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The Reilly's Integral Formula on Semi-Riemannian Manifolds with Nondegenerate Boundary

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ABSTRACT: In this paper, we obtained the Reilly's integral Formula on semi-Riemannian manifolds with nondegenerate boundary.

Key Words: Semi-Riemannian manifold, Nondegenerate boundary, Reilly's formula.

Contents

1. Introduction

In Riemannian geometry, integral formulas have been studied by many mathematicians [1] and their use many beatiful global results have been obtained. Perhaps the Reilly's formula is one of the most well known integral formula in Riemannian geometry as well as a very powerful tool for obtaining global results. Nonetheless, a Reilly's Formula in semi-Riemannian geometry has not been available. The main difficulty in stating an integral formula for semi-Riemannian manifolds is that the boundary may become degenerate at some points and hence there exists no well-defined unit outward normal at such points. Consequently there is no well defined induced volume form on the boundary.

Duggal was the first one who studied semi-Riemannian manifolds with boundary in one of his works on integral formulas in semi-Riemannian geometry [2]. In [2], Duggal defined a semi-Riemannian manifold to be regular if the usual form of integral formulas remains valid on it. In [3], Ünal defined nondegenerate boundary of a semi-Riemannian manifold and by making use of the volume form on the nondegenerate boundary, he obtained integral formulas.

In this paper, we define two type semi-Riemannian inner product. Using this definition we classify the boundaries.We define nondegenerate boundary of a semi-Riemannian manifold and we get Reilly's formula on the nondegenerate boundary. Of course, the validity of the Reilly's formula depends on some restrictions, namely, the degenerate part of the boundary must have measure zero. Finally, we obtain different results from Riemannian geometry.

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2. Basic Notions and Terminologies

Let R^n be n-dimensional real vector space. Semi-Riemannian inner product for n-dimensional real vector space R^n is defined as follows [4]:

$$\begin{aligned} \langle,\rangle_1 \quad : \quad & \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ (\overrightarrow{X},\overrightarrow{Y}) \quad & \to \left\langle \overrightarrow{X},\overrightarrow{Y} \right\rangle_1 = \sum_{i=1}^{v} x_i y_i - \sum_{j=v+1}^{n} x_j y_j \end{aligned} \tag{2.1}$$

or

$$\begin{aligned} \langle,\rangle_2 \quad : \quad & \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ (\overrightarrow{X},\overrightarrow{Y}) \quad & \to \left\langle \overrightarrow{X},\overrightarrow{Y} \right\rangle_2 = -\sum_{i=1}^{v} x_i y_i + \sum_{j=v+1}^{n} x_j y_j \end{aligned} \tag{2.2}$$

In addition, β_i and ε_i are defined as follows:

$$\beta_i = \begin{cases} 1 \quad , & \text{if} \quad 1 \leq i \leq v \\ -1 \quad , & \text{if} \quad v+1 \leq i \leq n \end{cases} \tag{2.3}$$

and

$$\varepsilon_i = \begin{cases} -1 \quad , & \text{if} \quad 1 \leq i \leq v \\ 1 \quad , & \text{if} \quad v+1 \leq i \leq n \end{cases} \tag{2.4}$$

Considering β_i and ε_i in Eq. (2.1) and Eq. (2.2), respectively, we get

$$\left\langle \overrightarrow{X},\overrightarrow{Y} \right\rangle_1 = \sum_{i=1}^{n} \beta_i x_i y_i \tag{2.5}$$

and

$$\left\langle \overrightarrow{X},\overrightarrow{Y} \right\rangle_2 = \sum_{i=1}^{n} \varepsilon_i x_i y_i \tag{2.6}$$

Here the functions of $\langle,\rangle_1$ and $\langle,\rangle_2$ are semi-Riemannian inner product in R^n and R^n_v is semi-Riemannian space which is united with the functions of $\langle,\rangle_1$ and $\langle,\rangle_2$.

For the sake of shortness, let's unite both of semi-Riemannian inner product definition and let's express this definition as follows:

If is written as the following,

$$\gamma_i = \begin{cases} \beta_i \quad , & \text{if} \quad \langle,\rangle_1 \\ \varepsilon_i \quad , & \text{if} \quad \langle,\rangle_2 \end{cases}$$

then semi-Riemannian inner product is as follows:

$$\left\langle \overrightarrow{X}, \overrightarrow{Y} \right\rangle = \sum_{i=1}^{n} \gamma_i x_i y_i \tag{2.7}$$

Throughout this paper, let M denote an n-dimensional semi-Riemannian manifold with metric $\langle , \rangle$ of index $0 \leq v \leq n$ and boundary ∂M. Then the open submanifold $\partial M^{'} = \partial M_{+} \cup \partial M_{-}$ of ∂M is called the nondegenerate boundary of $(M, \langle , \rangle)$. A vector $0 \neq v \in TM$ is respectively called spacelike, timelike and null if $\langle , \rangle > 0$, $\langle , \rangle < 0$, $\langle , \rangle = 0$. We will also assume that M is oriented and ∂M is oriented by the induced orientation. Also let dv be the semi-Riemannian volume element on M, that is, dv is an exterior n-form on M with

$$dv = \sqrt{|g|} dx_1 \wedge ... \wedge dx_n \tag{2.8}$$

for semi-Riemannian orthonormal basis $\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ . \ . \ . \ , \frac{\partial}{\partial x_n} \right\}$of $\chi(U)$ on $a \in U \subset M$, where $g = det(g_{ij})$ [5] and let e_n be the unit outward normal vector field on the nondegenerate boundary $\partial M^{'}$ of $(M, \langle , \rangle)$.

Let M be an n-dimensional semi-Riemannian manifold, $\wedge^k(M)$ be k-forms set defined on M and dv be volume element. Hence,

$$* \ : \wedge^k(M) \rightarrow \wedge^{n-k}(M)$$

If " $*$ " isomorphism holds the following equality for $\forall \alpha, \beta \in \wedge^k(M)$,

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, dv \tag{2.9}$$

then this transformation is called Hodge-star operator [6].

For n-dimensional semi-Riemannian manifold M, $gradf$ denotes the gradient of f and we define as [7]

$$gradf = \sum_{i=1}^{n} \gamma_i \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \tag{2.10}$$

In addition we define the Laplace operator on M as [7]

$$\Delta(f) = \sum_{i=1}^{n} \gamma_i f_{ii} \tag{2.11}$$

and also we define Hessian form of differentiable function f on M as [7]

$$H_f(u, v) \ = \ \langle \nabla_u gradf, v \rangle \tag{2.12}$$

where $H_f(e_i, e_j) = f_{ij}$.

In addition, we define the second fundamental form of vector fields U and V of nondegenerate boundary $\partial M^{'} = \partial M_{+} \cup \partial M_{-}$ as follows:

$$II(U, V) = \langle \nabla_U e_n, \ V \rangle$$

Definition 2.1. *Let M be an n-dimensional semi-Riemannian manifold and R be a Riemannian curvature tensor of M. Let $\{e_1, e_2, ..., e_n\}$ be a semi-Riemannian orthonormal basis of $T_p(M)$. Thus, one can write the following:*

$$\begin{aligned} Ric \quad &: \quad T_p(M) \times T_p(M) \to IR \\ (U,V) \quad &\to \quad Ric(U,V) = \sum_{i=1}^{n} \gamma_i \langle R(e_i,V)U, e_i \rangle \end{aligned} \tag{2.13}$$

where the curvature tensor field Ric is called Ricci curvature tensor field and also the value of $Ric(U,V)$ on $p\epsilon M$ is called Ricci curvature of M [7].

Taking U and V as follows in (2.13)

$$U = \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}, \qquad V = \sum_{j=1}^{n} v_j \frac{\partial}{\partial x_j} \epsilon \chi(M)$$

and defining Rij as follows

$$R_{ij} = \sum_{k=1}^{n} R_{ikj}^{k} \tag{2.14}$$

we have

$$Ric(U,V) = \sum_{i,j=1}^{n} R_{ij} u_i v_j. \tag{2.15}$$

Definition 2.2. *Let M be an n-dimensional semi-Riemannian manifold and $f\epsilon C^{\infty}(M,\mathbb{R})$. Hence the differential of f can be defined as follows:*

$$\begin{aligned} df_{|p} \quad &: \quad T_p(M) \to \mathbb{R} \\ \overrightarrow{X}_p \quad &\to \quad df_{|p}(\overrightarrow{X}_p) = \overrightarrow{X}_p[f] \end{aligned}$$

If $\{x_1, x_2, ..., x_n\}$ is local coordinate system on point p, then $\{dx_{1|p}, dx_{2|p}, ..., dx_{n|p}\}$ will be basis on $T_p^*(M)$. In addition there is the following relation among the components of the basis $\{dx_1, dx_2, ..., dx_n\}$

$$g^{ij} = \gamma_i \delta_{ij} = \langle dx_i, dx_j \rangle, \qquad 1 \le i,j \le n, \tag{2.16}$$

where $\{dx_1, dx_2, ..., dx_n\}$ is the dual basis of $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\}$ and also g^{ij} is the inverse matrix of g_{ij} [7].

Definition 2.3. *Let M be an n-dimensional semi-Riemannian manifold with boundary ∂M . Then the open subset ∂M_+ is called nondegenerate space-like boundary where unit outward normal is timelike and index of the induced nondegenerate metric is $v-1$ on ∂M_+.*

Definition 2.4. *Let M be an n-dimensional semi-Riemannian manifold with boundary ∂M. Then the open subset ∂M_- is called nondegenerate time-like boundary where unit outward normal is spacelike and index of the induced nondegenerate metric is v on ∂M_-.*

Remark 1. *Note that $\partial M = \partial M_+ \cup \partial M_- \cup \partial M_0$ and $\partial M_+,\ \partial M_-, \partial M_0$ are pairwise disjoint subsets of ∂M. Also notice that ∂M_+ and ∂M_- are open submanifolds of ∂M and $\partial M' = \partial M_+ \cup \partial M_-$ can be considered as the nondegenerate boundary of M.*

Definition 2.5. *Let M be an n-dimensional semi-Riemannian manifold with nondegenerate space-like boundary and time-like unit outward normal. In addition let $\{u_1, ..., u_{n-1}\}$ be an orthonormal basis of $T_a\partial M_+$ and $D = (n_1, ..., n_n)$ be a time-like unit outward normal of ∂M_+. Then*

$$w_{\partial M_+}(u_1, u_2, ..., u_{n-1}) = \det \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1} \\ D \end{bmatrix} = \langle u_1 \wedge u_2 \wedge ... \wedge u_{n-1}, D\rangle_1$$

the equality defined in the above is called volume element of ∂M_+.

Definition 2.6. *Let M be an n-dimensional semi-Riemannian manifold with nondegenerate time-like boundary and space-like unit outward normal. In addition let $\{u_1, ..., u_{n-1}\}$ be an orthonormal basis of $T_a\partial M_-$ and $D = (n_1, ..., n_n)$ be a space-like unit outward normal of ∂M_-. Then*

$$w_{\partial M_-}(u_1, u_2, ..., u_{n-1}) = \det \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1} \\ D \end{bmatrix} = \langle u_1 \wedge u_2 \wedge ... \wedge u_{n-1}, D\rangle_2$$

the equality defined in the above is called volume element of ∂M_-.

Definition 2.7. *Let U be an open set of semi-Riemannian manifold M and $w_1, w_2, ..., w_n$ be 1-forms on U. In addition, let w_j^i be connection coefficients. E. Cartan structure equations are defined as follows:*

1. *E. Cartan Structure Equation;*

$$dw_i = \sum_{j=1}^{n} \gamma_i w_j^i \wedge w_j, \qquad w_j^i + w_i^j = 0, \tag{2.17}$$

and

2. *E. Cartan Structure Equation;*

$$dw_j^i = \sum_{k=1}^{n} \gamma_k w_k^i \wedge w_j^k + \frac{1}{2} \sum_{k,l=1}^{n} \gamma_k \gamma_l R_{ijkl} w_k \wedge w_l \tag{2.18}$$

where R_{ijkl} is the component of the Riemannian-Christoffel curvature tensor [8].

Lemma 2.8. ***(Cartan's Lemma)*** *Let M be an n-dimensional manifold and w_i be 1-forms on M for $i = 1, 2, ..., n$. In addition, let λ_i be the other 1-forms. Suppose that λ_i and w_i are linearly independent. Then*

$$\sum_{i=1}^{n} w_i \wedge \lambda_i = 0.$$

Hence, for $1 \leq i, j \leq n$, $a_{ij} = a_{ji}$ and $a_{ij} \epsilon C^\infty(M, \mathbb{R})$, one can write the following [6]

$$\lambda_i = \sum_{j=1}^{n} a_{ij} w_j.$$

Theorem 2.9. *Let M be an n-dimensional semi-Riemannian manifold and U be open subset of M. In addition, let $w_1, w_2, ..., w_n$ be 1-forms on $U \subset M$ and f be any differentiable function. Then*

$$f_{ijk} - f_{ikj} = \sum_{l=1}^{n} \gamma_i \gamma_j R_{jil}^k f_l \tag{2.19}$$

where $i,\ j,\ k = 1, ..., n$.

Proof: Suppose that, $f \in C^\infty(M, \mathbb{R})$, $w_i \in \chi^*(M)$. The differential of f is as follows:

$$df = \sum_{i=1}^{n} f_i w_i$$

Here, by using exterior differential, we get

$$df_i \wedge w_i - \sum_{j=1}^{n} f_j dw_j = 0$$

Considering (2.18) equality here and making routine calculations, we have

$$\sum_{i=1}^{n} \left[df_i - \sum_{j=1}^{n} \gamma_i f_j dw_i^j \right] \wedge w_i = 0$$

In addition, from Lemma 2.8, there are f_{ij} functions on an open subset U of M. Hence, from $f_{ij} = f_{ji}$, the above equality can be written as follows:

$$df_i - \sum_{j=1}^{n}\gamma_i f_j dw_i^j = \sum_{j=1}^{n} f_{ij} w_j \tag{2.20}$$

Using exterior differential for the (2.20) equality, we obtain

$$\begin{aligned}&\sum_{j=1}^{n}(df_{ij} - \sum_{k=1}^{n}\gamma_i f_{kj} dw_i^k - \sum_{k=1}^{n}\gamma_i f_{ik} dw_j^k) \wedge w_j \\ &= \frac{1}{2}\sum_{l,k,j=1}^{n}\gamma_i\gamma_l R_{lij}^k w_k \wedge w_l\end{aligned} \tag{2.21}$$

From Lemma 2.8, there are f_{ijk} functions on an open subset U of M. Then we have

$$df_{ij} - \sum_{k=1}^{n}\gamma_i f_{kj} w_i^k - \sum_{k=1}^{n}\gamma_i f_{ik} w_j^k = \sum_{k=1}^{n} f_{ijk} w_k \tag{2.22}$$

On the other hand, let

$$f_{ijk} = f_{jik} \tag{2.23}$$

Here writing the similar equality of Eq. (2.22) for f_{ijk} and considering Eq. (2.21), we get

$$f_{ijk} - f_{ikj} = \sum_{l=1}^{n}\gamma_i\gamma_j R_{jil}^k f_l$$

This completes the proof of theorem. □

Definition 2.10. *Let M be an n-dimensional semi-Riemannian manifold and N be a semi-Riemannian submanifold with index $(v-1)$. Let's consider the following isometric immersion*

$$\tau : N \to M.$$

Owing to this immersion, local semi-Riemannian orthonormal basis $\{e_1, e_2, ..., e_{n-1}\}$ in the coordinate neighbourhood U of N transforms a local semi-Riemannian orthonormal basis $\{e_1, e_2, ..., e_n\}$. (Here, e_n is the time-like unit outward normal of N). In addition, 1-forms $\{w_1, w_2, ..., w_n\}$ which are the dual basis of local semi-Riemannian orthonormal basis $\{e_1, e_2, ..., e_n\}$ transform dual 1-forms which are defined as follows of N, under this immersion,

$$\theta_i = \tau^*(w_i), \qquad 1 \le i \le n, \tag{2.24}$$

and connection coefficients w^i_j, $1 \leq i,j \leq n$ *also transform dual connection coefficients* θ^i_j *of* N *which are defined as follows*

$$\theta^i_j = \tau^*(w^i_j), \qquad 1 \leq i,j \leq n. \tag{2.25}$$

That is, 1*-forms* w_i , $1 \leq i \leq n$ *under the transformation of* τ *transform* 1*-forms* $\theta_1, \theta_2, ..., \theta_{n-1}$ *on* N*, where*

$$\theta_n = 0 \tag{2.26}$$

This 1-forms are shortly called co-frame.

On the other hand, using exterior differential for (2.26), we have $d\theta_n = 0$*. In addition, let* $II_{ij} = II_{ji}$ *that are the components of second fundamental form of* N*. This* 1*-forms are defined as follows [1]:*

$$\theta^n_i = -\sum_{j=1}^{n-1} II_{ij}\theta_j \tag{2.27}$$

and

$$d\theta_i = \sum_{j=1}^{n-1} \beta_j \theta^i_j \wedge \theta_j \qquad , \qquad \theta^i_j = -\theta^j_i. \tag{2.28}$$

Definition 2.11. *Let* M *and* $\overline{M}$ *be an n-dimensional and (n-1)-dimensional semi-Riemannian manifold, respectively. Let's consider the following isometric immersion*

$$\tau : \overline{M} \to M.$$

Owing to this isometric immersion, local semi-Riemannian orthonormal basis $\{e_1, e_2, ..., e_{n-1}\}$ *in the coordinate neighbourhood* U *of* $\overline{M}$ *transforms a local semi-Riemannian orthonormal basis* $\{e_1, e_2, ..., e_n\}$ *of* M*. (Here,* e_n *is the space-like unit outward normal of* $\overline{M}$*). In addition,* 1*-forms* $\{w_1, w_2, ..., w_n\}$ *which are the dual basis of local semi-Riemannian orthonormal basis* $\{e_1, e_2, ..., e_n\}$ *transform dual* 1*-forms* θ_i *which are defined as follows of* $\overline{M}$*, under this immersion,*

$$\theta_i = \tau^*(w_i), \qquad 1 \leq i \leq n, \tag{2.29}$$

and connection coefficients w^i_j, $1 \leq i,j \leq n$*, also transform dual connection coefficients* θ^i_j *of* $\overline{M}$ *which are defined as follows*

$$\theta^i_j = \tau^*(w^i_j), \qquad 1 \leq i,j \leq n. \tag{2.30}$$

That is, 1-forms w_i, $1 \leq i \leq n$*, under the transformation of* τ *transform 1-forms* $\theta_1, \theta_2, ..., \theta_{n-1}$ *on* $\overline{M}$*, where*

$$\theta_n = 0 \tag{2.31}$$

This 1-forms are called semi-Riemannian co-frame.

On the other hand, using exterior differential for (2.31), we have $d\theta_n = 0$*. In addition, let* $II_{ij} = II_{ji}$ *that are the components of second fundamental form of* $\overline{M}$*. This* 1*-forms are defined as follows:*

$$\theta_i^n = -\sum_{j=1}^{n} II_{ij}\theta_j \tag{2.32}$$

and

$$d\theta_i = \sum_{j=1}^{n-1} \varepsilon_j \theta_j^i \wedge \theta_j \qquad\qquad \theta_j^i = -\theta_i^j. \tag{2.33}$$

Definition 2.12. *Let* M *be an* n*-dimensional semi-Riemannian manifold with non-degenerate space-like boundary* ∂M_+*. Let* $\{e_1, ..., e_{n-1}\}$ *be local semi-Riemannian orthonormal frame of* ∂M_+ *and* e_n *be a time-like unit outward normal of* ∂M_+*. Thus* $\{e_1, ..., e_n\}$ *is a local semi-Riemannian orthonormal frame of* M*. Let* $\{w_1, ..., w_n\}$ *be dual co-frame of this local semi-Riemannian orthonormal frame and* f *be a differentiable function on* M*.*

Now, getting ∂M_+ instead of N in definition 2.10 let's study the covariant derivative of function f. Firstly, let's compute f_i, f_{ij}, f_{ni} on a point of ∂M_+.

Hence let's take

$$z = \tau^*(f)$$

such that τ is inclusion transformation as follows:

$$\tau : \partial M_+ \to M$$

Thus, we can write the following:

$$dz = \tau^*(df) = \sum_{i=1}^{n-1} f_i \mid_{\partial M_+} \theta_i \tag{2.34}$$

Here, if

$$z_i = f_i \mid_{\partial M_+} \tag{2.35}$$

then

$$dz = \sum_{i=1}^{n-1} f_i \theta_i \tag{2.36}$$

Using exterior differential for (2.36), we get

$$\begin{aligned}
\sum_{j=1}^{n-1} z_{ij}\theta_j &= dz_i - \sum_{j=1}^{n-1}\beta_j z_j \theta_i^j \\
&= \tau^*(df_i - \sum_{j=1}^{n-1}\beta_j f_j w_i^j) \\
&= \tau^*(df_i - \sum_{j=1}^{n}\beta_j f_j w_i^j - f_n w_i^n) \\
&= \tau^*(df_i - \sum_{j=1}^{n}\beta_j f_j w_i^j) - f_n \tau^*(w_i^n)
\end{aligned}$$

Here, getting u instead of f_n and using (2.22) and (2.25), we get

$$= \tau^*(\sum_{j=1}^{n} f_{ij} w_j) - u\theta_i^n \tag{2.37}$$

In the last equality, considering (2.24) and (2.27), we have

$$\sum_{j=1}^{n-1} z_{ij}\theta_j = \sum_{j=1}^{n-1} f_{ij} \mid_{\partial M_+} \theta_j + u\sum_{j=1}^{n-1} II_{ij} \mid_{\partial M_+} \theta_j$$

or

$$f_{ij} \mid_{\partial M_+} = z_{ij} - uII_{ij} \tag{2.38}$$

Let's consider (2.34) equality for computing f_{ni} . According to this, we get

$$\begin{aligned}
\sum_{i=1}^{n-1} f_{ni} \mid_{\partial M_+} \theta_i &= \tau^*(\sum_{i=1}^{n} f_{ni} w_i) \\
&= \tau^*(df_n - \sum_{i=1}^{n}\beta_i f_i w_n^i) \\
&= \tau^*(df_n) - \tau^*(\sum_{i=1}^{n}\beta_i f_i w_n^i) \\
&= du - \sum_{i=1}^{n-1} z_i \tau^*(w_n^i) \\
&= du - \sum_{i=1}^{n-1} z_i \theta_n^i, \qquad \theta_n^i = -\theta_i^n \\
&= du + \sum_{i=1}^{n-1} z_i \theta_i^n
\end{aligned}$$

Using (2.27) in the last equality, we obtain

$$\sum_{i=1}^{n-1} f_{ni} \mid_{\partial M_+} \theta_i = du - \sum_{i,j=1}^{n-1} \beta_i z_i II_{ij} \theta_j$$

or

$$f_{ni} \mid_{\partial M_+} = u_i - \sum_{j=1}^{n-1} \beta_j z_j II_{ij} \tag{2.39}$$

Thus, the covariant derivatives of f_i, f_{ij}, f_{ni} are obtained for semi-Riemannian manifold with nondegenerate spacelike boundary ∂M_+.

Definition 2.13. *Let M be an n-dimensional semi-Riemannian manifold with non-degenerate time-like boundary ∂M_-. Let $\{e_1, ..., e_{n-1}\}$ be local semi-Riemannian orthonormal frame of ∂M_- and e_n be a space-like unit outward normal of ∂M_-. Thus $\{e_1, ..., e_n\}$ is a local semi-Riemannian orthonormal frame of M. Let $\{w_1, ..., w_n\}$ be dual co-frame of this local semi-Riemannian orthonormal frame and f be a differentiable function on M.*

Now, getting ∂M_- instead of $\overline{M}$ in definition 2.11 let's study the covariant derivative of function f. Firstly, let's compute f_i, f_{ij}, f_{ni} on a point of ∂M_-.

Hence let's take

$$z = \tau^*(f)$$

such that τ is inclusion transformation as follows:

$$\tau : \partial M_- \to M$$

Thus, we can write the following:

$$dz = \tau^*(df) = \sum_{i=1}^{n-1} f_i \mid_{\partial M_-} \theta_i \tag{2.40}$$

Here, if

$$z_i = f_i \mid_{\partial M_-} \tag{2.41}$$

then

$$dz = \sum_{i=1}^{n-1} f_i \theta_i \tag{2.42}$$

Using exterior differential for (2.42), we get

$$\begin{aligned}\sum_{j=1}^{n-1} z_{ij}\theta_j &= dz_i - \sum_{j=1}^{n-1}\varepsilon_j z_j \theta_i^j \\ &= \tau^*(df_i - \sum_{j=1}^{n-1}\varepsilon_j f_j w_i^j) \\ &= \tau^*(df_i - \sum_{j=1}^{n}\varepsilon_j f_j w_i^j + f_n w_i^n) \\ &= \tau^*(df_i - \sum_{j=1}^{n}\varepsilon_j f_j w_i^j) + f_n \tau^*(w_i^n)\end{aligned}$$

Here, getting u instead of f_n and considering (2.22), (2.29) and (2.30), we obtain

$$= \sum_{j=1}^{n-1} f_{ij} \mid_{\partial M_-} \theta_j + u\theta_i^n$$

and using (2.32), we get

$$= \sum_{j=1}^{n-1} f_{ij} \mid_{\partial M_-} \theta_j - u\sum_{j=1}^{n-1} II_{ij}\theta_j$$

or

$$f_{ij} \mid_{\partial M_-} = z_{ij} + uII_{ij} \tag{2.43}$$

Now, let's consider (2.40) equality for computing f_{ni}. According to this, we have

$$\begin{aligned}\sum_{i=1}^{n-1} f_{ni} \mid_{\partial M_-} \theta_i = \tau^*(\sum_{i=1}^{n} f_{ni} w_i) & \\ &= \tau^*(df_n - \sum_{i=1}^{n}\varepsilon_i f_i w_n^i) \\ &= \tau^*(df_n) - \tau^*(\sum_{i=1}^{n}\varepsilon_i f_i w_n^i) \\ &= du + \sum_{i=1}^{n-1}\varepsilon_i z_i \theta_i^n\end{aligned}$$

Using (2.32) in the last equality, we obtain

$$= du - \sum_{i,j=1}^{n-1}\varepsilon_i z_i II_{ij}\theta_j$$

or

$$f_{ni}\mid_{\partial M_-}= u_i - \sum_{j=1}^{n-1}\varepsilon_j z_j II_{ij}. \tag{2.44}$$

Theorem 2.14. *Let M be an n-dimensional semi-Riemannian manifold with nondegenerate boundary $\partial M'$ $(\partial M' = \partial M_+ \cup \partial M_-)$. In addition, let $D = (n_1, ..., n_n)$ be unit outward normal and $\{x_1, ..., x_n\}$ be semi-Riemannian coordinate system of M [4].*

i) *If M has ∂M_+ space-like boundary and D time-like unit outward normal, $w_{\partial M_+}$ volume element of ∂M_+ is as follows:*

$$w_{\partial M_+} = \sum_{j=1}^{n}(-1)^{j-1} n_j dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_n \tag{2.45}$$

or

$$-\beta_j . n_j . w_{\partial M_+} = (-1)^{j-1} dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_n, \ 1 \leq j \leq n \tag{2.46}$$

where β_j is as (2.3).

ii) *If M has ∂M_- time-like boundary and D space-like unit outward normal, $w_{\partial M_-}$ volume element of ∂M_- is as follows:*

$$w_{\partial M_-} = \sum_{j=1}^{n}(-1)^{j-1} n_j dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_n \tag{2.47}$$

or

$$\varepsilon_j . n_j . w_{\partial M_-} = (-1)^{j-1} dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_n, \ 1 \leq j \leq n \tag{2.48}$$

where ε_j is as Eq. (2.4).

Theorem 2.15. ***(Stokes Theorem)*** *Let M be an n-dimensional compact, orientable semi-Riemannian manifold with nondegenerate boundary $\partial M'$ $(\partial M' = \partial M_+ \cup \partial M_-)$. If $\alpha \epsilon \wedge^{n-1}(M)$, then [4]*

$$\int_M d\alpha = -\int_{\partial M_+} \alpha - \int_{\partial M_-} \alpha \tag{2.49}$$

Theorem 2.16. ***(Green Formula)*** *Let M be an n-dimensional compact, orientable semi-Riemannian manifold with nondegenerate boundary $\partial M'$ $(\partial M' = \partial M_+ \cup \partial M_-)$ and let f and h be two differentiable functions on M. Then [4],*

$$\int_M (\langle gradf, gradh \rangle + f\Delta h) dv = \int_{\partial M_+} f h_n w_{\partial M_+} - \int_{\partial M_-} f h_n w_{\partial M_-} \tag{2.50}$$

where dv, $w_{\partial M_+}$, $w_{\partial M_-}$ are the volume elements on M, ∂M_+, ∂M_-, respectively.

3. Reilly's Formula

Let M be an n-dimensional compact, orientable semi-Riemannian manifold with nondegenerate boundary $\partial M'(\partial M' = \partial M_+ \cup \partial M_-)$. Thus, for $w\epsilon \wedge^1 (M)$ we can write the following from Stokes theorem

$$-\int_{\partial M_+} * w - \int_{\partial M_-} * w = \int_M d * w \tag{3.1}$$

where $*$ is the Hodge-star operator. Throughout this paper, Δ_1 is the Laplace operator on ∂M_+ and Δ_2 is the Laplace operator on ∂M_-.

Theorem 3.1. *Let M be an n-dimensional compact orientable semi-Riemannian manifold with nondegenerate boundary $\partial M'(\partial M' = \partial M_+ \cup \partial M_-)$. Let f be a differentiable function on semi-Riemannian manifold M and dv be a volume element of M. Then*

$$\begin{aligned}
&\int_M [Ric(gradf, gradf) + \langle gradf, grad\Delta f\rangle + Hessf.Hessf]dv \qquad (3.2)\\
&= -\int_{\partial M_+} \sum_{i,j=1}^{n} \beta_j (f_j f_{ji})(-1)^{i-1}\sqrt{|g|}g^{ii} dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n\\
&\quad -\int_{\partial M_-} \sum_{i,j=1}^{n} \varepsilon_j (f_j f_{ji})(-1)^{i-1}\sqrt{|g|}g^{ii} dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n
\end{aligned}$$

Proof: i) Let's choose 1-form $w\epsilon \wedge^1 (M)$ as follows:

$$w = \sum_{i,j=1}^{n} \beta_j (f_j f_{ji}) dx_i$$

Here, considering the property of Hodge-star operator, we have

$$* w = \sum_{i,j=1}^{n} \beta_j (f_j f_{ji})(-1)^{i-1}\sqrt{|g|}g^{ii} dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n \tag{3.3}$$

Using exterior differential for (3.3) and using (2.16) and (2.8), we have

$$d * w = [\sum_{i,j=1}^{n} \beta_i\beta_j f_{ji} f_{ji} + \sum_{i,j=1}^{n} \beta_i\beta_j f_j f_{jii}]dv \tag{3.4}$$

Considering (2.12), (2.23) and (2.19) in (3.4), respectively, we get

$$d * w = [\ Hessf.Hessf + \sum_{i,j=1}^{n} R^i_{jij} f_j f_j + \sum_{i,j=1}^{n} \beta_i\beta_j]dv \tag{3.5}$$

Using (2.14) and (2.15) in the second term of (3.5) and using (2.10), (2.11) and (2.7) in the last term of (3.5), we obtain

$$d * w = [\ Hessf.Hessf + Ric(gradf, gradf) + \langle gradf, grad\Delta f\rangle]dv \tag{3.6}$$

Consequently, considering (3.3) and (3.6) in (3.1), we have

$$\begin{aligned}&\int_M [Ric(gradf, gradf) + \langle gradf, grad\Delta f\rangle + Hessf.Hessf]dv \\ &= -\int_{\partial M_+} \sum_{i,j=1}^{n} \beta_j (f_j f_{ji})(-1)^{i-1}\sqrt{|g|}g^{ii}dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n\end{aligned} \tag{3.7}$$

ii) Suppose that $w\epsilon \wedge^1 (M)$

$$w = \sum_{i,j=1}^{n} \varepsilon_j (f_j f_{ji})dx_i$$

Here, considering the property of Hodge-star operator, we have

$$* w = \sum_{i,j=1}^{n} \varepsilon_j (f_j f_{ji})(-1)^{i-1}\sqrt{|g|}g^{ii}dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n \tag{3.8}$$

Using exterior differential for (3.8) and using (2.16) and (2.8), we have

$$d * w = [\sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j f_{ji}f_{ji} + \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j f_j f_{jii}]dv \tag{3.9}$$

Writing (2.12) in the first term on the right hand of (3.9), we get

$$d * w = [\ Hessf.Hessf + \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j f_j f_{jii}]dv$$

Using (2.23) and (2.19) in the second term of the above equality, we have

$$d * w = [\ Hessf.Hessf + \sum_{i,j=1}^{n} R^i_{jij} f_j f_j + \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j f_j f_{iij}]dv \tag{3.10}$$

Considering (2.14) and (2.15) in the second term of (3.10) and (2.10), (2.11) and (2.7) in the last term of (3.10), we obtain

$$d * w = [\ Hessf.Hessf + Ric(gradf, gradf) + \langle gradf, grad\Delta f\rangle]dv$$

Consequently, substituting the above equality and (3.8) in (3.1), we get

$$\int_M [Ric(gradf, gradf) + \langle gradf, grad\Delta f\rangle + Hessf.Hessf]dv \tag{3.11}$$
$$= -\int_{\partial M_-} \sum_{i,j=1}^{n} \varepsilon_j (f_j f_{ji})(-1)^{i-1}\sqrt{|g|}g^{ii}dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n$$

Because of the fact that $\partial M = \partial M_+ \cup \partial M_-$, considering (3.7) and (3.11) together, we have (3.2). This completes the proof of theorem. □

Theorem 3.2. *Let M be an n-dimensional compact orientable semi-Riemannian manifold with nondegenerate boundary $\partial M'(\partial M' = \partial M_+ \cup \partial M_-)$. Let f be a differentiable function on semi-Riemannian manifold M and dv, $w_{\partial M_+}$, $w_{\partial M_-}$ be the volume elements of M, ∂M_+, ∂M_- respectively. Then,*

$$\int_M [Ric(gradf, gradf) + \langle gradf, grad\Delta f\rangle + Hessf.Hessf]dv$$
$$= \int_{\partial M_+} \sum_{j=1}^{n} \beta_j (f_j f_{jn}) w_{\partial M_+} - \int_{\partial M_-} \sum_{j=1}^{n} \varepsilon_j (f_j f_{jn}) w_{\partial M_-} \tag{3.12}$$

Proof: i) Using the first term on the right hand of (3.2) in Theorem 3.1, for $i = 1, 2, ..., n$, we have

$$\begin{aligned} * w \;=\; & \sum_{j=1}^{n} \beta_j (f_j f_{j1})(-1)^{1-1} g^{11}\sqrt{|g|}\widehat{dx_1} \wedge dx_2 \wedge ... \wedge dx_n \\ & + \sum_{j=1}^{n} \beta_j (f_j f_{j2})(-1)^{2-1} g^{22}\sqrt{|g|}dx_1 \wedge \widehat{dx_2} \wedge ... \wedge dx_n \\ & . \\ & . \\ & . \\ & + \sum_{j=1}^{n} \beta_j (f_j f_{jn})(-1)^{n-1} g^{nn}\sqrt{|g|}dx_1 \wedge dx_2 \wedge ... \wedge \widehat{dx_n} \end{aligned} \tag{3.13}$$

Considering (2.45) and (2.46) in (3.13), we get

$$* w = \sum_{j=1}^{n} \beta_j (f_j f_{jn}) g^{nn}\sqrt{|g|} w_{\partial M_+} \tag{3.14}$$

From (2.46), considering $\sqrt{|g|} = 1$ and $g^{nn} = -1$, we obtain

$$* w = -\sum_{j=1}^{n} \beta_j (f_j f_{jn}) w_{\partial M_+} \tag{3.15}$$

ii) Using the second term on the right hand of (3.2) in Theorem 3.1, for $i = 1, 2, ..., n$ we get

$$
\begin{aligned}
* w &= \sum_{j=1}^{n} \varepsilon_j (f_j f_{j1})(-1)^{1-1} g^{11} \sqrt{|g|}\widehat{dx_1} \wedge dx_2 \wedge ... \wedge dx_n \\
&+ \sum_{j=1}^{n} \varepsilon_j (f_j f_{j2})(-1)^{2-1} g^{22} \sqrt{|g|} dx_1 \wedge \widehat{dx_2} \wedge ... \wedge dx_n \\
&. \\
&. \\
&. \\
&+ \sum_{j=1}^{n} \varepsilon_j (f_j f_{jn})(-1)^{n-1} g^{nn} \sqrt{|g|} dx_1 \wedge dx_2 \wedge ... \wedge \widehat{dx_n}
\end{aligned}
\tag{3.16}
$$

Substituting (2.48) in (3.16), we have

$$
* w = \sum_{j=1}^{n} \varepsilon_j (f_j f_{jn}) g^{nn} \sqrt{|g|} w_{\partial M_-} \tag{3.17}
$$

Here, considering $\sqrt{|g|} = 1$ and $g^{nn} = 1$, we obtain

$$
* w = \sum_{j=1}^{n} \varepsilon_j (f_j f_{jn}) w_{\partial M_-} \tag{3.18}
$$

Substituting (3.15) and (3.18) on the right hand of (3.2), we get (3.12).
This completes the proof of theorem. □

Theorem 3.3. ***(Reilly's Formula)*** *Let M be an n-dimensional compact orientable semi-Riemannian manifold with nondegenerate boundary $\partial M'(\partial M' = \partial M_+ \cup \partial M_-)$. Let f be a differentiable function on semi-Riemannian manifold M. We define the following:*

$$
z_1 = f \mid_{\partial M_+}, \qquad u_1 = f_n, \qquad H_1 = \frac{1}{n-1} \sum_{i=1}^{n-1} \beta_i II_{ii} \tag{3.19}
$$

and

$$
z_2 = f \mid_{\partial M_-}, \qquad u_2 = f_n, \qquad H_2 = \frac{1}{n-1} \sum_{i=1}^{n-1} \varepsilon_i II_{ii} \tag{3.20}
$$

Hence

$$\int\limits_{M}[(\Delta f)^2 - Ric(gradf, gradf) - Hessf.Hessf]dv$$
$$= \{\int\limits_{\partial M_+}[(\Delta_1 z)u - (n-1)Hu^2 - \langle gradz, gradu\rangle_1$$
$$+II(gradz, gradz)]w_{\partial M_+}\}$$
$$-\{\int\limits_{\partial M_-}[(\Delta_2 z)u + (n-1)Hu^2 - \langle gradz, gradu\rangle_2$$
$$+II(gradz, gradz)]w_{\partial M_-}\}$$

where $w_{\partial M_+}$, $w_{\partial M_-}$, dv *are the volume elements of* ∂M_+, ∂M_- *and* M, *respectively.*

Proof: Considering (2.50) in (3.12) and making routine calculations, we have

$$\int\limits_{M}[(\Delta f)^2 - Ric(gradf, gradf) - Hessf.Hessf]dv$$
$$= \int\limits_{\partial M_+}[f_n\sum_{i=1}^{n-1}\beta_i f_{ii} - \sum_{i=1}^{n-1}\beta_i f_i f_{in}]w_{\partial M_+} \tag{3.21}$$
$$- \int\limits_{\partial M_-}[f_n\sum_{i=1}^{n-1}\varepsilon_i f_{ii} - \sum_{i=1}^{n-1}\varepsilon_i f_i f_{in}]w_{\partial M_-}$$

Writing the value of f_{ii} from (2.38), we have

$$f_{ii} = z_{ii} - uII_{ii}$$

Summing up the last equality over $i = 1, 2, ..., n-1$ and multiplying it with β_i and f_n, we obtain

$$f_n\sum_{i=1}^{n-1}\beta_i f_{ii} = (\sum_{i=1}^{n-1}\beta_i z_{ii} - u\sum_{i=1}^{n-1}\beta_i II_{ii})f_n \tag{3.22}$$

Considering (2.11) and (3.19) in (3.22), we have

$$f_n\sum_{i=1}^{n-1}\beta_i f_{ii} = (\Delta_1 z - (n-1)Hu)u \tag{3.23}$$

Now, let's compute the second term on the right hand of (3.21). Multiplying (2.39) with $\beta_i f_i$ and summing up it for $i = 1, 2, ..., n-1$, we get

$$\sum_{i=1}^{n-1}\beta_i f_i f_{ni} = \sum_{i=1}^{n-1}\beta_i u_i f_i - \sum_{i,j=1}^{n-1}\beta_i\beta_j II_{ij}z_j f_i$$

or

$$\sum_{i=1}^{n-1} f_i f_{ni} = \langle gradz, gradu \rangle_1 - II(gradz, gradz) \tag{3.24}$$

Also, in the third term on the right hand of (3.21), the value of f_{ii} can be written as the following from (2.43),

$$f_{ii} = z_{ii} + uII_{ii} \tag{3.25}$$

Multiplying (3.25) with ε_i and summing up it over $i = 1, 2, ..., n-1$, we have

$$\sum_{i=1}^{n-1} \varepsilon_i f_{ii} = \sum_{i=1}^{n-1} \varepsilon_i z_{ii} + u \sum_{i=1}^{n-1} \varepsilon_i II_{ii}$$

Here, considering (2.11) and (3.20), we get

$$f_n \sum_{i=1}^{n-1} \varepsilon_i f_{ii} = (\Delta_2 z + (n-1)Hu)u \tag{3.26}$$

Finally, the fourth term on the right hand of (3.21) can be written as the following from (2.44),

$$f_{ni} = u_i - \sum_{j=1}^{n-1} \varepsilon_j z_j II_{ij}$$

Multiplying the last equality with $\varepsilon_i f_i$ and summing up it over $i = 1, 2, ..., n-1$, we obtain

$$\sum_{i=1}^{n-1} \varepsilon_i f_i f_{ni} = \sum_{i=1}^{n-1} \varepsilon_i f_i u_i - u \sum_{i,j=1}^{n-1} \varepsilon_i \varepsilon_j f_i z_j II_{ij}$$

or

$$\sum_{i=1}^{n-1} \varepsilon_i f_i f_{ni} = \langle gradz, gradu \rangle_2 - II(gradz, gradz) \tag{3.27}$$

Thus, substituting (3.23), (3.24), (3.27) and (3.26) in (3.21), we obtain

$$\begin{aligned}
&\int_M [(\Delta f)^2 - Ric(gradf, gradf) - Hessf.Hessf]dv \\
&= \{ \int_{\partial M_+} [(\Delta_1 z)u - (n-1)Hu^2 - \langle gradz, gradu\rangle_1 \\
&+II(gradz, gradz)]w_{\partial M_+}\} \\
&-\{ \int_{\partial M_-} [(\Delta_2 z)u + (n-1)Hu^2 - \langle gradz, gradu\rangle_2 \\
&+II(gradz, gradz)]w_{\partial M_-}\}
\end{aligned}$$

This completes the proof of theorem. □

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