Bol. Soc. Paran. Mat. (3s.) **v. 31** 1 (2013): 145–152.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v31i1.15406

On the Gauss Map of Ruled Surfaces of Type II in 3-Dimensional Pseudo-Galilean Space

Alper Osman Öğrenmiş and Mahmut Ergüt

ABSTRACT: In this paper, ruled surfaces of type II in a three-dimensional Pseudo-Galilean space are given. By studying its Gauss map and Laplacian operator, we obtain a classification of ruled surfaces of type II in a three-dimensional Pseudo-Galilean space.

Key Words: Pseudo-Galilean Space, Ruled Surfaces, Gauss Map.

Contents

1. Introduction

The classification of submanifold in Euclidean or Non-Euclidean spaces has been of particular interest for Geometers. In 1983, the authors classified spacelike ruled surfaces in a three-dimensional Minkowski space $\mathbb{R}^3_1$ [13], and De Woestijne [11] extended it to the Lorentz version in 1998. In late 1970s Chen [4,5] introduced the notion of Euclidean immersions of finite type. In a framework of theory of finite type, Chen and Piccinni [6] characterized the submanifold satisfying the condition $\Delta G = \lambda G$ ($\lambda \in \mathbb{R}$), where Δ in the Laplacian of the induced metric and G the Gauss map for submanifold. Submanifold of Euclidean and pseudo-Euclidean spaces with finite type Gauss map also studied by following geometers. (cf. [7,9,12,15,16], etc.) On the otherhand, for the Gauss map of a surface in a three-dimensional Euclidean space $\mathbb{R}^3$ the following theorem is proved by Boikoussis and Blair [8].

Theorem 1.1. *The only ruled surfaces in $\mathbb{R}^3$ whose Gauss map ξ satisfies*

$$\Delta\xi = A\xi, \quad A \in Mat(3, \mathbb{R}) \tag{1.1}$$

are locally the plane and the circular cylinder.

Also, for the Lorentz version Choi [15] investigated ruled surfaces with non-null base curve satisfying the condition (1.1) in a three-dimensional Minkowski space $\mathbb{R}^3_1$.

2000 *Mathematics Subject Classification:* 53B30,53A35

It seems to be interesting to investigate the Pseudo-Galilean version of the above theorem. Now, let G_3^1 be a three-dimensional Pseudo-Galilean space with standard coordinate system $\{\mathbf{X}_A\}$. Let S_m^1 (resp. H_m) be an m-dimensional de Sitter space (resp. a hyperbolic space) in a $m+1-$dimensional Pseudo-Galilean space G_{m+1}^1. We denote by $\mathbf{X}_m(\varepsilon)$ a de Sitter space $S_m^1(1)$ or a hyperbolic space $H_m(-1)$, according as $\varepsilon = 1$ or $\varepsilon = -1$. Let $\mathbf{X}$ be a surface in G_3^1 and ξ be a unit vector field normal to $\mathbf{X}$. Then, for any point ρ in $\mathbf{X}$, we can regard $\xi(\rho)$ as a point in $H_2(-1)$ or $S_2^1(1)$ by translating parallelly to the origin in ambient space G_3^1, according as $\varepsilon = \mp 1$. The map ξ of $\mathbf{X}$ into $\mathbf{X}_2(\varepsilon)$ is called a Gauss map of $\mathbf{X}$ in G_3^1. Then we prove the following

Theorem 1.2. *The only ruled surfaces of type II in G_3^1 whose Gauss map $\xi : \mathbf{X} \to \mathbf{X}_2(\varepsilon)$ satisfies (1.1) are locally the following spaces:*

1. *G_2^1 and $S_1^1 \times \mathbb{R}$ if $\varepsilon = -1$*
2. *G_2^1 and $H_1 \times \mathbb{R}$ if $\varepsilon = -1$.*

The theorem is proved in section 3.

2. Ruled Surfaces

First of all, we recall fundamental properties in three-dimensional Pseudo-Galilean space.

Differential geometry of the Galilean space G_3 has been largely developed in O. Röschel's paper [14].

The Pseudo-Galilean geometry is one of the real Cayley-Klein geometries (of projective signature $(0,0,+,-)$, explained in [10]. The absolute of the Pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$ where w is the ideal (absolute) plane, f is line in w and I is the fixed hyperbolic involution of points of f.

The group of motions of G_3^1 is a six-parameter group given (in affine coordinates) by

$$\begin{aligned} \overline{x} &= a + x \\ \overline{y} &= b + cx + ych\varphi + zsh\varphi \\ \overline{z} &= d + ex + ysh\varphi + zch\varphi. \end{aligned}$$

As in [1], Pseudo-Galilean scalar product g can be written as

$$g(v_1, v_2) = \begin{cases} x_1 x_2 & \text{if } x_1 \neq 0 \vee x_2 \neq 0 \\ y_1 y_2 - z_1 z_2 & \text{if } x_1 = 0 \wedge x_2 = 0 \end{cases} \tag{2.1}$$

where $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2)$.

It leaves invariant the Pseudo-Galilean norm of the vector $v = (x, y, z)$ defined by

$$\|v\| = \begin{cases} x, \ x \neq 0 \\ \sqrt{|y^2 - z^2|}, \ x = 0 \text{ [2].} \end{cases} \tag{2.2}$$

A vector $v = (x, y, z)$ is said to be non-isotropic if $x \neq 0$. All unit non-isotropic vectors are the form $(1, y, z)$. For isotropic vectors $x = 0$ holds. There are four types of isotropic vectors: spacelike $(y^2 - z^2 > 0)$, timelike $(y^2 - z^2 < 0)$ and two types of lightlike $(y = \pm z)$ vectors. A non-lightlike isotropic vector is a unit vector if $y^2 - z^2 = \pm 1$ [3].

A trihedron $(T_0; e_1, e_2, e_3)$, with a proper origin

$$T_0(x_0, y_0, z_0) \sim (1 : x_0 : y_0 : z_0),$$

is orthonormal in Pseudo-Galilean sense if the vectors e_1, e_2, e_3 are of the following form:

$$e_1 = (1, y_1, z_1),\ e_2 = (0, y_2, z_2),\ e_3 = (0, \varepsilon z_2, \varepsilon y_2),$$

with $y_2^2 - z_2^2 = \delta$, where ε, δ is $+1$ or -1 [3].

Such trihedron $(T_0; e_1, e_2, e_3)$ is called positively oriented if for its vectors $\det(e_1, e_2, e_3) = 1$ holds, i.e. if $y_2^2 - z_2^2 = \varepsilon$ [3].

Let $\mathbf{X}$ be a ruled surface, $r \geq 1$, in G_3^1 given by its parametrization

$$\mathbf{X}(u, v) = r(u) + v\mathbf{a}(u), \quad v \in \mathbb{R} \tag{2.3}$$

where r is the directrix (parametrized by the Pseudo-Galilean arc length) and $\mathbf{a}$ is a unit generator vector field. according to the position of the striction curve with respect to the absolute figure, we have three types of skew ruled surfaces in G_3^1.

Throughout this paper, we assume that all objects are smooth unless otherwise mentioned. Now, we define a ruled surface of type II in G_3^1.

A ruled surface of type II in G_3^1 is a surface parametrized by

$$\begin{aligned} \mathbf{X}(u, v) &= \mathbf{r}(u) + v\mathbf{a}(u) \\ y, z, a_2, a_3 &\in C^2, \quad u \in I \subseteq \mathbb{R}, \quad v \in \mathbb{R} \\ \left| y'^2 - z'^2 \right| &= 1, \quad y' a_2' - z' a_3' = 0. \end{aligned} \tag{2.4}$$

It is a surface whose striction curve $r(u) = (0, y(u), z(u))$ lies in pseudo-Euclidean plane, u is the arc-lenght on the striction curve, and whose generators $\mathbf{a}(u) = (1, a_2(u), a_3(u))$ are non-isotropic. In particular, if $\mathbf{a}(u)$ is constant, then it said to be cylindrical, and if it is not so, then the surface is said to be non-cylindrical. Since our discussion is local, we may assume that we always have $\mathbf{a}'(u) \neq 0$ in the non-cylindrical case. That is, the direction of the rulings is always changing. [3]

The natural basis $\{\mathbf{X}_u, \mathbf{X}_v\}$ along the coordinate curves are given by

$$\begin{aligned} \mathbf{X}_u &= dx(\frac{\partial}{\partial u}) = \mathbf{r}' + v\mathbf{a}' \\ \mathbf{X}_v &= dx(\frac{\partial}{\partial v}) = \mathbf{a}. \end{aligned}$$

Accordingly we see

$$\begin{aligned} g(\mathbf{X}_u, \mathbf{X}_u) &= g(\mathbf{r}^{'}, \mathbf{r}^{'}) + 2vg(\mathbf{r}^{'}, \mathbf{a}^{'}) + v^2 g(\mathbf{a}^{'}, \mathbf{a}^{'}) \\ g(\mathbf{X}_u, \mathbf{X}_v) &= 0 \\ g(\mathbf{X}_v, \mathbf{X}_v) &= g(\mathbf{a}, \mathbf{a}). \end{aligned}$$

For the components g_{ij} of the Pseudo-Galilean metric g we denote (g^{ij}) (resp. g) the inverse matrix (resp. the determinant) of matrix (g_{ij}). Then the Laplacian Δ is given by

$$\Delta = -\frac{1}{\sqrt{|g|}} \sum \frac{\partial}{\partial x^i} (\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j}). \tag{2.5}$$

3. Cylindrical Ruled Surfaces

In this section we are concerned with cylindrical ruled surfaces. Let $\mathbf{X}$ be a cylindrical ruled surface swept out by the vector field $\mathbf{a}$ along the base curve r in G_3^1. That is, $r = r(u)$ is isotropic curve and $\mathbf{a} = \mathbf{a}(u)$ is a non-isotropic unit constant vector along orthogonal to r. Then the cylindrical ruled surface $\mathbf{X}$ is only or type II. And $\mathbf{X}$ is parametrized by

$$\mathbf{X}(u, v) = \mathbf{r}(u) + v\mathbf{a}(u), \quad u \in I \subseteq \mathbb{R}, \quad v \in \mathbb{R}.$$

Then we get $g(\xi, \xi) = \varepsilon \ (= \mp 1)$. Let $\mathbf{X}_2(\varepsilon)$ be a two-dimensional space form as follows:

$$\mathbf{X}_2(\varepsilon) = \begin{cases} S_2(1) \text{ in } G_3^1 \text{ if } \varepsilon = -1 \\ H_2(-1) \text{ in } G_3^1 \text{ if } \varepsilon = 1. \end{cases}$$

Then, for any point x in $\mathbf{X}$, $\xi(x)$ can be regarded as a point in $\mathbf{X}_2(\varepsilon)$ and the map $\xi : \mathbf{X} \to \mathbf{X}_2(\varepsilon)$ is the Gauss map of $\mathbf{X}$ into $\mathbf{X}_2(\varepsilon)$.

We give here theorem of ruled surface of type II whose Gauss map satisfies

$$\Delta\xi = A\xi, \quad A \in Mat(3, \mathbb{R}). \tag{3.1}$$

Theorem 3.1. *The only cylindrical ruled surface of type II in G_3^1 whose Gauss map satisfies the condition (3.1) are locally the plane and the hyperbolic cylinder (resp. the Pseudo-Galilean plane and Pseudo-Galilean circular cylinder).*

Proof: Let $\mathbf{X}$ be a cylindrical ruled surface of type II in G_3^1 parametrized by

$$\begin{aligned} \mathbf{X}(u, v) &= \mathbf{r}(u) + v\mathbf{a}(u) \\ y, z, a_2, a_3 &\in C^2, \quad u \in I \subseteq \mathbb{R}, \quad v \in \mathbb{R} \\ \left| y^{'2} - z^{'2} \right| &= 1, \quad y^{'} a_2^{'} - z^{'} a_3^{'} = 0 \end{aligned} \tag{3.2}$$

where **a** is a unit constant vector along the curve r orthogonal to it. That is, it satisfies $g(r^{'}, \mathbf{a}) = 0$, $g(\mathbf{a}, \mathbf{a}) = 1$. Acting a Pseudo-Galilean transformation, we may assume that $\mathbf{a} = (1, 0, 0)$ without loss of generality. Then r may be regarded as the plane curve $r(u) = (0, y(u), z(u))$ parametrized by arc length:

$$g(\mathbf{r}^{'}, \mathbf{r}^{'}) = y^{'2} - z^{'2} = \varepsilon \quad (\varepsilon = \mp 1). \tag{3.3}$$

The Gauss map ξ is given by $\xi = (0, \varepsilon z^{'}(u), \varepsilon y^{'}(u))$. It is unit normal to $\mathbf{X}$. Since the induced Pseudo-Galilean metric g is given by $g_{11} = \varepsilon$, $g_{12} = 0$ and $g_{22} = 1$, the Laplacian of ξ is given by $\Delta\xi = (0, \varepsilon z^{'''}, \varepsilon y^{'''})$ from (2.5). Thus, from the condition (3.1) we have the following system of differential equation:

$$\begin{aligned} 0 &= a_{12} z^{'} + a_{13} y^{'} \\ \varepsilon z^{'''} &= a_{22} z^{'} + a_{23} y^{'} \\ \varepsilon y^{'''} &= a_{32} z^{'} + a_{33} y^{'}. \end{aligned} \tag{3.4}$$

where $A = (a_{ij})$ is the constant matrix.

Now, in order to prove this theorem we may solve this equation and obtain the solution y and z. First we consider that $\varepsilon = -1$. So we get $g(\mathbf{r}^{'}, \mathbf{r}^{'}) = y^{'2} - z^{'2} = -1$. Accordingly we can parametrize as follows:

$$y^{'} = \sinh\theta, \quad z^{'} = \cosh\theta \tag{3.5}$$

where $\theta = \theta(u)$. Differentiating (3.5), we obtain

$$\begin{aligned} y^{''} &= \theta^{'} \cosh\theta, \quad y^{'''} = \theta^{''} \cosh\theta + (\theta^{'})^2 \sinh\theta \\ z^{''} &= \theta^{'} \sinh\theta, \quad z^{'''} = \theta^{''} \sinh\theta + (\theta^{'})^2 \cosh\theta \end{aligned} \tag{3.6}$$

By (3.4), (3.5) and (3.6) we have

$$\begin{aligned} -\theta^{''} \sinh\theta - (\theta^{'})^2 \cosh\theta &= a_{22} \cosh\theta + a_{23} \sinh\theta \\ -\theta^{''} \cosh\theta - (\theta^{'})^2 \sinh\theta &= a_{32} \cosh\theta + a_{33} \sinh\theta, \end{aligned}$$

which give

$$\theta^{''} = (a_{22} - a_{33}) \cosh\theta \sinh\theta + a_{23} \sinh^2\theta - a_{32} \cosh^2\theta \tag{3.7}$$

$$(\theta^{'})^2 = (a_{32} - a_{23}) \cosh\theta \sinh\theta + a_{33} \sinh^2\theta - a_{22} \cosh^2\theta. \tag{3.8}$$

Differentiating (3.8), we get

$$2\theta^{'}\theta^{''} = \theta^{'} \left\{ (a_{22} - a_{33})(\cosh^2\theta + \sinh^2\theta) + 2(a_{33} - a_{22}) \sinh\theta \cosh\theta \right\}.$$

Substituting (3.7) into this equation, we get

$$\theta' \left\{4(a_{22} - a_{33}) \cosh\theta \sinh\theta + (a_{23} - 3a_{32}) \cosh^2\theta + (3a_{23} - a_{32}) \sinh^2\theta\right\} = 0. \tag{3.9}$$

We suppose that $\theta' \neq 0$. By (3.4) and (3.9) we get

$$a_{22} = a_{33} \text{ and } a_{12} = a_{13} = a_{23} = a_{32} = 0, \tag{3.10}$$

because $\sinh\theta \cosh\theta$, $\sinh^2\theta$ and $\cosh^2\theta$ are linearly independent functions of $\theta = \theta(u)$. Combining the above equations with (3.8) gives

$$\theta = \mp\frac{1}{r}u + c_0, \quad r\rangle 0, \quad c_0 \in \mathbb{R}$$

where $-\frac{1}{r^2} = a_{22} = a_{33}$. Accordingly we have

$$\begin{aligned} y &= \mp r\cosh\theta + c_{1,} \quad c_1 \in \mathbb{R} \\ z &= \mp r\sinh\theta + c_{2,} \quad c_2 \in \mathbb{R}. \end{aligned}$$

This representation gives us to

$$(y - c_1)^2 - (z - c_2)^2 = r^2, \quad r\rangle 0.$$

We denote by S_1^1 $(r,(c_1, c_2))$ the pseudo-circle centered at (c_1, c_2) with radius r in the Pseudo-Galilean plane G_2^1 (the $yz-$plane). By the above equation the curve r is contained in S_1^1 $(r,(c_1, c_2))$ and hence the ruled surface $\mathbf{X}$ is contained in the Pseudo-Galilean circular cylinder $S_1^1 \times \mathbb{R}$.

Now, we consider that $\varepsilon = 1$. So we get $g(\mathbf{r}', \mathbf{r}') = y'^2 - z'^2 = 1$. Accordingly we can parametrize as follows:

$$y' = \cosh\theta, \quad z' = \sinh\theta$$

where $\theta = \theta(u)$. By similar discussion to that of the above $\varepsilon = -1$ we can get

$$\theta' \left\{4(a_{22} - a_{33}) \cosh\theta \sinh\theta + (3a_{23} - a_{32}) \cosh^2\theta + (a_{23} - 3a_{32}) \sinh^2\theta\right\} = 0. \tag{3.11}$$

We suppose that $\theta' \neq 0$. By (3.4) and (3.11) we get

$$a_{22} = a_{33} \text{ and } a_{12} = a_{13} = a_{23} = a_{32} = 0,$$

which yields that

$$\theta = \mp\frac{1}{r}u + c_0, \quad r\rangle 0, \quad c_0 \in \mathbb{R}$$

where $\frac{1}{r^2} = a_{22} = a_{33}$. Accordingly we have

$$\begin{aligned} y &= \mp r\sinh\theta + c_{1,} \quad c_1 \in \mathbb{R} \\ z &= \mp r\cosh\theta + c_{2,} \quad c_2 \in \mathbb{R}. \end{aligned}$$

This representation gives us to

$$(y-c_1)^2-(z-c_2)^2=-r^2,\quad r\rangle 0.$$

We denote by H_1 $(r,(c_1,c_2))$ the hyperbolic circle centered at (c_1,c_2) with radius r in the Pseudo-Galilean plane G_2^1 (the $yz-$plane). By the above equation the curve r is contained in H_1 $(r,(c_1,c_2))$ and hence the ruled surface $\mathbf{X}$ is contained in the hyperbolic cylinder $H_1\times\mathbb{R}$.

Example 3.2. *A Pseudo-Galilean circular cylinder*

$$S_1^1\times\mathbb{R}=\left\{(x,y,z)\in G_3^1 : y^2-z^2=r^2,\quad r\rangle 0\right\}$$

is a cylindrical ruled surface of type II with base curve $\mathbf{r}(u)=(0,r\sinh\frac{u}{r},r\cosh\frac{u}{r})$ *and director curve* $\mathbf{a}(u)=(1,0,0)$*. The Gauss map is given by*

$$\xi=(0,\cosh\frac{u}{r},\sinh\frac{u}{r})$$

and the Laplacian $\Delta\xi$ *of the Gauss map* ξ *can be expressed as*

$$\Delta\xi=-\frac{1}{r^2}\xi.$$

Hence the Pseudo-Galilean circular cylinder satisfies (3.1) with

$$A=\begin{bmatrix} a_{11} & 0 & 0\\ a_{21} & -\frac{1}{r^2} & 0\\ a_{31} & 0 & -\frac{1}{r^2}\end{bmatrix}.$$

Example 3.3. *A hyperbolic cylinder*

$$H_1\times\mathbb{R}=\left\{(x,y,z)\in G_3^1 : y^2-z^2=-r^2,\ r\rangle 0\right\}$$

is a cylindrical ruled surface of type II with base curve $\mathbf{r}(u)=(0,r\sinh\frac{u}{r},r\cosh\frac{u}{r})$ *and director curve* $\mathbf{a}(u)=(1,0,0)$*. The Gauss map is given by*

$$\xi=(0,\sinh\frac{u}{r},\cosh\frac{u}{r})$$

and the Laplacian $\Delta\xi$ *of the Gauss map* ξ *can be expressed as*

$$\Delta\xi=\frac{1}{r^2}\xi.$$

Hence the hyperbolic cylinder satisfies (3.1) with

$$A=\begin{bmatrix} a_{11} & 0 & 0\\ a_{21} & \frac{1}{r^2} & 0\\ a_{31} & 0 & \frac{1}{r^2}\end{bmatrix}.$$

References

1. B. Divjak, Curves in pseudo-Galilean geometry, *Annales Univ. Sci.Budapest.* 41(1998), 117-128.
2. B. Divjak and Z.M.Sipus, Minding isometries of ruled surfaces in pseudo-Galilean space, *Journal of Geometry* 77(2003),35-47.
3. B. Divjak and Z.M.Sipus,Special curves on ruled surfaces in Galilean and pseudo-Galilean spaces, *Acta Math. Hungar.* 98(3) (2003),203-215.
4. B.-Y. Chen, On submanifolds of finite type, *Soochow J. Math.* 9 (1983) 65-81.
5. B.-Y. Chen, Total Mean Curvature and Submanifolds of Finite Type, *World Scientific* Singapore, 1984.
6. B.-Y. Chen and P. Piccinni, Submanifolds with finite type Gauss map, *Bull. Austral. Math. Soc.* 35 (1987), 161-186.
7. C. Baikoussis, B.-Y. Chen, L. Verstraelen, Surfaces with Finite Type Gauss Map, *Geometry of Topology of Submanifolds: IV, World Scientific* Singapore, 1992, pp.214-216.
8. C. Baikoussis and D.E. Blair, On the Gauss map of ruled surfaces, *Glasgow Math. J.* 34 (1992) 355-359.
9. C. Baikoussis, B.-Y. Chen, L. Verstraelen, Ruled surfaces and tubes with finite type Gauss map, *Tokyo J. Math.* 16 (1993) 341-348.
10. E.Molnar, The projective interpretation of the eight 3-dimensional homogeneous geometries, *Beitr. Algebra Geom.* 38 (1987),261-288.
11. I.V. de Woestijne, Minimal surfaces in the 3-dimensional Minkowski space, *Geometry of Topology of Submanifolds: II, World Scientific* Singapore, 1990, pp.344-369.
12. L.J. Alías, A. Ferrández, P. Lucas, M.A. Mcroño, On the Gauss map of B-scrolls, *Tsukuba J. Math.* 22 (1998) 371-377.
13. O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space L^3, *Tokyo J. Math.* 6 (1983) 297-309.
14. O. Röschel, Die Geometrie Des Galileischen Raumes, *Berichte der Math.-Stat. Sektion im Forschungszentrum Graz* Ber. 256, 1986, 1-20.
15. S.M. Choi, On The Gauss map ruled surfaces in a three-dimensional Minkowski space, *Tsukuba J. Math.* 19 (1995) 285-304.
16. Y.H. Kim, D.W. Yoon, Ruled surfaces with finite type Gauss map in Minkowski spaces, *Soochow J. Math.* 26 (2000) 85-96.

Alper Osman Öğrenmiş
Fırat University,
Faculty of Science,
Department of Mathematics,
23119 Elazığ, Turkey
E-mail address: ogrenmisalper@gmail.com

and

Mahmut Ergüt
Fırat University,
Faculty of Science,
Department of Mathematics,
23119 Elazığ, Turkey
E-mail address: mergut@firat.edu.tr