
where K_λ is an integer such that $|\lambda| < K_\lambda$. This completes the proof. □

Theorem 4.3. *Let f, f_1 and f_2 be modulus functions, q, q_1 and q_2 be seminorms, and s, s_1 and $s_2 \geq 0$. Then,*
(1)$\Gamma^2\left(\Delta_v^m, f_1, p, q, s\right) \subseteq \Gamma^2\left(\Delta_v^m, f \circ f_1, p, q, s\right)$,
(2)$\Gamma^2\left(\Delta_v^m, f_1, p, q, s\right) \bigcap \Gamma^2\left(\Delta_v^m, f_2, p, q, s\right) \subseteq \Gamma^2\left(\Delta_v^m, f_1 + f_2, p, q, s\right)$,
(3) $\Gamma^2\left(\Delta_v^m, f, p, q_1, s\right) \bigcap \Gamma^2\left(\Delta_v^m, f, p, q_2, s\right) \subseteq \Gamma^2\left(\Delta_v^m, f, p, q_1 + q_2, s\right)$,
(4) If q_1 is stronger than q_2, then $\Gamma^2\left(\Delta_v^m, f, p, q_1, s\right) \subseteq \Gamma^2\left(\Delta_v^m, f, p, q_2, s\right)$,
(5) If $s_1 \leq s_2$, then $\Gamma^2\left(\Delta_v^m, f, p, q, s_1\right) \subseteq \Gamma^2\left(\Delta_v^m, f, p, q, s_2\right)$,

Proof: Let $S_{mn} = (mn)^{-s}\left[f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}} \to 0, (m, n \to \infty)$
Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. Now we write

$$I_1 = \left\{(m, n) \in \mathbb{N} : f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right) \leq \delta\right\},$$

$$I_2 = \left\{(m, n) \in \mathbb{N} : f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right) > \delta\right\},$$

If $x \in \Gamma^2\left(\Delta_v^m, f_1, p, q, s\right)$, then for $f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right) > \delta$,

$$f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right) < f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\delta^{-1} < 1 + \left[f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\delta^{-1}\right]$$

where $m, n \in I_2$ and $[u]$ denotes the integer part of u. Given $\epsilon > 0$, by the definition of f, we have for $f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right) > \delta$,
$f\left(f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right) \leq \left(1 + \left[f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\delta^{-1}\right]\right) f(1)$
$\leq 2f(1)\left(f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right)\delta^{-1}$ and hence,

$$(mn)^{-s}\left[f\left(f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right)\right]^{p_{mn}} \leq \left[2f(1)\delta^{-1}\right]^H S_{mn} < \epsilon, (m, n \in I_2) \tag{2}$$

and $m, n > m_2 n_2$.
If $x \in \Gamma^2\left(\Delta_v^m, f_1, p, q, s\right)$, for $\left(f_1\left(f\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right)\right) \leq \delta$,
$\left(f\left(f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right)\right) < \epsilon$, where $(m, n) \in I_1$. Therefore, given $\epsilon > 0$ if $m, n \in I_2$, we have

$$(mn)^{-s}\left[f\left(f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right)\right]^{p_{mn}} \leq (mn)^{-s}\, max\left(\epsilon^{inf p_{mn}}, \epsilon^{sup_{mn} p_{mn}}\right) < \epsilon, \tag{3}$$

$(m, n \in I_1), mn > m_1 n_1$
From (2) and (3) for every $m, n > max\left\{(m_1 n_1), (m_2 n_2)\right\}$,

$$(mn)^{-s}\left[f\left(f_1\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right)\right]^{p_{mn}} < \epsilon.$$

Hence, $x \in \Gamma^2(\Delta_v^m, f \circ f_1, p, q, s)$. Thus, $\Gamma^2(\Delta_v^m, f_1, p, q, s) \subseteq \Gamma^2(\Delta_v^m, f \circ f_1, p, q, s)$
(2) It follows from the inequality

$$(mn)^{-s}\Big[(f_1+f_2)\Big((|\Delta_v^m x_{mn}|)^{\frac{1}{m+n}}\Big)\Big]^{p_{mn}} \leq D\,(mn)^{-s}\Big[f_1\Big(q\,(|\Delta_v^m x_{mn}|)^{\frac{1}{m+n}}\Big)\Big]^{p_{mn}} + D\,(mn)^{-s}\Big[f_2\Big(q\,(|\Delta_v^m x_{mn}|)^{\frac{1}{m+n}}\Big)\Big]^{p_{mn}}.$$

Since (3),(4) and (5) can be established by the same way, we omit the detail. □

Proposition 4.4. *The following inclusion relations hold:*
(1) $\Gamma^2(\Delta_v^m, p, q, s) \subseteq \Gamma^2(\Delta_v^m, f, p, q, s)$,
(2) $\Gamma^2(\Delta_v^m, f, p, q) \subseteq \Gamma^2(\Delta_v^m, f, p, q, s)$,
(3) $\Gamma^2(\Delta_v^m, p, q) \subseteq \Gamma^2(\Delta_v^m, p, q, s)$.

The proof of the inclusions in (1)-(3) is routine verification. So, we leave it to the reader.

Proposition 4.5. *If* $q_1 \cong q_2$, *then* $\Gamma^2(\Delta_v^m, f, p, q_1, s) = \Gamma^2(\Delta_v^m, f, p, q_2, s)$

Theorem 4.6. *For any two sequences* $p = (p_{mn})$ *and* $t = (t_{mn})$ *of strictly positive real numbers and for any two semi norms* q_1 *and* q_2 *on* X, *the spaces* $\Gamma^2(\Delta_v^m, f, p, q_1, s)$ *and* $\Gamma^2(\Delta_v^m, f, p, q_2, s)$ *are not disjoint.*

Proof: Since the zero element belongs to each of the above classes of double sequences, the intersection is non empty. □

Theorem 4.7. *For any two sequences* (p_{mn}) *and* (t_{mn}), *we have* $\Gamma^2(\Delta_v^m, f, t, q) \subset \Gamma^2(\Delta_v^m, f, p, q)$ *if and only if* $liminf \frac{p_{mn}}{t_{mn}} > 0$.

Proof: If we take $y_{mn} = f\left(q\,(|\Delta_v^m x_{mn}|)^{\frac{1}{m+n}}\right)$ for all $m, n \in \mathbb{N}$. □

Theorem 4.8. *For any two sequences* (p_{mn}) *and* (t_{mn}), *the spaces* $\Gamma^2(\Delta_v^m, f, t, q)$ *and* $\Gamma^2(\Delta_v^m, f, p, q)$ *are identical if and only if* $liminf \frac{p_{mn}}{t_{mn}} > 0$ *and if and only if* $liminf \frac{t_{mn}}{p_{mn}} > 0$.

Theorem 4.9. $\Gamma^2(\Delta_v^m, f, p, q, s)$ *is not solid for* $m > 0$
To prove that the space $\Gamma^2(\Delta_v^m, f, p, q, s)$ *is not solid, in general, we give the following counter-example: Let* $X = \mathbb{C}, f(x) = x, q(x) = |x|, \alpha_{mn} = (-1)^{mn}, s = 0$, $v = (v_{mn})$ *with* $v = (v_{mn}) = p_{mn} = 1$ *for all* $m, n \in \mathbb{N}$. *Then,*
$|x_{mn}|^{\frac{1}{m+n}} = (mn)^{m-1} \in \Gamma^2(\Delta_v^m, f, p, q, s)$, *but* $(\alpha_{mn} x_{mn}) \notin \Gamma^2(\Delta_v^m, f, p, q, s)$.

Theorem 4.10. $\Gamma^2(\Delta_v^m, f, p, q, s)$ *is not sequence algebra*

Proof: Let $q(x) = |x|, f(x) = x, s = 0, v = (v_{mn}) = \begin{pmatrix} 1, & 1, & ... & 1 \\ 1, & 1, & ... & 1 \\ . & & & \\ . & & & \\ . & & & \\ 1, & 1, & ... & 1 \end{pmatrix}$ and $p_{mn} = 1$ for all $m, n \in \mathbb{N}$.

Consider $|x_{mn}|^{\frac{1}{m+n}} = (mn)^{m-1}$ and $|y_{mn}|^{\frac{1}{m+n}} = (mn)^{m-1}$, then $x, y \in \Gamma^2(\Delta_v^m, f, p, q, s)$ and $x \cdot y \notin \Gamma^2(\Delta_v^m, f, p, q, s)$. □

Theorem 4.11. *The space $\Gamma^2(\Delta_v^m, f, p, q, s)$ is not convergence free in general*

Proof: To prove that the space $\chi^2(\Delta_v^m, f, p, q, s)$ is not convergence free, in general, we give the following counter-example: Consider the sequences $(\Delta_v^m x_{mn})$, $(\Delta_v^m y_{mn}) \in \Gamma^2(\Delta_v^m, f, p, q, s)$ defined by $(\Delta_v^m x_{mn}) = \left(\frac{1}{m+n}\right)^{m+n}$ and $(\Delta_v^m y_{mn}) = \left(\frac{m-n}{m+n}\right)^{m+n}$ for all $m, n \in \mathbb{N}$. Then,
$(mn)^{-s}\left[f\left(q\left(\frac{1}{(m+n)}\right)\right)\right]^{p_{mn}} \to 0, \quad as \quad m, n \to \infty$, which implies that $(\Delta_v^m x_{mn}) \to 0$, $as\, m, n \to \infty$. Similarly, $(mn)^{-s}\left[f\left(q\left(\frac{m-n}{(m+n)}\right)\right)\right]^{p_{mn}} \to 0 \quad as \quad m, n \to \infty$. But, $\{\Delta_v^m y_{mn}\}$ does not tends to zero, as $m, n \to \infty$. This step completes the proof. □

Theorem 4.12. *Let f be a modulus function. Then $\Gamma^2(\Delta_v^m, f, p, q, s) \subseteq \Lambda^2(\Delta_v^m, f, p, q, s)$ and the inclusions are strict*

Proof:
$(mn)^{-s}\left[f\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}} \le D\,(mn)^{-s}\left[f\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}}$
Then, there exists an integer K such that
$(mn)^{-s}\left[f\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}} \le D\,(mn)^{-s}\left[f\left(q\left(|\Delta_v^m x_{mn}|\right)^{\frac{1}{m+n}}\right)\right]^{p_{mn}} + max\left[1, (K)^H\right].$
Therefore, $x \in \Lambda^2(\Delta_v^m, f, p, q, s)$. □

Example 4.13. *Let $q(x) = |x|, f(x) = 0, s = 0, v = (v_{mn}) = \begin{pmatrix} 1, & 1, & ... & 1 \\ 1, & 1, & ... & 1 \\ . & & & \\ . & & & \\ . & & & \\ 1, & 1, & ... & 1 \end{pmatrix}$ and $p_{mn} = 1$ for all $m, n \in \mathbb{N}$. Then $x = (mn)^{m(m+n)} = (mn)^{m^2+mn} \in \Lambda^2(\Delta_v^m, f, p, q, s)$, but $x \notin \Gamma^2(\Delta_v^m, f, p, q, s)$. Since $|\Delta_v^m (mn)^m|^{\frac{1}{m+n}} = (-1)^m \cdot m!$.*

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